# Graduate Texts in Mathematics

Serge Lang

# Real and Functional Analysis

Third Edition

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Serge Lang

# Real and Functional Analysis

Third Edition

With 37 Illustrations

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# Foreword

This book is meant as a text for a first year graduate course in analysis. Any standard course in undergraduate analysis will constitute sufficient preparation for its understanding, for instance, my *Undergraduate Analysis*. I assume that the reader is acquainted with notions of uniform convergence and the like.

In this third edition, I have reorganized the book by covering integration before functional analysis. Such a rearrangement fits the way courses are taught in all the places I know of. I have added a number of examples and exercises, as well as some material about integration on the real line (e.g. on Dirac sequence approximation and on Fourier analysis), and some material on functional analysis (e.g. the theory of the Gelfand transform in Chapter XVI). These upgrade previous exercises to sections in the text.

In a sense, the subject matter covers the same topics as elementary calculus, viz. linear algebra, differentiation and integration. This time, however, these subjects are treated in a manner suitable for the training of professionals, i.e. people who will use the tools in further investigations, be it in mathematics, or physics, or what have you.

In the first part, we begin with point set topology, essential for all analysis, and we cover the most important results.

I am selective here, since this part is regarded as a tool, especially Chapters I and II. Many results are easy, and are less essential than those in the text. They have been given in exercises, which are designed to acquire facility in routine techniques and to give flexibility for those who want to cover some of them at greater length. The point set topology simply deals with the basic notions of continuity, open and closed sets, connectedness, compactness, and continuous functions. The chapter

concerning continuous functions on compact sets properly emphasizes results which already mix analysis and uniform convergence with the language of point set topology.

In the second part, Chapters IV and V, we describe briefly the two basic linear spaces of analysis, namely Banach spaces and Hilbert spaces.

The next part deals extensively with integration.

We begin with the development of the integral. The fashion has been to emphasize positivity and ordering properties (increasing and decreasing sequences). I find this excessive. The treatment given here attempts to give a proper balance between $L^1$-convergence and positivity. For more detailed comments, see the introduction to Part Three and Chapter VI.

The chapters on applications of integration and distributions provide concrete examples and choices for leading the course in other directions, at the taste of the lecturer. The general theory of integration in measured spaces (with respect to a given positive measure) alternates with chapters giving specific results of integration on euclidean spaces or the real line. Neither is slighted at the expense of the other. In this third edition, I have added some material on functions of bounded variation, and I have emphasized convolutions and the approximation by Dirac sequences or families even more than in the previous editions, for instance, in Chapter VIII, §2.

For want of a better place, the calculus (with values in a Banach space) now occurs as a separate part after dealing with integration, and before the functional analysis.

The differential calculus is done because at best, most people will only be acquainted with it only in euclidean space, and incompletely at that. More importantly, the calculus in Banach spaces has acquired considerable importance in the last two decades, because of many applications like Morse theory, the calculus of variations, and the Nash–Moser implicit mapping theorem, which lies even further in this direction since one has to deal with more general spaces than Banach spaces. These results pertain to the geometry of function spaces. Cf. the exercises of Chapter XIV for simpler applications.

The next part deals with functional analysis. The purpose here is twofold. We place the linear algebra in an infinite dimensional setting where continuity assumptions are made on the linear maps, and we show how one can "linearize" a problem by taking derivatives, again in a setting where the theory can be applied to function spaces. This part includes several major spectral theorems of analysis, showing how we can extend to the infinite dimensional case certain results of finite dimensional linear algebra. The compact and Fredholm operators have applications to integral operators and partial differential elliptic operators (e.g. in papers of Atiyah–Singer and Atiyah–Bott).

Chapters XIX and XXIX, on unbounded hermitian operators, combine

both the linear algebra and integration theory in the study of such operators. One may view the treatment of spectral measures as providing an example of general integration theory on locally compact spaces, whereby a measure is obtained from a functional on the space of continuous functions with compact support.

I find it appropriate to introduce students to differentiable manifolds during this first year graduate analysis course, not only because these objects are of interest to differential geometers or differential topologists, but because global analysis on manifolds has come into its own, both in its integral and differential aspects. It is therefore desirable to integrate manifolds in analysis courses, and I have done this in the last part, which may also be viewed as providing a good application of integration theory.

A number of examples are given in the text but many interesting examples are also given in the exercises (for instance, explicit formulas for approximations whose existence one knows abstractly by the Weierstrass–Stone theorem; integral operators of various kinds; etc). The exercises should be viewed as an integral part of the book. Note that Chapters XIX and XX, giving the spectral measure, can be viewed as providing an example for many notions which have been discussed previously: operators in Hilbert space, measures, and convolutions. At the same time, these results lead directly into the real analysis of the working mathematician.

As usual, I have avoided as far as possible building long chains of logical interdependence, and have made chapters as logically independent as possible, so that courses which run rapidly through certain chapters, omitting some material, can cover later chapters without being logically inconvenienced.

The present book can be used for a two-semester course, omitting some material. I hope I have given a suitable overview of the basic tools of analysis. There might be some reason to include other topics, such as the basic theorems concerning elliptic operators. I have omitted this topic and some others, partly because the appendices to my $SL_2(\mathbf{R})$ constitutes a sub-book which contains these topics, and partly because there is no time to cover them in the basic one year course addressed to graduate students.

The present book can also be used as a reference for basic analysis, since it offers the reader the opportunity to select various topics without reading the entire book. The subject matter is organized so that it makes the topics available to as wide an audience as possible.

There are many very good books in intermediate analysis, and interesting research papers, which can be read immediately after the present course. A partial list is given in the Bibliography. In fact, the determination of the material included in this *Real and Functional Analysis* has been greatly motivated by the existence of these papers and books, and by the need to provide the necessary background for them.

Finally, I thank all those people who have made valuable comments and corrections, especially Keith Conrad, Martin Mohlenkamp, Takesi Yamanaka, and Stephen Chiappari, who reviewed the book for Springer-Verlag.

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# Contents

# PART ONE

# General Topology

CHAPTER I

# Sets

## I, §1. SOME BASIC TERMINOLOGY

We assume that the reader understands the meaning of the word "set", and in this chapter, summarize briefly the basic properties of sets and operations between sets. We denote the empty set by $\emptyset$. A subset $S'$ of $S$ is said to be **proper** if $S' \neq S$. We write $S' \subset S$ or $S \supset S'$ to denote the fact that $S'$ is a subset of $S$.

Let $S$, $T$ be sets. A **mapping** or **map** $f: T \to S$ is an association which to each element $x \in T$ associates an element of $S$, denoted by $f(x)$, and called the **value** of $f$ at $x$, or the **image** of $x$ under $f$. If $T'$ is a subset of $T$, we denote by $f(T')$ the subset of $S$ consisting of all elements $f(x)$ for $x \in T'$. The association of $f(x)$ to $x$ is denoted by the special arrow

$$x \mapsto f(x).$$

We usually reserve the word function for a mapping whose values are in the real or complex numbers. The **characteristic function** of a subset $S'$ of $S$ is the function $\chi$ such that $\chi(x) = 1$ if $x \in S'$ and $\chi(x) = 0$ if $x \notin S'$. We often write $\chi_{S'}$ for this function.

Let $X$, $Y$ be sets. A map $f: X \to Y$ is said to be **injective** if for all $x$, $x' \in X$ with $x \neq x'$ we have $f(x) \neq f(x')$. We say that $f$ is **surjective** if $f(X) = Y$, i.e. if the image of $f$ is all of $Y$. We say that $f$ is **bijective** if it is both injective and surjective. As usual, one should index a map $f$ by its set of arrival and set of departure to have absolutely correct notation, but this is too clumsy, and the context is supposed to make it clear what these sets are. For instance, let $\mathbf{R}$ denote the real numbers, and $\mathbf{R}'$ the

real numbers $\geqq 0$. The map

$$f_{\mathbf{R}}^{\mathbf{R}}: \mathbf{R} \to \mathbf{R}$$

given by $x \mapsto x^2$ is not surjective, but the map

$$f_{\mathbf{R}'}^{\mathbf{R}}: \mathbf{R} \to \mathbf{R}'$$

given by the same formula is surjective.

If $f: X \to Y$ is a map and $S$ a subset of $X$, we denote by

$$f|S$$

the restriction of $f$ to $S$, namely the map $f$ viewed as a map defined only on $S$. For instance, if $f: \mathbf{R} \to \mathbf{R}'$ is the map $x \mapsto x^2$, then $f$ is not injective, but $f|\mathbf{R}'$ is injective. We often let $f_S = f\chi_S$ be the function equal to $f$ on $S$ and 0 outside $S$.

A composite of injective maps is injective, and a composite of surjective maps is surjective. Hence a composite of bijective maps is bijective.

We denote by $\mathbf{Q}$, $\mathbf{Z}$ the sets of rational numbers and integers respectively. We denote by $\mathbf{Z}^+$ the set of positive integers (integers $> 0$), and similarly by $\mathbf{R}^+$ the set of positive reals. We denote by $\mathbf{N}$ the set of natural numbers (integers $\geqq 0$), and by $\mathbf{C}$ the complex numbers. A mapping into $\mathbf{R}$ or $\mathbf{C}$ will be called a **function**.

Let $S$ and $I$ be sets. By a **family of elements of** $S$, **indexed by** $I$, one means simply a map $f: I \to S$. However, when we speak of a family, we write $f(i)$ as $f_i$, and also use the notation $\{f_i\}_{i \in I}$ to denote the family.

**Example 1.** Let $S$ be the set consisting of the single element 3. Let $I = \{1, \ldots, n\}$ be the set of integers from 1 to $n$. A family of elements of $S$, indexed by $I$, can then be written $\{a_i\}_{i=1,\ldots,n}$ with each $a_i = 3$. Note that a family is different from a subset. The same element of $S$ may receive distinct indices.

A family of elements of a set $S$ indexed by positive integers, or non-negative integers, is also called a **sequence**.

**Example 2.** A sequence of real numbers is written frequently in the form

$$\{x_1, x_2, \ldots\} \qquad \text{or} \qquad \{x_n\}_{n \geqq 1}$$

and stands for the map $f: \mathbf{Z}^+ \to \mathbf{R}$ such that $f(i) = x_i$. As before, note that a sequence can have all its elements equal to each other, that is

$$\{1, 1, 1, \ldots\}$$

is a sequence of integers, with $x_i = 1$ for each $i \in \mathbf{Z}^+$.

We define a **family of sets indexed by a set** $I$ in the same manner, that is, a family of sets indexed by $I$ is an assignment

$$i \mapsto S_i$$

which to each $i \in I$ associates a set $S_i$. The sets $S_i$ may or may not have elements in common, and it is conceivable that they may all be equal. As before, we write the family $\{S_i\}_{i \in I}$.

We can define the intersection and union of families of sets, just as for the intersection and union of a finite number of sets. Thus, if $\{S_i\}_{i \in I}$ is a family of sets, we define the **intersection** of this family to be the set

$$\bigcap_{i \in I} S_i$$

consisting of all elements $x$ which lie in all $S_i$. We define the **union**

$$\bigcup_{i \in I} S_i$$

to be the set consisting of all $x$ such that $x$ lies in some $S_i$.

If $S$, $S'$ are sets, we define $S \times S'$ to be the set of all pairs $(x, y)$ with $x \in S$ and $y \in S'$. We can define finite products in a similar way. If $S_1$, $S_2, \ldots$ is a sequence of sets, we define the product

$$\prod_{i=1}^{\infty} S_i$$

to be the set of all sequences $(x_1, x_2, \ldots)$ with $x_i \in S_i$. Similarly, if $I$ is an indexing set, and $\{S_i\}_{i \in I}$ a family of sets, we define the product

$$\prod_{i \in I} S_i$$

to be the set of all families $\{x_i\}_{i \in I}$ with $x_i \in S_i$.

Let $X$, $Y$, $Z$ be sets. We have the formula

$$(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z).$$

To prove this, let $(w, z) \in (X \cup Y) \times Z$ with $w \in X \cup Y$ and $z \in Z$. Then $w \in X$ or $w \in Y$. Say $w \in X$. Then $(w, z) \in X \times Z$. Thus

$$(X \cup Y) \times Z \subset (X \times Z) \cup (Y \times Z).$$

Conversely, $X \times Z$ is contained in $(X \cup Y) \times Z$ and so is $Y \times Z$. Hence their union is contained in $(X \cup Y) \times Z$, thereby proving our assertion.

We say that two sets $X$, $Y$ are **disjoint** if their intersection is empty. We say that a union $X \cup Y$ is **disjoint** if $X$ and $Y$ are disjoint. Note that if $X$, $Y$ are disjoint, then $(X \times Z)$ and $(Y \times Z)$ are disjoint.

We can take products with arbitrary families. For instance, if $\{X_i\}_{i \in I}$ is a family of sets, then

$$\left(\bigcup_{i \in I} X_i\right) \times Z = \bigcup_{i \in I} (X_i \times Z).$$

If the family $\{X_i\}_{i \in I}$ is disjoint (that is $X_i \cap X_j$ is empty if $i \neq j$ for $i$, $j \in I$), then the sets $X_i \times Z$ are also disjoint.

We have similar formulas for intersections. For instance,

$$(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z).$$

We leave the proof to the reader.

Let $X$ be a set and $Y$ a subset. The **complement** of $Y$ in $X$, denoted by $\mathscr{C}_X Y$, or $X - Y$, is the set of all elements $x \in X$ such that $x \notin Y$. If $Y$, $Z$ are subsets of $X$, then we have the following formulas:

$$\mathscr{C}_X(Y \cup Z) = \mathscr{C}_X Y \cap \mathscr{C}_X Z,$$
$$\mathscr{C}_X(Y \cap Z) = \mathscr{C}_X Y \cup \mathscr{C}_X Z.$$

These are essentially reformulations of definitions. For instance, suppose $x \in X$ and $x \notin (Y \cup Z)$. Then $x \notin Y$ and $x \notin Z$. Hence $x \in \mathscr{C}_X Y \cap \mathscr{C}_X Z$. Conversely, if $x \in \mathscr{C}_x Y \cap \mathscr{C}_X Z$, then $x$ lies neither in $Y$ nor in $Z$, and hence $x \in \mathscr{C}_X(Y \cup Z)$. This proves the first formula. We leave the second to the reader. Exercise: Formulate these formulas for the complement of the union of a family of sets, and the complement of the intersection of a family of sets.

Let $A$, $B$ be sets and $f\colon A \to B$ a mapping. If $Y$ is a subset of $B$, we define $f^{-1}(Y)$ to be the set of all $x \in A$ such that $f(x) \in Y$. It may be that $f^{-1}(Y)$ is empty, of course. We call $f^{-1}(Y)$ the **inverse image** of $Y$ (under $f$). If $f$ is injective, and $Y$ consists of one element $y$, then $f^{-1}(\{y\})$ is either empty or has precisely one element.

The following statements are easily proved:

If $f\colon A \to B$ is a map, and $Y$, $Z$ are subsets of $B$, then

$$f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z),$$
$$f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z).$$

More generally, if $\{Y_i\}_{i \in I}$ is a family of subsets of $B$, then

$$f^{-1}\left(\bigcup_{i \in I} Y_i\right) = \bigcup_{i \in I} f^{-1}(Y_i),$$

and similarly for the intersection. Furthermore, if we denote by $Y - Z$ the set of all elements $y \in Y$ and $y \notin Z$, then

$$f^{-1}(Y - Z) = f^{-1}(Y) - f^{-1}(Z).$$

In particular,

$$f^{-1}(\mathscr{C}_B Z) = \mathscr{C}_A f^{-1}(Z).$$

Thus the operation $f^{-1}$ commutes with all set theoretic operations.

## I, §2. DENUMERABLE SETS

Let $n$ be a positive integer. Let $J_n$ be the set consisting of all integers $k$, $1 \leqq k \leqq n$. If $S$ is a set, we say that $S$ has $n$ elements if there is a bijection between $S$ and $J_n$. Such a bijection associates with each integer $k$ as above an element of $S$, say $k \mapsto a_k$. Thus we may use $J_n$ to "count" $S$. Part of what we assume about the basic facts concerning positive integers is that if $S$ has $n$ elements, then the integer $n$ is uniquely determined by $S$.

One also agrees to say that a set has 0 elements if the set is empty.

We shall say that a set $S$ is **denumerable** if there exists a bijection of $S$ with the set of positive integers $\mathbf{Z}^+$. Such a bijection is then said to **enumerate** the set $S$. It is a mapping

$$n \mapsto a_n$$

which to each positive integer $n$ associates an element of $S$, the mapping being injective and surjective.

If $D$ is a denumerable set, and $f: S \to D$ is a bijection of some set $S$ with $D$, then $S$ is also denumerable. Indeed, there is a bijection $g: D \to \mathbf{Z}^+$, and hence $g \circ f$ is a bijection of $S$ with $\mathbf{Z}^+$.

Let $T$ be a set. A **sequence** of elements of $T$ is simply a mapping of $\mathbf{Z}^+$ into $T$. If the map is given by the association $n \mapsto x_n$, we also write the sequence as $\{x_n\}_{n \geqq 1}$, or also $\{x_1, x_2, \ldots\}$. For simplicity, we also write $\{x_n\}$ for the sequence. Thus we think of the sequence as prescribing a first, second, ..., $n$-th element of $T$. We use the same braces for sequences as for sets, but the context will always make our meaning clear.

**Examples.** The even positive integers may be viewed as a sequence $\{x_n\}$ if we put $x_n = 2n$ for $n = 1, 2, \ldots$. The odd positive integers may also be viewed as a sequence $\{y_n\}$ if we put $y_n = 2n - 1$ for $n = 1, 2, \ldots$. In each case, the sequence gives an enumeration of the given set.

We also use the word sequence for mappings of the natural numbers into a set, thus allowing our sequences to start from 0 instead of 1. If we

need to specify whether a sequence starts with the 0-th term or the first term, we write

$$\{x_n\}_{n \geqq 0} \qquad \text{or} \qquad \{x_n\}_{n \geqq 1}$$

according to the desired case. Unless otherwise specified, however, we always assume that a sequence will start with the first term. Note that from a sequence $\{x_n\}_{n \geqq 0}$ we can define a new sequence by letting $y_n = x_{n-1}$ for $n \geqq 1$. Then $y_1 = x_0$, $y_2 = x_1$, .... Thus there is no essential difference between the two kinds of sequences.

Given a sequence $\{x_n\}$, we call $x_n$ the $n$-th term of the sequence. A sequence may very well be such that all its terms are equal. For instance, if we let $x_n = 1$ for all $n \geqq 1$, we obtain the sequence $\{1, 1, 1, \ldots\}$. Thus there is a difference between a sequence of elements in a set $T$, and a subset of $T$. In the example just given, the set of all terms of the sequence consists of one element, namely the single number 1.

Let $\{x_1, x_2, \ldots\}$ be a sequence in a set $S$. By a **subsequence** we shall mean a sequence $\{x_{n_1}, x_{n_2}, \ldots\}$ such that $n_1 < n_2 < \cdots$. For instance, if $\{x_n\}$ is the sequence of positive integers, $x_n = n$, the sequence of even positive integers $\{x_{2n}\}$ is a subsequence.

An enumeration of a set $S$ is of course a sequence in $S$.

A set is **finite** if the set is empty, or if the set has $n$ elements for some positive integer $n$. If a set is not finite, it is called **infinite**.

Occasionally, a map of $J_n$ into a set $T$ will be called a **finite sequence** in $T$. A finite sequence is written as usual,

$$\{x_1, \ldots, x_n\} \qquad \text{or} \qquad (x_i)_{i=1,\ldots,n}.$$

When we need to specify the distinction between finite sequences and maps of $\mathbf{Z}^+$ into $T$, we call the latter infinite sequences. Unless otherwise specified, we shall use the word "sequence" to mean infinite sequence.

**Proposition 2.1.** *Let $D$ be an infinite subset of $\mathbf{Z}^+$. Then $D$ is denumerable, and in fact there is a unique enumeration of $D$, namely $\{k_1, k_2, \ldots\}$ such that*

$$k_1 < k_2 < \cdots < k_n < k_{n+1} < \cdots.$$

*Proof.* We let $k_1$ be the smallest element of $D$. Suppose inductively that we have defined $k_1 < \cdots < k_n$ in such a way that any element $k$ in $D$ which is not equal to $k_1, \ldots, k_n$ is $> k_n$. We define $k_{n+1}$ to be the smallest element of $D$ which is $> k_n$. Then the map $n \mapsto k_n$ is the desired enumeration of $D$.

**Corollary 2.2.** *Let $S$ be a denumerable set and $D$ an infinite subset of $S$. Then $D$ is denumerable.*

*Proof.* Given an enumeration of $S$, the subset $D$ corresponds to a subset of $\mathbf{Z}^+$ in this enumeration. Using Proposition 2.1 we conclude that we can enumerate $D$.

**Proposition 2.3.** *Every infinite set contains a denumerable subset.*

*Proof.* Let $S$ be a infinite set. For every non-empty subset $T$ of $S$, we select a definite element $a_T$ in $T$. We then proceed by induction. We let $x_1$ be the chosen element $a_S$. Suppose that we have chosen $x_1, \ldots, x_n$ having the property that for each $k = 2, \ldots, n$ the element $x_k$ is the selected element in the subset which is the complement of $\{x_1, \ldots, x_{k-1}\}$. We let $x_{n+1}$ be the selected element in the complement of the set $\{x_1, \ldots, x_n\}$. By induction, we thus obtain an association $n \mapsto x_n$ for all positive integers $n$, and since $x_n \neq x_k$ for all $k < n$ it follows that our association is injective, i.e. gives an enumeration of a subset of $S$.

**Proposition 2.4.** *Let $D$ be a denumerable set, and $f: D \to S$ a surjective mapping. Then $S$ is denumerable or finite.*

*Proof.* For each $y \in S$, there exists an element $x_y \in D$ such that $f(x_y) = y$ because $f$ is surjective. The association $y \mapsto x_y$ is an injective mapping of $S$ into $D$, because if $y, z \in S$ and $x_y = x_z$, then

$$y = f(x_y) = f(x_z) = z.$$

Let $g(y) = x_y$. The image of $g$ is a subset of $D$ and is denumerable. Since $g$ is a bijection between $S$ and its image, it follows that $S$ is denumerable or finite.

**Proposition 2.5.** *Let $D$ be a denumerable set. Then $D \times D$ (the set of all pairs $(x, y)$ with $x, y \in D$) is denumerable.*

*Proof.* There is a bijection between $D \times D$ and $\mathbf{Z}^+ \times \mathbf{Z}^+$, so it will suffice to prove that $\mathbf{Z}^+ \times \mathbf{Z}^+$ is denumerable. Consider the mapping of $\mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ given by

$$(m, n) \mapsto 2^n 3^m.$$

In view of Proposition 2.1, it will suffice to prove that this mapping is injective. Suppose $2^n 3^m = 2^r 3^s$ for positive integers $n, m, r, s$. Say $r < n$. Dividing both sides by $2^r$, we obtain

$$2^k 3^m = 3^s$$

with $k = n - r \geqq 1$. Then the left-hand side is even, but the right-hand side is odd, so the assumption $r < n$ is impossible. Similarly, we cannot

have $n < r$. Hence $r = n$. Then we obtain $3^m = 3^s$. If $m > s$, then $3^{m-s} = 1$ which is impossible. Similarly, we cannot have $s > m$, whence $m = s$. Hence our map is injective, as was to be proved.

**Proposition 2.6.** *Let $\{D_1, D_2, \ldots\}$ be a sequence of denumerable sets. Let $S$ be the union of all sets $D_i$ $(i = 1, 2, \ldots)$. Then $S$ is denumerable.*

*Proof.* For each $i = 1, 2, \ldots$ we enumerate the elements of $D_i$, as indicated in the following notation:

$$\begin{gathered} D_1 \colon \{x_{11}, x_{12}, x_{13}, \ldots\} \\ D_2 \colon \{x_{21}, x_{22}, x_{23}, \ldots\} \\ \vdots \\ D_i \colon \{x_{i1}, x_{i2}, x_{i3}, \ldots\} \\ \vdots \end{gathered}$$

The map $f \colon \mathbf{Z}^+ \times \mathbf{Z}^+ \to D$ given by

$$f(i, j) = x_{ij}$$

is then a surjective map of $\mathbf{Z}^+ \times \mathbf{Z}^+$ onto $S$. By Proposition 2.4, it follows that $S$ is denumerable.

**Corollary 2.7.** *Let $F$ be a non-empty finite set and $D$ a denumerable set. Then $F \times D$ is denumerable. If $S_1$, $S_2$, $\ldots$ are a sequence of sets, each of which is finite or denumerable, then the union $S_1 \cup S_2 \cup \cdots$ is denumerable or finite.*

*Proof.* There is an injection of $F$ into $\mathbf{Z}^+$ and a bijection of $D$ with $\mathbf{Z}^+$. Hence there is an injection of $F \times D$ into $\mathbf{Z}^+ \times \mathbf{Z}^+$ and we can apply Corollary 2.2 and Proposition 2.6 to prove the first statement. One could also define a surjective map of $\mathbf{Z}^+ \times \mathbf{Z}^+$ onto $F \times D$. As for the second statement, each finite set is contained in some denumerable set, so that the second statement follows from Propositions 2.1 and 2.6.

For convenience, we shall say that a set is **countable** if it is either finite or denumerable.

## I, §3. ZORN'S LEMMA

In order to deal efficiently with infinitely many sets simultaneously, one needs a special property. To state it, we need some more terminology.

Let $S$ be a set. An **ordering** (also called partial ordering) of (or on) $S$

is a relation, written $x \leqq y$, among some pairs of elements of $S$, having the following properties.

**ORD 1.** *We have* $x \leqq x$.

**ORD 2.** *If* $x \leqq y$ *and* $y \leqq z$ *then* $x \leqq z$.

**ORD 3.** *If* $x \leqq y$ *and* $y \leqq x$ *then* $x = y$.

We sometimes write $y \geqq x$ for $x \leqq y$. Note that we don't require that the relation $x \leqq y$ or $y \leqq x$ hold for every pair of elements $(x, y)$ of $S$. Some pairs may not be comparable. If the ordering satisfies this additional property, then we say that it is a **total ordering**.

**Example 1.** Let $G$ be a group. Let $S$ be the set of subgroups. If $H$, $H'$ are subgroups of $G$, we define

$$H \leqq H'$$

if $H$ is a subgroup of $H'$. One verifies immediately that this relation defines an ordering on $S$. Given two subgroups, $H$, $H'$ of $G$, we do not necessarily have $H \leqq H'$ or $H' \leqq H$.

**Example 2.** Let $R$ be a ring, and let $S$ be the set of left ideals of $R$. We define an ordering in $S$ in a way similar to the above, namely if $L$, $L'$ are left ideals of $R$, we define

$$L \leqq L'$$

if $L \subset L'$.

**Example 3.** Let $X$ be a set, and $S$ the set of subsets of $X$. If $Y$, $Z$ are subsets of $X$, we define $Y \leqq Z$ if $Y$ is a subset of $Z$. This defines an ordering on $S$.

In all these examples, the relation of ordering is said to be that of inclusion.

In an ordered set, if $x \leqq y$ and $x \neq y$ we then write $x < y$.

Let $A$ be an ordered set, and $B$ a subset. Then we can define an ordering on $B$ by defining $x \leqq y$ for $x, y \in B$ to hold if and only if $x \leqq y$ in $A$. We shall say that it is the ordering on $B$ **induced** by the ordering on $A$, or is the **restriction** to $B$ of the partial ordering of $A$.

Let $S$ be an ordered set. By a **least** element of $S$ (or a **smallest** element) one means an element $a \in S$ such that $a \leqq x$ for all $x \in S$. Similarly, by a **greatest element** one means an element $b$ such that $x \leqq b$ for all $x \in S$.

By a **maximal element** $m$ of $S$ one means an element such that if $x \in S$ and $x \geqq m$, then $x = m$. Note that a maximal element need not be a greatest element. There may be many maximal elements in $S$, whereas if a greatest element exists, then it is unique (proof?).

Let $S$ be an ordered set. We shall say that $S$ is **totally ordered** if given $x, y \in S$ we have necessarily $x \leqq y$ or $y \leqq x$.

**Example 4.** The integers **Z** are totally ordered by the usual ordering. So are the real numbers.

Let $S$ be an ordered set, and $T$ a subset. An **upper bound** of $T$ (in $S$) is an element $b \in S$ such that $x \leqq b$ for all $x \in T$. A **least upper bound** of $T$ in $S$ is an upper bound $b$ such that if $c$ is another upper bound, then $b \leqq c$. We shall say that $S$ is **inductively ordered** if every non-empty totally ordered subset has an upper bound.

We shall say that $S$ is **strictly** inductively ordered if every non-empty totally ordered subset has a least upper bound.

In Examples 1, 2, 3, in each case, the set is strictly inductively ordered. To prove this, let us take Example 1. Let $T$ be a non-empty totally ordered subset of the set of subgroups of $G$. This means that if $H, H' \in T$, then $H \subset H'$ or $H' \subset H$. Let $U$ be the union of all sets in $T$. Then:

(1) $U$ is a subgroup. *Proof*: If $x, y \in U$, there exist subgroups $H, H' \in T$ such that $x \in H$ and $y \in H'$. If, say, $H \subset H'$, then both $x, y \in H'$ and hence $xy \in H'$. Hence $xy \in U$. Also, $x^{-1} \in H'$, so $x^{-1} \in U$. Hence $U$ is a subgroup.

(2) $U$ is an upper bound for each element of $T$. *Proof*: Every $H \in T$ is contained in $U$, so $H \leqq U$ for all $H \in T$.

(3) $U$ is a least upper bound for $T$. *Proof*: Any subgroup of $G$ which contains all the subgroups $H \in T$ must then contain their union $U$.

The proof that the sets in Examples 2, 3 are strictly inductively ordered is entirely similar.

We can now state the property mentioned at the beginning of the section.

**Zorn's Lemma.** *Let $S$ be a non-empty inductively ordered set. Then there exists a maximal element in $S$.*

Zorn's lemma could be just taken as an axiom of set theory. However, it is not psychologically completely satisfactory as an axiom, because its statement is too involved, and one does not visualize easily the existence of the maximal element asserted in that statement. We show how one can prove Zorn's lemma from other properties of sets which everyone would immediately grant as acceptable psychologically.

From now on to the end of the proof of Theorem 3.1, we let $A$ be a non-empty partially ordered and strictly inductively ordered set. We recall that **strictly inductively ordered** means that every non-empty totally ordered subset has a least upper bound. We assume given a map

$f: A \to A$ such that for all $x \in A$ we have $x \leqq f(x)$. We could call such a map an **increasing** map.

Let $a \in A$. Let $B$ be a subset of $A$. We shall say that $B$ is **admissible** if:

(1) $B$ contains $a$.
(2) We have $f(B) \subset B$.
(3) Whenever $T$ is a totally ordered subset of $B$, the least upper bound of $T$ in $A$ lies in $B$.

Then $B$ is also strictly inductively ordered, by the induced ordering of $A$. We shall prove:

**Theorem 3.1 (Bourbaki).** *Let $A$ be a non-empty partially ordered and strictly inductively ordered set. Let $f: A \to A$ be an increasing mapping. Then there exists an element $x_0 \in A$ such that $f(x_0) = x_0$.*

*Proof.* Suppose that $A$ were totally ordered. By assumption, it would have a least upper bound $b \in A$, and then

$$b \leqq f(b) \leqq b,$$

so that in this case, our theorem is clear. The whole problem is to reduce the theorem to that case. In other words, what we need to find is a totally ordered admissible subset of $A$.

If we throw out of $A$ all elements $x \in A$ such that $x$ is not $\geqq a$, then what remains is obviously an admissible subset. Thus without loss of generality, we may assume that $A$ has a least element $a$, that is $a \leqq x$ for all $x \in A$.

Let $M$ be the intersection of all admissible subsets of $A$. Note that $A$ itself is an admissible subset, and that all admissible subsets of $A$ contain $a$, so that $M$ is not empty. Furthermore, $M$ is itself an admissible subset of $A$. To see this, let $x \in M$. Then $x$ is in every admissible subset, so $f(x)$ is also in every admissible subset, and hence $f(x) \in M$. Hence $f(M) \subset M$. If $T$ is a totally ordered non-empty subset of $M$, and $b$ is the least upper bound of $T$ in $A$, then $b$ lies in every admissible subset of $A$, and hence lies in $M$. It follows that $M$ is the smallest admissible subset of $A$, and that any admissible subset of $A$ contained in $M$ is equal to $M$.

We shall prove that $M$ is totally ordered, and thereby prove Theorem 3.1.

[First we make some remarks which don't belong to the proof, but will help in the understanding of the subsequent lemmas. Since $a \in M$, we see that $f(a) \in M$, $f \circ f(a) \in M$, and in general $f^n(a) \in M$. Furthermore,

$$a \leqq f(a) \leqq f^2(a) \leqq \cdots.$$

If we had an equality somewhere, we would be finished, so we may assume that the inequalities hold. Let $D_0$ be the totally ordered set $\{f^n(a)\}_{n \geqq 0}$. Then $D_0$ looks like this:

$$a < f(a) < f^2(a) < \cdots < f^n(a) < \cdots .$$

Let $a_1$ be the least upper bound of $D_0$. Then we can form

$$a_1 < f(a_1) < f^2(a_1) < \cdots$$

in the same way to obtain $D_1$, and we can continue this process, to obtain

$$D_1, D_2, \ldots .$$

It is clear that $D_1$, $D_2$, ... are contained in $M$. If we had a precise way of expressing the fact that we can establish a never-ending string of such denumerable sets, then we would obtain what we want. The point is that we are now trying to prove Zorn's lemma, which is the natural tool for guaranteeing the existence of such a string. However, given such a string, we observe that its elements have two properties: If $c$ is an element of such a string and $x < c$, then $f(x) \leqq c$. Furthermore, there is no element between $c$ and $f(c)$, that is if $x$ is an element of the string, then $x \leqq c$ or $f(c) \leqq x$. We shall now prove two lemmas which show that elements of $M$ have these properties.]

Let $c \in M$. We shall say that $c$ is an **extreme point** of $M$ if whenever $x \in M$ and $x < c$, then $f(x) \leqq c$. For each extreme point $c \in M$ we let

$$M_c = \text{set of } x \in M \text{ such that } x \leqq c \text{ or } f(c) \leqq x.$$

Note that $M_c$ is not empty because $a$ is in it.

**Lemma 3.2.** *We have $M_c = M$ for every extreme point $c$ of $M$.*

*Proof.* It will suffice to prove that $M_c$ is an admissible subset. Let $x \in M_c$. If $x < c$ then $f(x) \leqq c$ so $f(x) \in M_c$. If $x = c$ then $f(x) = f(c)$ is again in $M_c$. If $f(c) \leqq x$, then $f(c) \leqq x \leqq f(x)$, so once more $f(x) \in M_c$. Thus we have proved that $f(M_c) \subset M_c$.

Let $T$ be a totally ordered subset of $M_c$ and let $b$ be the least upper bound of $T$ in $A$. Since $M$ is admissible, we have $b \in M$. If all elements $x \in T$ are $\leqq c$, then $b \leqq c$ and $b \in M_c$. If some $x \in T$ is such that $f(c) \leqq x$, then

$$f(c) \leqq x \leqq b,$$

and so $b$ is in $M_c$. This proves our lemma.

**Lemma 3.3.** *Every element of $M$ is an extreme point.*

*Proof.* Let $E$ be the set of extreme points of $M$. Then $E$ is not empty because $a \in E$. It will suffice to prove that $E$ is an admissible subset. We first prove that $f$ maps $E$ into itself. Let $c \in E$. Let $x \in M$ and suppose $x < f(c)$. We must prove that

$$f(x) \leqq f(c).$$

By Lemma 3.2, $M = M_c$, and hence we have $x < c$, or $x = c$, or $f(c) \leqq x$. This last possibility cannot occur because $x < f(c)$. If $x < c$ then

$$f(x) \leqq c \leqq f(c).$$

If $x = c$ then $f(x) = f(c)$, and hence $f(E) \subset E$.

Next let $T$ be a totally ordered subset of $E$. Let $b$ the least upper bound of $T$ in $A$. We must prove that $b \in E$. Let $x \in M$ and $x < b$. We must show that $f(x) \leqq b$. If for all $c \in E$ we have $f(c) \leqq x$, then $c \leqq f(c) \leqq x$ for all $c \in E$, whence $x$ is an upper bound for $E$, whence $b \leqq c$ and $b \in E$. Otherwise, since $M_c = M$ for all $c \in E$, we must therefore have $x \leqq c$ for some $c \in E$. If $x < c$, then $f(x) \leqq c \leqq b$, and if $x = c$, then

$$f(x) = f(c) \in E$$

by what has already been proved, and so $f(x) \leqq b$. This proves that $b \in E$, that $E$ is admissible, and thus proves Lemma 3.3.

We now see trivially that $M$ is totally ordered. For let $x, y \in M$. Then $x$ is an extreme point of $M$ by Lemma 3.3, and $y \in M_x$ so $y \leqq x$ or

$$x \leqq f(x) \leqq y,$$

thereby proving that $M$ is totally ordered. As remarked previously, this concludes the proof of Theorem 3.1.

We shall obtain Zorn's lemma essentially as a corollary of Theorem 3.1. We first obtain Zorn's lemma in a slightly weaker form.

**Corollary 3.4.** *Let $A$ be a non-empty strictly inductively ordered set. Then $A$ has a maximal element.*

*Proof.* Suppose that $A$ does not have a maximal element. Then for each $x \in A$ there exists an element $y_x \in A$ such that $x < y_x$. Let $f: A \to A$ be the map such that $f(x) = y_x$ for all $x \in A$. Then $A$, $f$ satisfy the hypotheses of Theorem 3.1 and applying Theorem 3.1 yields a contradiction.

The only difference between Corollary 3.4 and Zorn's lemma is that in Corollary 3.4, we assume that a non-empty totally ordered subset has a *least* upper bound, rather than an upper bound. It is, however, a simple

matter to reduce Zorn's lemma to the seemingly weaker form of Corollary 3.4. We do this in the second corollary.

**Corollary 3.5 (Zorn's Lemma).** *Let $S$ be a non-empty inductively ordered set. Then $S$ has a maximal element.*

*Proof.* Let $A$ be the set of non-empty totally ordered subsets of $S$. Then $A$ is not empty since any subset of $S$ with one element belongs to $A$. If $X$, $Y \in A$, we define $X \leqq Y$ to mean $X \subset Y$. Then $A$ is partially ordered, and is in fact strictly inductively ordered. For let $T = \{X_i\}_{i \in I}$ be a totally ordered subset of $A$. Let

$$Z = \bigcup_{i \in I} X_i.$$

Then $Z$ is totally ordered. To see this, let $x$, $y \in Z$. Then $x \in X_i$ and $y \in X_j$ for some $i, j \in I$. Since $T$ is totally ordered, say $X_i \subset X_j$. Then $x$, $y \in X_j$ and since $X_j$ is totally ordered, $x \leqq y$ or $y \leqq x$. Thus $Z$ is totally ordered, and is obviously a least upper bound for $T$ in $A$. By Corollary 3.4, we conclude that $A$ has a maximal element $X_0$. This means that $X_0$ is a maximal totally ordered subset of $S$ (non-empty). Let $m$ be an upper bound for $X_0$ in $S$. Then $m$ is the desired maximal element of $S$. For if $x \in S$ and $m \leqq x$, then $X_0 \cup \{x\}$ is totally ordered, whence equal to $X_0$ by the maximality of $X_0$. Thus $x \in X_0$ and $x \leqq m$. Hence $x = m$, as was to be shown.

CHAPTER II

# Topological Spaces

This chapter develops the standard properties of topological spaces. Most of these properties do not go beyond the level of a convenient language. In the text proper, we have given precisely those results which are used very frequently in all analysis. In the exercises, we give additional results, of which some just give routine practice and others give more special results. To incorporate all this material in the text proper would be extremely oppressive and would obscure the principal lines of thought inherent in the basic aspects of the subject. The reader can always be referred to Bourbaki [Bo] or Kelley [Ke] for encyclopaedic treatments.

## II, §1. OPEN AND CLOSED SETS

Let $X$ be a set. By a **topology** on $X$ we mean a collection $\mathscr{T}$ of subsets called the **open sets** of the topology, satisfying the following conditions:

**TOP 1.** *The empty set and $X$ itself are open.*

**TOP 2.** *A finite intersection of open sets is open.*

**TOP 3.** *An arbitrary union of open sets is open.*

**Example 1.** Let $X$ be any set. If we define an open set to be the empty set or $X$ itself, we have a topology on $X$, which is definitely not interesting.

**Example 2.** Let $X$ be a set, and define every subset to be open. In particular, each element of $X$ constitutes an open set. Again we have a

topology, which is called the **discrete topology** on $X$. A space with the discrete topology is called a discrete space. It does not look as if this topology were any more interesting than that of Example 1, but in fact it does occur in practice.

**Example 3.** Let $X = \mathbf{R}$ be the set of real numbers. Define a subset $U$ of $\mathbf{R}$ to be open if for each point $x$ in $U$ there exists an open interval $J$ containing $x$ and contained in $U$. The three axioms of a topology are easily verified. This topology is called the **ordinary** topology.

**Example 4.** Generalization of Example 3, and used very frequently in analysis. We recall that a **normed vector space** (over the real numbers) is a vector space $E$ together with a function on $E$ denoted by $x \mapsto |x|$ (real valued) such that:

**NVS 1.** *We have* $|x| \geqq 0$ *and* $= 0$ *if and only if* $x = 0$.

**NVS 2.** *If* $c \in \mathbf{R}$ *and* $x \in E$, *then* $|cx| = |c||x|$.

**NVS 3.** *If* $x, y \in E$, *then* $|x + y| \leqq |x| + |y|$.

Similarly, one defines the notion of normed vector space over the complex numbers. The axioms are the same, except that we then take the number $c$ to be complex in **NVS 2**.

By an **open ball** $B$ in $E$ centered at a point $v$, and of radius $r > 0$, we mean the set of all $x \in E$ such that $|x - v| < r$. We denote such a ball by $B_r(v)$. We define a set $U$ to be open in $E$ if for each point $r \in U$ there exists an open ball $B$ centered at $x$ and contained in $U$. Again it is easy to verify that this defines a topology, also called the **ordinary** topology of the normed vector space. It is but an exercise to verify that an open ball is indeed an open set of this topology.

Let $\{x_n\}$ be a sequence in a normed vector space $E$. This sequence is said to be **Cauchy** if given $\varepsilon$ (always assumed $> 0$) there exists $N$ such that for all $m, n \geqq N$ we have

$$|x_m - x_n| < \varepsilon.$$

This sequence is said to **converge** to an element $x$ if given $\varepsilon$, there exists $N$ such that for all $n > N$ we have

$$|x - x_n| < \varepsilon.$$

### Examples of Normed Vector Spaces

**The sup norm.** Let $S$ be a set. A map $f: S \to F$ of $S$ into a normed vector space $F$ is said to be **bounded** if there exists a number $C > 0$ such

that $|f(x)| \leqq C$ for all $x \in S$. If $f$ is bounded, define

$$\|f\|_S = \|f\| = \sup_{x \in S} |f(x)|,$$

sup meaning least upper bound. It can be easily shown that the set of bounded maps $B(S, F)$ of $S$ into $F$ is a vector space, and that $\| \ \|$ is a norm on this space, called the **sup norm**.

**The $L^1$-Norm.** Let $E$ be the space of continuous functions on $[0, 1]$. For $f \in E$ define

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Then $\| \ \|_1$ is a norm on $E$, called the $L^1$-norm. This norm will be a major object of study when we do integration later, in a general context.

Much of this book is devoted to studying the convergence of sequences for one or the other of the above two norms. For instance, consider the sup norm. A sequence of maps $\{f_n\}$ is said to be **uniformly Cauchy** on $S$ if given $\varepsilon$ there exists $N$ such that for all $m, n > N$ we have

$$\|f_n - f_m\|_S < \varepsilon.$$

It is said to be **uniformly convergent** to a map $f$ if given $\varepsilon$ there exists $N$ such that for all $n \geqq N$ we have

$$\|f_n - f\|_S < \varepsilon.$$

In the second example, we would use the expressions $L^1$-Cauchy and $L^1$-convergent instead of uniformly Cauchy and uniformly convergent, if we replace the sup norm by the $L^1$-norm in these definitions.

Up to a point, one can generalize the notion of subset of a normed vector space as follows. Let $X$ be a set. A **distance function** (also called a **metric**) on $X$ is a map $(x, y) \mapsto d(x, y)$ from $X \times X$ into $\mathbf{R}$ satisfying the following conditions:

**DIS 1.** *We have $d(x, y) \geqq 0$ for all $x, y \in X$, and $= 0$ if and only if $x = y$.*

**DIS 2.** *For all $x, y$, we have $d(x, y) = d(y, x)$.*

**DIS 3.** *For all $x, y, z$, we have*

$$d(x, z) \leqq d(x, y) + d(y, z).$$

A set with a metric is called a **metric space**. We can then define open

balls just as we did in the case of normed vector spaces, and also define a topology in a metric space just as we did for a normed vector space. Every open set is then a union of open balls. This topology is said to be determined by the metric.

In a normed vector space, we can define the **distance** between elements $x, y$ to be $d(x, y) = |x - y|$. It is immediately verified that this is a metric on the space. Conversely, the reader will see in Exercise 5 how a metric space can be embedded naturally in a normed vector space, in a manner preserving the metric, so that the "generality" of metric spaces is illusory. For convenience, we also make here the following definition: If $A$, $B$ are **subsets** of a normed vector space, we define their **distance** to be

$$d(A, B) = \inf |x - y|, \qquad x \in A,\ y \in B.$$

Basic theorems concerning subsets of normed vector spaces hold just as well for metric spaces. However, almost all metric spaces which arise naturally (and certainly all of those in this course) occur in a normed vector space with a natural linear structure. There is enough of a change of notation from $|x - y|$ to $d(x, y)$ to warrant carrying out proofs with the norm notation rather than the other.

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set $X$. One verifies at once that they are equal if and only if the following condition is satisfied: For each $x \in X$ and each set $U$ open in $\mathcal{T}$ containing $x$, there exists a set $U'$ open in $\mathcal{T}'$ such that $x \in U' \subset U$, and conversely, given $U'$ open in $\mathcal{T}'$ containing $x$, there exists $U$ open in $\mathcal{T}$ such that $x \in U \subset U'$.

**Example.** The reader will verify easily that two norms $|\ |_1$ and $|\ |_2$ on a vector space $E$ give rise to the same topology if and only if they satisfy the following condition: There exist $C_1, C_2 > 0$ such that for all $x \in E$ we have

$$C_1 |x|_1 \leqq |x|_2 \leqq C_2 |x|_1 .$$

If this is the case, the norms are called **equivalent**.

Just to fix terminology, we define the **closed** ball centered at $v$ and of radius $r \geqq 0$ to be the set of all $x \in E$ such that

$$|x - v| \leqq r.$$

We define the **sphere** centered at $v$, of radius $r$, to be the set of points $x$ such that

$$|x - v| = r.$$

**Warning.** In some books, what we call a ball is called a sphere. This is not good terminology, and the terminology used here is now essentially universally adopted.

Examples of normed vector spaces are given in the exercises. The standard properties of subsets of normed vector spaces having to do with limits are also valid in metric spaces (cf. Exercise 5). We can define balls and spheres in metric spaces just as in normed vector spaces. We can also define the notion of **Cauchy sequence** in a metric space $X$ as usual (again cf. Exercise 5), and $X$ is said to be **complete** if every Cauchy sequence converges, i.e. has a limit in $X$.

**Example 5.** Let $G$ be a group. We define a subset $U$ of $G$ to be open if for each element $x \in U$ there exists a subgroup $H$ of $G$, of finite index, such that $xH$ is contained in $U$. It is a simple exercise in algebra to show that this defines a topology, which is called the **profinite** topology.

**Example 6.** Let $R$ be a commutative ring (which according to standard conventions has a unit element). We define a subset $U$ of $R$ to be open if for each $x \in U$ there exists an ideal $J$ in $R$ such that $x + J$ is contained in $U$. It is a simple exercise in algebra to show that this defines a topology, which is called the **ideal** topology.

**Note.** The topologies of Examples 5 and 6 will not occur in any significant way in this course, and may thus be disregarded by anyone uninterested in this type of algebra.

A set together with a topology is called a **topological space**. In this chapter we develop a large number of basic trivialities about topological spaces, and except for the numbered theorems, it is recommended that readers work out the proofs for all other assertions by themselves, even though we have given most of them.

The duality between intersections and unions with respect to taking the complement of a subset allows us to define a topology by means of the complements of open sets, called **closed sets**. In any topological space, the closed sets satisfy the following conditions:

**CL 1.** *The empty set and the whole space are closed.*

**CL 2.** *The finite union of closed sets is closed.*

**CL 3.** *The arbitrary intersection of closed sets is closed.*

The first condition is clear, and the other two come from the fact that the complement of the union of subsets is equal to the intersection of their complements, and that the complement of the intersection of subsets is equal to the union of their complements.

Conversely, given a collection $\mathscr{F}$ of subsets of a set $X$ (not yet a topological space), we say that it defines a topology on $X$ by means of

closed sets if its elements satisfy the three conditions **CL 1, 2, 3**. We can then define an open set to be the complement of a set in $\mathcal{F}$.

**Example 7.** Let $X = \mathbf{R}^n$. Let $f(x_1, \ldots, x_n)$ be a polynomial in $n$ variables. A point $a = (a_1, \ldots, a_n)$ in $\mathbf{R}^n$ is called a **zero** of $f$ if $f(a) = 0$. We define a subset $S$ of $\mathbf{R}^n$ to be closed if there exists a family $\{f_i\}_{i \in I}$ of polynomials in $n$ variables (with real coefficients) such that $S$ consists precisely of the common zeros of all $f_i$ in the family (in other words, all points $a \in \mathbf{R}^n$ such that $f_i(a) = 0$ for all $i$). The reader may assume here the result that, for any such closed set $S$, there exists a finite number of polynomials $f_1, \ldots, f_r$ such that $S$ is already the set of zeros of the set $\{f_1, \ldots, f_r\}$. It is easy to prove that we have defined a topology by means of closed sets, and this topology is called the **Zariski** topology on $\mathbf{R}^n$. It is a topology which is adjusted to the study of algebraic sets, that is sets which are zeros of polynomials. It will not reappear in this course, and again a disinterested reader may omit it. It does become important in subsequent courses, however. In 2-space, a closed set consists of a finite number of points and algebraic curves. In 3-space, a closed set consists of a finite number of points, algebraic curves, and algebraic surfaces.

Let $X$ be a topological space, and $S$ a subset. A point $x \in X$ is said to be **adherent** to $S$ if given an open set $U$ containing $x$, there is some point of $S$ lying in $U$. In particular, every element of $S$ is adherent to $S$. A point of $X$ is called a **boundary** point of $S$ if every open set containing this point also contains a point of $S$ and a point not in $S$. Thus an adherent point of $S$ which does not lie in $S$ is a boundary point of $S$. An **interior** point of $S$ is a point of $S$ which does not lie in the boundary of $S$. The set $\text{Int}(S)$ of interior points of $S$ is open.

A *subset $S$ of $X$ is closed if and only if it contains all its boundary points.* This follows at once from the definitions.

By the **closure** of a subset $S$ of $X$ we mean the union of $S$ and all its boundary points. The closure of $S$, denoted by $\bar{S}$, *is therefore the set of adherent points of $S$.* It is also immediately verified that *$\bar{S}$ is closed, and is equal to the intersection of all closed sets containing $S$.* In particular, we have

$$\bar{\bar{S}} = \bar{S}.$$

As an exercise, the reader should prove that for subsets $S$, $T$ of $X$ we have:

$$\bar{S} \cup \bar{T} = \overline{S \cup T} \qquad \text{and} \qquad \overline{S \cap T} \subset \bar{S} \cap \bar{T}.$$

Equality does not necessarily hold in the formula on the right. (Example?)

A subset $S$ of a space $X$ is said to be **dense** (in $X$) is $\bar{S} = X$. For instance, the rationals are dense in the reals.

Let $X$ be a topological space and $S$ a subset. We define a topology on $S$ by prescribing a subset $V$ of $S$ to be open in $S$ if there exists an open set $U$ in $X$ such that $V = U \cap S$. The conditions for a topology on $S$ are immediately verified, and this topology is called the **induced** topology. With this topology, $S$ is called a **subspace**.

**Note.** A subset of $S$ which is open in $S$ may not be open in $X$. For instance, the real line is open in itself, but definitely not open in $\mathbf{R}^2$. Similarly for closed sets. On the other hand, if $U$ is an open subset of $X$, then a subset of $U$ is open in $U$ in the induced topology if and only if it is open in $X$. Similarly, if $S$ is a closed subset of $X$, a subset of $S$ is closed in $S$ if and only if it is closed in $X$.

If $P$ is a certain property of certain topological spaces (e.g. connected, or compact as we shall define later), then we say that a subset has property $P$ if it has this property as a subspace.

A topology on a set is often defined by means of a base for the open sets. By a **base** for the open sets we mean a collection $\mathscr{B}$ of open sets such that any open set $U$ is a union (possibly infinite) of elements of $\mathscr{B}$. There is an easy criterion for a collection of subsets to be a base for a topology. Let $X$ be a set and $\mathscr{B}$ a collection of subsets satisfying:

**B 1.** *Every element of $X$ lies in some set in $\mathscr{B}$.*

**B 2.** *If $B$, $B'$ are in $\mathscr{B}$ and $x \in B \cap B'$ then there exists some $B''$ in $\mathscr{B}$ such that $x \in B''$ and $B'' \subset B \cap B'$.*

If $\mathscr{B}$ satisfies these two conditions, then there exists a unique topology whose open sets are the unions of sets in $\mathscr{B}$. Indeed, such a topology is uniquely determined, and it exists because we can define a set to be open if it is a union of sets in $\mathscr{B}$. The axioms for open sets are trivially verified.

**Example.** The open balls in a normed vector space form a base for the ordinary topology of that space.

**Example.** Let $X$ be a set and let $\mathscr{U}$, $\mathscr{V}$ be topologies on $X$, that is collections of open sets satisfying the axioms for a topology. We say that $\mathscr{V}$ is a **refinement** of $\mathscr{U}$, or that $\mathscr{U}$ is **coarser** than $\mathscr{V}$, if every set open in $\mathscr{U}$ is also open in $\mathscr{V}$. Thus $\mathscr{U}$ has fewer open sets than $\mathscr{V}$ ("fewer" in the weak sense since $\mathscr{U}$ may be equal to $\mathscr{V}$).

Let $Y$ be a topological space and let $\mathscr{F}$ be a family of mappings $f: X \to Y$ of $X$ into $Y$. Let $\mathscr{B}$ be the family of all subsets of $X$ consisting

of the sets $f^{-1}(W)$, where $W$ is open in $Y$ and $f$ ranges over $\mathscr{F}$. Then we leave to the reader the verification of the following facts:

1. $\mathscr{B}$ is a base for a topology on $X$, i.e. satisfies conditions **B 1**, **B 2**.
2. This topology is the coarsest topology (the one with the fewest open sets) such that every map $f \in \mathscr{F}$ is continuous.

We call this topology the **weak topology** on $X$ determined by $\mathscr{F}$.

For an application of the weak topology, see Chapter IV, §1 and also the appendix of Chapter IV.

There is a generalization of the weak topology as follows. Instead of considering one space $Y$, we consider a family of spaces $\{Y_i\}$, for $i$ ranging in some index set. We let $\mathscr{F}$ be a family of mappings $f_i: X \to Y_i$. We let $\mathscr{B}$ be the family of all subsets of $X$ consisting of finite intersections of sets $f_i^{-1}(U_i)$ where $U_i$ is open in $Y_i$. Then again it is easily verified that $\mathscr{B}$ is a base for a topology, called the **weak topology determined by the family** $\mathscr{F}$. The product topology defined below will provide an example of this more general case, when the family $\mathscr{F}$ is the family of projections on the factors of a product.

A topological space is said to be **separable** if it has a countable base. (By countable we mean finite or denumerable.) Exercises on separable spaces designed to acquaint the reader with them, and essentially all trivial, are given at the end of the chapter. It is easy to see that the real numbers have a countable base. Indeed, we can take for basis elements the open intervals of rational radius, centered at rational points. Similarly, $\mathbf{R}^n$ has a countable base.

**Note.** In most cases, the property defining separability is equivalent with the property that there exists a countable dense subset (cf. Exercise 15), and this second property is sometimes used to define separability. We find our definition to be more useful but the reader is warned on the discrepancy with some other texts.

An open set containing a point $x$ is called an open **neighborhood** of this point. By a **neighborhood** of $x$ we mean any set containing an open set containing $x$. In a normed vector space, one speaks of an $\varepsilon$-neighborhood of a point $x$ as being a ball of radius $\varepsilon$ centered at $x$.

Let $X$, $Y$ be topological spaces. A map $f: X \to Y$ is said to be **continuous** if the inverse image of an open set (in $Y$) is open in $X$. In other words, if $V$ is open in $Y$ then $f^{-1}(V)$ is open in $X$. Equivalently, we see that a map $f$ is continuous if and only if the inverse image of a closed set is closed.

**Proposition 1.1.** *Let $E$, $F$ be normed vector spaces and let $f: E \to F$ be a map. This map is continuous if and only if the usual* $(\varepsilon, \delta)$ *definition is satisfied at every point of $E$.*

We prove one of the two implications. Assume that $f$ is continuous and let $x \in E$. Given $\varepsilon$, let $V$ be the open ball of radius $\varepsilon$ centered at $f(x)$. The open set $U = f^{-1}(V)$ contains an open ball $B$ of radius $\delta$ centered at $x$ for some $\delta$. In particular, if $y \in E$ and $|x - y| < \delta$, then $f(y) \in V$ and $|f(y) - f(x)| < \varepsilon$. This proves the $(\varepsilon, \delta)$ property. The converse is equally clear and is left to the reader.

Actually, this $(\varepsilon, \delta)$ property can be formulated analogously in arbitrary topological spaces, as follows: The map $f: X \to Y$ is said to be **continuous at a point** $x \in X$ if given a neighborhood $V$ of $f(x)$ there exists a neighborhood $U$ of $x$ such that $f(U) \subset V$. It is then verified at once that $f$ is continuous if and only if it is continuous at every point.

**Proposition 1.2.** *Let $X$ be a metric space (or a subset of a normed vector space) and let $f: X \to E$ be a map into a normed vector space. Then $f$ is continuous if and only if the following condition is satisfied. Let $\{x_n\}$ be a sequence in $X$ converging to a point $x$. Then $\{f(x_n)\}$ converges to $f(x)$.*

The proof will be left as an exercise to the reader.

*A composite of continuous maps is continuous.*

Indeed, if $f: X \to Y$ and $g: Y \to Z$ are continuous maps and $V$ is open in $Z$, then

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$$

is seen to be open.

As usual, we observe that a continuous image of an open set is not necessarily open.

A continuous map $f: X \to Y$ which admits a continuous inverse map $g: Y \to X$ is called a **homeomorphism**, or **topological isomorphism**. It is clear that a composite of homeomorphisms is also a homeomorphism. As usual, we observe that a continuous bijective map need not be a homeomorphism. In fact, later in this course, we meet many examples of vector spaces with two different norms on them such that the identity map is continuous but not bicontinuous.

Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let

$$X = \prod_{i \in I} X_i$$

be their product. We define a topology on $X$, called the **product topology**, by characterizing a subset $U$ of $X$ to be open if for each $x \in U$ there exists a finite number of indices $i_1, \ldots, i_n$ and open sets $U_{i_1}, \ldots, U_{i_n}$ in the spaces $X_{i_1}, \ldots, X_{i_n}$ respectively such that

$$x \in U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_k} X_i \subset U.$$

The product for $i \neq i_k$ is taken for all indices $i$ unequal to $i_1, \ldots, i_n$. In other words, we can say that the product topology is the one having as a base all sets of the form

$$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_k} X_i.$$

Such sets have arbitrary open sets at a finite number of components, and the full space at all other components.

The product topology is the unique topology with the fewest open sets in $X$ which makes each projection map

$$\pi_i \colon X \to X_i$$

continuous. Indeed, for each open set $U_j$ in $X_j$, the set

$$\pi_j^{-1}(U_j) = U_j \times \prod_{i \neq j} X_i,$$

must be open if $\pi_j$ is continuous, and our previous assertion follows. In other words, it is the weak topology determined by the family of all projections on the factors.

More generally, given a set and a family of mappings of this set into topological spaces, one can define a unique topology on the set making all these mappings continuous, and having the fewest open sets doing this, namely the weak topology. If $S$ is a set, and

$$\{f_i \colon S \to Y_i\}_{i \in I}$$

is a family of maps into topological spaces $Y_i$, then the map

$$f \colon S \to \prod_{i \in I} Y_i$$

such that $f(x) = \{f_i(x)\}$ is continuous for this topology.

**Example 8.** We can give $\mathbf{R}^n$ the product topology, which is called the ordinary topology. We define the sup norm on $\mathbf{R}^n$ by

$$\|x\| = \max |x_i|$$

if $x = (x_1, \ldots, x_n)$ is given in terms of its coordinates. Then the topology determined by this norm is clearly the same as the product topology.

**Remark.** A map $f: X \to Y$ which maps open sets onto open sets is said to be **open**. A map which maps closed sets onto closed sets is said to be **closed**. A continuous map need not be either. For instance, the graph of the tangent is closed in the plane, but the projection map on the $x$-axis maps it on an open interval:

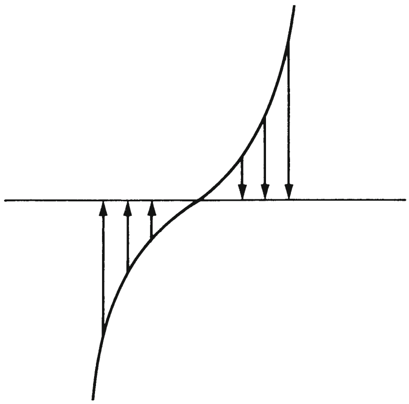


Figure 2.1

The map which folds the plane over the real axis maps the open plane on the closed half plane. If $f: X \to Y$ is continuous and bijective, then a necessary and sufficient condition that $f$ be a homeomorphism is that $f$ be open. This is simply a rephrasing of the continuity of the inverse mapping $f^{-1}$.

## II, §2. CONNECTED SETS

A topological space $X$ is said to be **connected** if it is not possible to express $X$ as a union of two disjoint non-empty open sets. Of course, we can formulate the definition in terms of closed sets instead of open sets.

The reader's intuition of connectedness probably comes from the possibility of connecting two points of a set by a path. We shall discuss the relation between this notion and the general notion later, after developing first some basic properties of connected sets.

**Proposition 2.1.** *Let $f: X \to Y$ be a continuous map. If $X$ is connected then the image of $X$ is connected.*

*Proof.* Without loss of generality we may assume that $Y$ is the image of $f$. Suppose that $Y$ is not connected, so that we can write $Y = U \cup V$ where $U$, $V$ are open, non-empty, and disjoint. Then

$$X = f^{-1}(U) \cup f^{-1}(V),$$

which is impossible. This proves our assertion.

**Proposition 2.2.** *A topological space $X$ is connected if and only if every continuous map of $X$ into a discrete space having at least two elements is constant.*

*Proof.* Assume that $X$ is connected, and that $f$ is a continuous map of $X$ into a discrete space with at least two elements. If $f$ is not constant, we can write the image of $f$ as a union of two disjoint non-empty sets, open by definition, and this contradicts our previous result. Conversely, suppose that we can write $X = U \cup V$ as a disjoint union of non-empty open sets. Let $p$, $q$ be two distinct objects and let the set $\{p, q\}$ have the discrete topology. If we define

$$f: X \to \{p, q\}$$

to be the map such that

$$f(U) = \{p\} \qquad \text{and} \qquad f(V) = \{q\},$$

then $f$ is continuous and not constant, as was to be shown.

Observe that our proof shows that instead of taking a discrete space having at least two points, we can take a space with exactly two points in characterizing a connected set, as we have just done.

**Proposition 2.3.** *Let $X$ be a topological space and let $\{S_i\}_{i \in I}$ be a family of subspaces which are connected. If they have a point in common then their union is connected.*

*Proof.* Let $a$ lie in the intersection of all $S_i$. If we can write

$$\bigcup S_i = U \cup V,$$

where $U$, $V$ are open in this union, then $S_i \cap U$ and $S_i \cap V$ are open in $S_i$ for each $i$ and hence $S_i \subset U$ or $S_i \subset V$. If for some $i$ we have $S_i \subset U$, then $a \in U$ and consequently we must have $S_i \subset U$ for all $i$, thus proving our assertion.

As a consequence of the preceding statement, we define the **connected component** of a point $a$ in $X$ to be the union of all connected subspaces of $X$ containing $a$. This component is actually not empty, because the set consisting of $a$ alone is connected.

**Proposition 2.4.** *Let $X$ be a topological space and $S$ a connected subset. Then the closure of $S$ is connected. In fact, if $S \subset T \subset \bar{S}$, then $T$ is connected.*

*Proof.* Left to the reader.

**Corollary 2.5.** *The connected component of a point is closed.*

*Proof.* Clear.

As promised, we now discuss the relation between the naive notion of connectedness and the general notion. Let $X$ be a topological space. We say that $X$ is **arcwise connected** if given two points $x$, $y$ in $X$ there exists a piecewise continuous path from $x$ to $y$. By a **piecewise continuous path**, we mean a sequence of continuous maps $\{\alpha_1, \ldots, \alpha_r\}$, where each

$$\alpha_i \colon [a_i, b_i] \to X$$

is a continuous map defined on a closed interval $[a_i, b_i]$ such that

$$\alpha_i(b_i) = \alpha_{i+1}(a_{i+1}).$$

We say that this path goes from $x$ to $y$ if

$$\alpha_1(a_1) = x \qquad \text{and} \qquad \alpha_r(b_r) = y.$$

Of course, if such a path exists, then it is easy to define just one continuous map

$$\alpha \colon [a, b] \to X$$

from some interval $[a, b]$ into $X$ such that $\alpha(a) = x$ and $\alpha(b) = y$. One can even take the interval $[a, b]$ to be $[0, 1]$.

**Proposition 2.6.** *Any interval of real numbers is connected.*

*Proof.* We give the proof for a closed interval $J = [a, b]$ and leave the other cases (open, half-open, infinite intervals) as exercises. Suppose that we can write $J = A \cup B$ where $A$, $B$ are closed, disjoint, and non-empty. Say that $a \in A$. Let $c$ be the greatest lower bound of $B$. Then $c$ lies in

the closure of $B$ and since $B$ is closed, $c \in B$, so $c \neq a$. For any $x \in J$ with $a < x < c$, we must have $x \in A$ since $c$ is a lower bound for $B$. Since $A$ is closed, and since $c$ lies in the closure of the interval $a < x < c$, it follows that $c$ lies in $A$, a contradiction which proves our assertion.

**Proposition 2.7.** *If a topological space is arcwise connected, then it is connected.*

*Proof.* Let $X$ be arcwise connected and suppose that we can write $X$ as a disjoint union of non-empty open sets $U$, $V$. Let $x \in U$ and $y \in V$. There exists a continuous map $\alpha: J \to X$ from a closed interval into $X$ starting at $x$ and ending at $y$. Then $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ express $J$ as a disjoint union of non-empty disjoint sets which are open in $J$, a contradiction.

*The converse of the preceding result is false.* For instance the subset of the plane consisting of the $y$-axis and the graph of the curve $y = \sin(1/x)$ is connected but not arcwise connected. In practice, however, most ordinary sets which are connected are also arcwise connected, and the sort of pathology which arises from $\sin(1/x)$ is just that: pathology. In Exercise 12, you will prove that *an open subset of a normed vector space is connected if and only if it is arcwise connected.*

**Theorem 2.8.** *Let $\{X_i\}_{i \in I}$ be a family of connected topological spaces. Then the product*

$$X = \prod_{i \in I} X_i$$

*is connected.*

*Proof.* Let $f: X \to \{p, q\}$ be a continuous map of $X$ into a discrete space consisting of two points. We must show that $f$ is constant. Let $a \in X$ and say that $f(a) = p$. Then $f^{-1}(p)$ contains an open neighborhood of $a$ of the form

$$U = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_k} X_i.$$

Let $b$ be any other point of $X$ and write $a$, $b$ in terms of their coordinates:

$$a = (a_{i_1}, \ldots, a_{i_n}, \ldots),$$

$$b = (b_{i_1}, \ldots, b_{i_n}, \ldots).$$

Let

$$z = \left(a_{i_1}, \ldots, a_{i_n}, (b_i)_{i \neq i_1, \ldots, i_n}\right)$$

so that the coordinates of $z$ are the same as those of $a$ for $i_1, \ldots, i_n$ and the same as those of $b$ for the other indices. Then $z \in U$ and $f(z) = p$.

Consider the composite of maps

$$X_{i_1} \xrightarrow{g} X \xrightarrow{f} \{p, q\},$$

where $g$ is the injective mapping such that

$$g(x_{i_1}) = \big(x_{i_1}, a_{i_2}, \ldots, a_{i_n}, (b_i)_{i \neq i_1, \ldots, i_n}\big).$$

Then $g$ is continuous, so is $f \circ g$, and since the continuous image of a connected set is connected, it follows that $f \circ g$ is constant on $X_{i_1}$. In particular, $f \circ g(a_{i_1}) = f(z) = p$, and also

$$f\big(b_{i_1}, a_{i_2}, \ldots, a_{i_n}, (b_i)_{i \neq i_1, \ldots, i_n}\big) = p.$$

We now perform the same trick, replacing $a_{i_2}$ by $b_{i_2}$, ..., and $a_{i_n}$ by $b_{i_n}$. We then see that $f(b) = p$, thus proving that $f$ is constant, which proves the theorem.

**Corollary 2.9.** *Euclidean n-space* $\mathbf{R}^n$ *is connected, and so is the product of any number of intervals.*

## II, §3. COMPACT SPACES

Let $X$ be a set and $\{S_\alpha\}_{\alpha \in A}$ a family of subsets. We say that this family is a **covering** of $X$ if its union is equal to $X$. If $X$ is a topological space, and $\{U_\alpha\}_{\alpha \in A}$ is a covering, we say it is an **open covering** if each $U_\alpha$ is open. If $\{S_\alpha\}_{\alpha \in A}$ is a covering of $X$, we define a **subcovering** to be a covering $\{S_\beta\}_{\beta \in B}$ where $B$ is a subset of $A$. In particular, a finite subcovering of $\{S_\alpha\}$ is a covering $\{S_{\alpha_1}, \ldots, S_{\alpha_n}\}$.

Let $X$ be a topological space. We shall say that $X$ is **compact** if any open covering of $X$ has a finite subcovering. As usual, we can express a dual condition relative to closed sets. Let $\{F_\alpha\}_{\alpha \in A}$ be a family of subsets of $X$. We say that this family has the **finite intersection property** if any finite intersection

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$$

is not empty.

**Proposition 3.1.** *A topological space $X$ is compact if and only if, for any family $\{F_\alpha\}_{\alpha \in A}$ of closed sets having the finite intersection property, the intersection*

$$\bigcap_{\alpha \in A} F_\alpha$$

*is not empty.*

*Proof.* Assume that $X$ is compact and let $\{F_\alpha\}$ be a family of closed sets having the finite intersection property. Suppose that the intersection of this family is empty. Then the complements $\mathscr{C}F_\alpha$ form an open covering of $X$, and there is a finite subcovering by open sets $\{\mathscr{C}F_{\alpha_1}, \ldots, \mathscr{C}F_{\alpha_n}\}$. Taking the complement, we conclude that the intersection

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$$

is empty, which is a contradiction, thus proving the finite intersection property. The converse is equally clear.

**Proposition 3.2.** *A continuous image of a compact set is compact.*

*Proof.* Let $X$ be compact, and let $f: X \to Y$ be a continuous map, which is surjective. Let $\{V_\alpha\}$ be an open covering of $Y$. Then $\{f^{-1}(V_\alpha)\}$ is an open covering of $X$, and there is a finite subcovering

$$\{f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_n})\}.$$

It follows that $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ is a covering of $Y$, as was to be shown.

**Proposition 3.3.** *A closed subspace of a compact space is compact.*

*Proof.* Let $X$ be a compact space and $S$ a closed subspace. Let $\{U_\alpha\}$ be a covering of $S$ by open sets in $X$. Let $U$ be the complement of $S$ in $X$. Then $\{U_\alpha\}$ together with $U$ form an open covering of $X$, having a finite subcovering

$$\{U_{\alpha_1}, \ldots, U_{\alpha_n}, U\}.$$

Since $U$ is disjoint from $S$, it follows that already $U_{\alpha_1}, \ldots, U_{\alpha_n}$ cover $S$, thus proving our assertion.

The converse of the preceding assertion is almost true but not quite. A topological space $X$ is said to be **Hausdorff** if given points $x, y \in X$ and $x \neq y$ there exist disjoint open sets $U$, $V$ such that $x \in U$ and $y \in V$. If $X$ is Hausdorff, then each point of $X$ is obviously closed.

**Proposition 3.4.** *A compact subspace of a Hausdorff space is closed.*

*Proof.* Let $S$ be a compact subset of the Hausdorff space $X$. We prove that its complement is open. Let $x$ be in the complement. For each $y \in S$ there exist disjoint open sets $U_y$, $V_y$ such that $x \in U_y$ and $y \in V_y$. The family $\{V_y\}_{y \in S}$ covers $S$ and there is a finite subcovering

$$\{V_{y_1}, \ldots, V_{y_n}\}.$$

Then the intersection $U_{y_1} \cap \cdots \cap U_{y_n}$ is open, contains $x$, and is contained in the complement of $S$, thus proving what we want.

A topological space $X$ is said to be **normal** if it is Hausdorff, and if given two disjoint closed sets $A$, $B$ in $X$ there exist disjoint open sets $U$, $V$ such that $A \subset U$ and $B \subset V$.

**Proposition 3.5.** *A compact Hausdorff space is normal. In fact, if $A$, $B$ are compact subsets of a Hausdorff space, and are disjoint, there exist disjoint open sets $U$, $V$ such that $A \subset U$ and $B \subset V$.*

*Proof.* The proof is similar to the previous one, and involves merely one further application of the same principle. Using the same trick as in this previous proof, we know that for each $x \in A$ there exist disjoint open sets $U_x$, $W_x$ such that $x \in U_x$ and $B \subset W_x$. (One would take the finite union of the open sets $V_{y_1}, \ldots, V_{y_n}$ to obtain $W_x$ in the analogous situation.) The family of open sets $\{U_x\}_{x \in A}$ covers $A$, and there exists a finite subcovering

$$\{U_{x_1}, \ldots, U_{x_m}\}.$$

The open sets $U_{x_1} \cup \cdots \cup U_{x_m}$ and $W_{x_1} \cap \cdots \cap W_{x_m}$ solve our problem.

In the case of Hausdorff spaces, or normal spaces, we say also that points (or closed sets) can be **separated** by open sets. The properties of being Hausdorff or normal are thus called separation properties.

It is clear that a subspace of a Hausdorff space is Hausdorff. The analogous statement for normal spaces is not necessarily true (cf. Kelley [Ke], Exercise F, p. 132).

The general notion of a compact space is, in many practical cases, equivalent with another notion with which the reader is probably already familiar. We call a space $X$ **sequentially compact** if it has the **Weierstrass–Bolzano property**, namely every sequence $\{x_n\}$ in $X$ has a point of accumulation (a point $c$ such that given an open neighborhood $U$ of $c$, there exist infinitely many $n$ such that $x_n \in U$). As usual, an equivalent condition is that an infinite subset of $X$ has a point of accumulation. It is an exercise to prove:

**Proposition 3.6.** *If a topological space has a countable base, then it is compact if and only if it is sequentially compact.*
(Cf. Exercise 19.)

The preceding criterion will not be used in this book.

**Proposition 3.7.** *Compactness implies sequential compactness.*

*Proof.* Let $X$ be compact. It will suffice to prove that an infinite subset of $X$ has a point of accumulation. Suppose that this is not the case, and let $S$ be an infinite subset. Given $x \in X$, there exists an open set $U_x$ containing $x$ but containing only a finite number of the elements of $S$. The family $\{U_x\}_{x \in X}$ covers $X$. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be a finite subcovering. We conclude that there is only a finite number of elements of $S$ lying in the finite union

$$U_{x_1} \cup \cdots \cup U_{x_n}.$$

This is a contradiction, which proves our assertion.

The converse is true under important and rather general conditions, as shown in the next theorem.

**Theorem 3.8.** *Let $S$ be a subset of a metric space, or of a normed vector space.*

(i) *$S$ is compact if and only if $S$ is sequentially compact.*
(ii) *$S$ is compact if and only if $S$ is complete, and given $r > 0$ there exists a finite number of open balls of radius $r$ which cover $S$.*

*Proof.* We have already proved that compactness implies sequential compactness. Conversely, assume that $S$ is sequentially compact. Then certainly $S$ is complete, and we shall prove that the other condition stated in (ii) is satisfied. Suppose it is not. Let $r > 0$. Let $x_1 \in S$ and let $B_1$ be the open ball of radius $r$ centered at $x_1$. Then $B_1$ does not contain $S$, and there is some $x_2 \in S$, $x_2 \notin B_1$. Proceeding inductively, suppose that we have found open balls $B_1, \ldots, B_n$ of radius $r$, and points $x_1, \ldots, x_n$ with $x_i \in B_i$ such that $x_{k+1}$ does not lie in $B_1 \cup \cdots \cup B_k$. We can then find $x_{n+1}$ which does not lie in $B_1 \cup \cdots \cup B_n$, and we let $B_{n+1}$ be the open ball of radius $r$ centered at $x_{n+1}$. Let $v$ be a point of accumulation of the sequence $\{x_n\}$. By definition, there exist positive integers $m$, $k$ with $k > m$ such that

$$|x_k - v| < r/2$$

and

$$|x_m - v| < r/2.$$

Then $|x_k - x_m| < r$ and this contradicts the property of our sequence $\{x_n\}$ because $x_k$ lies in the ball $B_m$. This proves that $S$ satisfies the condition of (ii).

Now assume this condition. Let $\{U_i\}_{i \in I}$ be an open covering of $S$, and suppose that there is no finite subcovering. We construct a sequence $\{x_n\}$ in $S$ inductively as follows. We know that $S$ is covered by a finite number of closed balls of radius $\frac{1}{2}$. Hence there exists at least one closed

ball $C_1$ of radius $\frac{1}{2}$ such that $C_1 \cap S$ is not covered by a finite number of $U_i$. We let $x_1$ be a point of $C_1 \cap S$. Suppose that we have obtained a sequence of closed balls

$$C_1 \supset \cdots \supset C_n$$

such that $C_n$ has radius $1/2^n$, with a point $x_n \in C_n \cap S$, and such that $C_n \cap S$ is not covered by a finite number of $U_i$. Since $S$ itself can be covered by a finite number of closed balls of radius $1/(2^{n+1})$, it follows that $C_n \cap S$ can also be so covered, and hence there exists a closed ball $C_{n+1}$ of radius $1/(2^{n+1})$ and such that $C_{n+1} \cap S$ cannot be covered by a finite number of $U_i$. We let $x_{n+1}$ be a point of $C_{n+1} \cap S$. This constructs our sequence as desired. We see that $\{x_n\}$ is a Cauchy sequence in $S$, which coverges to a point $x$ in $S$. But $x$ lies in some $U_{i_0}$ which contains $C_n$ for all sufficiently large $n$, a contradiction which proves our theorem.

A subset $S$ of a metric space, or a normed vector space, which can be covered by a finite number of open balls of given radius $r > 0$ is said to be **totally bounded**. We can phrase (ii) by saying that $S$ is compact if and only if it is complete and totally bounded. A subset of a topological space is said to be **relatively compact** if its closure is compact. From (ii) we get a convenient criterion for relative compactness.

**Corollary 3.9.** *Let $S$ be a subset of a complete normed vector space. Assume that given $r > 0$ there exists a finite covering of $S$ by balls of radius $r$. Then $S$ is relatively compact.*

*Proof.* The closure $\bar{S}$ of $S$ has the same property, because if $S$ is covered by a finite number of balls of radius $r/2$, then the closure of $S$ is covered by a finite number of balls of radius $r$ (centered at the same points). Also $\bar{S}$ is complete. Hence we conclude that the closure of $S$ is compact.

As an application of Theorem 3.8, we recall that a closed (bounded) interval in **R** has the Weierstrass–Bolzano property. Hence it is compact, and therefore so is any closed bounded subset of **R** (being a closed subset of a compact set). The converse is also true, since a compact set is closed, and must be bounded, otherwise one can find an infinite sequence tending to infinity, and not having a point of accumulation.

One can also prove the compactness of a closed interval directly from the least upper bound axiom, as follows. Let $a < b$, and let $\{U_i\}_{i \in I}$ be an open covering of $[a, b]$. Let $S$ be the set of all $x \in [a, b]$ such that $[a, x]$ admits a finite subcovering. Then $S$ is not empty (because $a \in S$) and is bounded from above by $b$. Let $c$ be its least upper bound. Then $c \in U_{i_0}$ for some index $i_0$. If $a < c$, select a number $t$ with $a < t < c$ such that

the interval $[t, c]$ is contained in $U_{i_0}$. If $a = c$, let $t = a$. Then $[a, t]$ can be covered by a finite number of sets $U_i$, say $U_{i_1}, \ldots, U_{i_n}$. If $c \neq b$, then $U_{i_0}, U_{i_1}, \ldots, U_{i_n}$ cover an interval $[a, c']$ with $c' > c$, a contradiction, proving that $c = b$ and that $[a, b]$ is compact.

One can generalize to arbitrary compact sets some standard theorems on closed intervals, e.g.:

**Proposition 3.10.** *Let $A$ be a compact set, and $f: A \to \mathbf{R}$ a continuous function on $A$. Then $f$ has a maximum (a point $c \in A$ such that $f(c) \geqq f(x)$ for all $x \in A$).*

*Proof.* The image $f(A)$ is compact, so closed and bounded. The least upper bound of $f(A)$ lies in $f(A)$, thus proving our assertion.

If $A$ is a subset of a normed vector space, and if $f: A \to F$ is a continuous map into some normed vector space $F$, then we say that $f$ is **uniformly** continuous on $A$ if given $\varepsilon$ there exists $\delta$ such that whenever $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. We recall the theorem from elementary analysis that:

**Proposition 3.11.** *Let $A$ be a compact subset of a normed vector space. If $f: A \to F$ is a continuous map into a normed vector space, then $f$ is uniformly continuous. In fact, if $A$ is contained in a subset $S$ of a normed vector space, if $f$ is defined on $S$ and continuous on $A$, then given $\varepsilon$ there exists $\delta$ such that if $x \in A$ and $y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.*

We recall the proof briefly. Given $\varepsilon$, for each $x \in A$ we let $r(x) > 0$ be such that if $|y - x| < r(x)$, then $|f(y) - f(x)| < \varepsilon$. We can cover $A$ by open balls $B_i$ of radius

$$\delta_i = r(x_i)/2,$$

centered at $x_i$ ($i = 1, \ldots, n$). We let $\delta = \min \delta_i$. If $x \in A$, then for some $i$ we have $|x - x_i| < r(x_i)/2$. If $|y - x| < \delta$, then $|y - x_i| < r(x_i)$ so that

$$\begin{aligned} |f(y) - f(x)| &\leqq |f(y) - f(x_i)| + |f(x_i) - f(x)| \\ &< 2\varepsilon, \end{aligned}$$

as was to be shown.

The preceding definition of uniform continuity, and the result just proved, are of course valid for metric spaces, with the usual notation $d(x, y)$ replacing $|x - y|$. The property which we proved, and which is

slightly stronger than uniform continuity on $A$, will be called **relative** uniform continuity (relative to $S$, that is).

The only non-trivial theorem of this section is the theorem that a product of compact spaces is compact. In situations when one can use sequences, and one takes a finite product of spaces, however, the proof is immediate. For instance, let $E$, $F$ be normed vector spaces, and let $S$, $T$ be compact subsets of $E$, $F$, respectively. Let $\{z_n\}$ be a sequence in $S \times T$, and write $z_n = (x_n, y_n)$ with $x_n \in E$ and $y_n \in F$. We can find a subsequence $\{x_{n_i}\}$ converging to a point $a$ in $S$. We can then find a subsequence $\{y_{n_{i_k}}\}$ converging to a point $b$ in $F$. Then the sequence $\{z_{n_{i_k}}\}$ converges to $(a, b)$ so that $S \times T$ is sequentially compact.

The idea for this proof is to project on the coordinates, and from coordinatewise convergence, get the convergence in the product space. However, if we do it for an infinite product, the above proof seems to fail because we may exhaust all the indices before being through with the proof. One can still formulate the basic idea so that it essentially carries over to the most general case. Part of the difficulty in doing this is that the points of accumulation in the various coordinate spaces are not uniquely determined. Thus one must find a set theoretic device which chooses simultaneously a point of accumulation in all coordinate spaces. The proof below is due to Bourbaki.

**Theorem 3.12 (Tychonoff's Theorem).** *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of compact spaces. Then the product*

$$X = \prod_{\alpha \in A} X_\alpha$$

*is compact.*

*Proof.* Let $\mathscr{F} = \{F_i\}_{i \in I}$ be a family of closed subsets of the product, having the finite intersection property. The family of subsets of $X$ (not necessarily closed) containing our given family $\mathscr{F}$ and having the finite intersection property is ordered by ascending inclusion. One verifies immediately by taking the usual union that it is inductively ordered. It is therefore contained in a maximal family $\mathscr{F}^*$ having the finite intersection property. Let

$$\pi_\alpha \colon X \to X_\alpha$$

be the projection on the $\alpha$-th factor. For each $\alpha$, the family of closed sets

$$\{\overline{\pi_\alpha(F)}\}, \qquad F \in \mathscr{F}^*,$$

has the finite intersection property, and consequently there exists an

element $x_\alpha$ in each set $\overline{\pi_\alpha(F)}$ for all $F \in \mathscr{F}^*$. Let $x = (x_\alpha)$. We contend that $x$ belongs to all sets $F \in \mathscr{F}^*$. This will prove our theorem.

To prove our contention, we observe that the intersection of a finite number of sets in $\mathscr{F}^*$ also lies in $\mathscr{F}^*$ because of the maximality of $\mathscr{F}^*$. Let $U$ be an open set of $X$ containing $x$, of the form

$$U = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_i} X_\alpha$$

with each $U_{\alpha_i}$ open in $X_{\alpha_i}$. Then $U_{\alpha_i}$ contains $x_{\alpha_i}$ for all $i$, and therefore $U_{\alpha_i}$ contains a point of $\pi_{\alpha_i}(F)$ for all $F \in \mathscr{F}^*$. Hence

$$\pi_{\alpha_i}^{-1}(U_{\alpha_i}) = U_{\alpha_i} \times \prod_{\alpha \neq \alpha_i} X_\alpha$$

contains a point of $F$ for each $F \in \mathscr{F}^*$. Because of the maximality of $\mathscr{F}^*$ with respect to the finite intersection property, it follows that

$$\pi_{\alpha_i}^{-1}(U_{\alpha_i})$$

belongs to $\mathscr{F}^*$, and hence the finite intersection of these sets for

$$i = 1, \ldots, n$$

also belongs to $\mathscr{F}^*$. But this finite intersection is nothing else but our set $U$, and hence $U$ intersects each $F$ in $\mathscr{F}^*$, so a fortiori each $F \in \mathscr{F}$. Hence $x$ lies in the closure of each $F \in \mathscr{F}$, whence $x \in F$ for all $F \in \mathscr{F}$, as was to be shown.

**Corollary 3.13.** *A subset of* $\mathbf{R}^n$ *is compact if and only if it is closed and bounded.*

*Proof.* Let $S$ be a subset of $\mathbf{R}^n$ and assume first that $S$ is closed and bounded. Then $S$ is contained in the product of a finite number of closed intervals, and is therefore a closed subset of a compact space. It is thus compact. Conversely, if it is compact, it is closed, and it must be bounded; otherwise, one can find a sequence of elements in $S$ going out to infinity, and not having a point of accumulation.

**Corollary 3.14.** *All norms on* $\mathbf{R}^n$ *are equivalent.*

*Proof.* Let $\| \ \|$ be the sup norm, and $| \ |$ any other norm. It will suffice to prove that these two norms are equivalent. If $e_1, \ldots, e_n$ are the usual unit vectors of $\mathbf{R}^n$, then for $x = x_1 e_1 + \cdots + x_n e_n$ we get

$$|x| \leqq |x_1||e_1| + \cdots + |x_n||e_n| \leqq C\|x\|$$

with $C = n \cdot \max |e_i|$. This proves one of the desired inequalities, and also shows that the other norm is continuous, because

$$||x| - |y|| \leqq |x - y| \leqq C\|x - y\|.$$

Let $S_1$ be the unit sphere centered at the origin for the sup norm. Then $S_1$ is closed and bounded, so compact, and the other norm has a minimum on $S_1$, say at $v$. Thus for any $x \in \mathbf{R}^n$ we get

$$\left|\frac{x}{\|x\|}\right| \geqq |v|, \qquad \text{and hence} \qquad |v|\,\|x\| \leqq |x|.$$

This yields the other inequality, and proves our corollary.

Using coordinates, we see that Corollary 3.14 also applies to a finite dimensional vector space. A closed subset of a complete metric space is complete, and a complete subset of a metric space is closed. We conclude that a finite dimensional subspace of a normed vector space is complete, and therefore closed.

A space $X$ is said to be **locally compact** if every point has a compact neighborhood. For instance, $\mathbf{R}^n$ is locally compact, and so is any finite dimensional vector space. It is clear that a normed vector space is locally compact if and only if the closed unit ball is compact. (If the space is locally compact, then some closed ball of radius $r > 0$ is compact, and hence the unit ball is compact by multiplication with a positive number.)

**Corollary 3.15 (F. Riesz).** *A normed vector space is locally compact if and only if it is finite dimensional.*

*Proof.* Let $E$ be a locally compact normed vector space, and let $B$ be the closed ball of radius 1 centered at 0. We can find a finite number of points $x_1, \ldots, x_n \in B$ such that $B$ is covered by the open balls of radius $\frac{1}{2}$ centered at these points. We contend that $x_1, \ldots, x_n$ generate $E$. Let $F$ be the subspace generated by $x_1, \ldots, x_n$. Then $F$ is finite dimensional, hence closed in $E$ as a trivial consequence of Corollary 3.14. Suppose that $x \in E$ and $x \notin F$. Let

$$d(x, F) = \inf_{y \in F} |x - y|.$$

Drawing a closed ball around $x$ intersecting $F$, and using the fact that the intersection of $F$ and this ball is compact, we conclude that there is some $z \in F$ such that $d(x, F) = |x - z|$, and we have $x - z \neq 0$ since $F$ is closed in $E$. Then there is some $x_i$ such that

$$\left|\frac{x - z}{|x - z|} - x_i\right| < \frac{1}{2}$$

and consequently that

$$|x - z - |x - z|x_i| < \frac{|x - z|}{2}.$$

However, $z + |x - z|x_i$ lies in $F$, and by definition of $z$ such that

$$d(x, F) = |x - z|$$

we conclude that the left-hand side is $\geqq |x - z|$. This is a contradiction which proves our corollary.

Let $X$ be a locally compact Hausdorff space. One can construct a compact space by adjoining to $X$ a point "at infinity" as follows. Let $p$ be some point not in $X$ and let $X'$ be the union of $X$ and $\{p\}$. We define a base of open sets in $X'$ by throwing into this base all subsets of $X$ which are open in $X$, and the complements in $X'$ of compact sets in $X$. That this defines a base is clear, and one also verifies at once that $X'$ is then compact. It is called the **one point compactification of** $X$.

It is easy to see that the one point compactification of $\mathbf{R}$ is homeomorphic to a circle. The one point compactification of the plane $\mathbf{R}^2$ is homeomorphic to the sphere. In general, the one point compactification of $\mathbf{R}^n$ is homeomorphic to the $n$-sphere (i.e. the set of all $x \in \mathbf{R}^{n+1}$ such that $|x| = 1$, where $| \ |$ is the euclidean norm).

## II, §4. SEPARATION BY CONTINUOUS FUNCTIONS

We are concerned throughout this section with a normal space $X$ and the manner by which one can separate two disjoint closed sets by means of a continuous function.

**Lemma 4.1.** *Let $X$ be a normal space. If $A$ is closed in $X$ and $A \subset U$ is contained in an open set $U$, then there exists an open set $U_1$ such that*

$$A \subset U_1 \subset \overline{U}_1 \subset U.$$

*Proof.* Let $B$ be the complement of $U$. By the definition of normality, there exist disjoint open sets $U_1$, $V_1$ such that $A \subset U_1$ and $B \subset V_1$. It is clear that $U_1$ satisfies our requirements.

**Theorem 4.2 (Urysohn's Lemma).** *Let $X$ be a normal space and let $A$, $B$ be disjoint closed subsets. Then there exists a continuous function $f$ on $X$ with values in the interval $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.*

*Proof.* In a metric space, which is the most important in practice, one can give a trivial proof. Cf. Exercise 7. We now give the proof in general. Let $U_1$ be the complement of $B$ so that $A \subset U_1$. We find $U_{1/2}$ such that

$$A \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_1.$$

We then find $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_{3/4} \subset \overline{U}_{3/4} \subset U_1.$$

Inductively, for each integer $k$ with $0 \leqq k \leqq 2^n$, we find $U_{k/2^n}$ such that if $r < s$, then $U_r \subset \overline{U}_r \subset U_s$. We then define the function $f$ by

$$f(x) = 1 \qquad \text{if } x \in B,$$
$$f(x) = \inf \text{ of all } r \text{ such that } x \in U_r \text{ if } x \notin B.$$

It is then essentially clear that $f$ is continuous. We carry out the details. It will suffice to prove that for numbers $a$, $b$ such that $0 < a \leqq 1$ and $0 \leqq b < 1$ the inverse images of the half-open intervals

$$f^{-1}[0, a) \qquad \text{and} \qquad f^{-1}(b, 1]$$

are open. In fact, we have

$$f^{-1}[0, a) = \bigcup_{r<a} U_r$$

because $f(x) < a$ if and only if $x$ lies in some $U_r$ with $r < a$. Similarly, we have $f(x) > b$ if and only if $x \notin \overline{U}_r$ for some $r > b$, so that

$$f^{-1}(b, 1] = \bigcup_{r>b} \mathscr{C}\overline{U}_r.$$

This proves our theorem.

Since a compact Hausdorff space is normal. Urysohn's lemma applies in this case. One needs it frequently in the locally compact case in the following form.

**Corollary 4.3.** *Let $X$ be a locally compact Hausdorff space, and $K$ a compact subset. There exists a continuous function $g$ on $X$ which is 1 on $K$ and which is equal to 0 outside a compact set.*

*Proof.* Each $x \in K$ has an open neighborhood $V_x$ with compact closure. A finite number of such neighborhoods $V_{x_1}, \dots, V_{x_n}$ covers $K$. Let

$$V = V_{x_1} \cup \cdots \cup V_{x_n}.$$

Then the closure of $V$ is compact. There exists a continuous function $g \geqq 0$ on $\overline{V}$ (compact Hausdorff, hence normal) which is 1 on $K$ and 0 outside $V$, i.e. 0 on $\overline{V} \cap \mathscr{C}V$. We define $g$ to be 0 on the complement of $\overline{V}$ in $X$. Then $g$ is continuous at every point in the complement of $\overline{V}$, and as function on $X$ is also continuous on $\overline{V}$. This proves our corollary.

**Theorem 4.4 (Tietze Extension Theorem).** *Let $A$ be a closed subset of a normal space $X$ and let $f$ be a continuous (real valued) function on $A$. Then there exists a continuous function $f^*$ on $X$ whose restriction to $A$ is equal to $f$. If $f$ has values in $[0, 1]$, then we can choose $f^*$ to have values in $[0, 1]$ also.*

*Proof.* Assume first that $f$ has values in $[0, 1]$. If $A$, $B$ are disjoint closed subsets of $X$, we denote by $g_{A,B}$ a function with values in $[0, 1]$ such that $g(A) = 0$ and $g(B) = 1$. Such a function exists by Theorem 4.2.

We shall now define functions $f_n$ on $A$ and $g_n$ on $X$.
We let $f_0 = f$ and define sets $A_0$, $B_0$ by the conditions:

$$A_0 = \{x \in A \text{ such that } f(x) \leqq \tfrac{1}{3}\},$$
$$B_0 = \{x \in A \text{ such that } f(x) \geqq \tfrac{2}{3}\}.$$

We let $g_0 = \frac{1}{3} g_{A_0, B_0}$ and define $f_1 = f_0 - g_0$. Inductively, suppose that we have defined $f_n$; we have

$$A_n = \{x \in A \text{ such that } f_n(x) \leqq (\tfrac{1}{3})(\tfrac{2}{3})^n\},$$
$$B_n = \{x \in A \text{ such that } f_n(x) \geqq (\tfrac{2}{3})(\tfrac{2}{3})^n\}.$$

We then define

$$g_n = (\tfrac{1}{3})(\tfrac{2}{3})^n g_{A_n, B_n}$$

and let $f_{n+1} = f_n - g_n$. (Here of course, we understand by $g_n$ its restriction to $A$.) Then in particular:

$$f_{n+1} = f - (g_0 + \cdots + g_n).$$

We have

$$(*) \qquad 0 \leqq g_n \leqq \tfrac{1}{3}(\tfrac{2}{3})^n \quad \text{and} \quad 0 \leqq f_n \leqq (\tfrac{2}{3})^n.$$

The first inequality is clear. The second is proved by induction. It is clear for $n = 0$. Let $n > 0$. One distinguishes the three cases in which for a given $x \in A$ we have $x \in A_n$, or $x \notin A_n$ but $x \notin B_n$, or $x \in B_n$. The desired inequality of $f_n$ is then obvious in each case, using the inductive hypothesis.

From our inequalities (*), we then conclude that the series

$$g_0 + g_1 + \cdots + g_n + \cdots$$

converges pointwise, and furthermore converges to $f$ on $A$. The uniform bounds imply at once that the limit function is continuous, thus proving Theorem 4.4, when $f$ has values in $[0, 1]$.

**Remark 1.** The restriction to the interval $[0, 1]$ is of course unnecessary, and the theorem extends at once to any other closed bounded interval, for instance by mapping such an interval linearly on $[0, 1]$.

Now suppose that $f$ is unbounded. Using the arctangent map we reduce the theorem to the case when $f$ takes values in the open interval $(-1, 1)$ and we must then know that the extension can be so chosen that its values also lie in the open interval $(-1, 1)$. Let $B$ be the closed set where the extension $f^*$ (which we have constructed with values in $[-1, 1]$) takes on the values 1 or $-1$. Then $A$ and $B$ are disjoint, so that by Urysohn's lemma there exists a continuous function $h$ on $X$ with values in $[0, 1]$ such that $h$ is 1 on $A$ and 0 on $B$. Then $hf^*$ has values in the open interval $(-1, 1)$, as desired. This concludes the proof of Theorem 4.4.

**Remark 2.** The theorem also holds in the complex case dealing separately with the real and imaginary parts. The extra condition on the restriction of the values can then be formulated analogously by requiring that

$$\|f^*\| \leqq \|f\|.$$

Indeed, suppose that we have extended $f$ to a bounded continuous complex valued function $g$. Let $b = \|f\|$. Let $h$ be the function such that $h(z) = z$ if $|z| \leqq b$, and $h(z) = bz/|z|$ if $|z| > b$. Then $h$ is continuous, $\|h\| \leqq b$, and $h \circ g$ fulfills our requirement.

## II, §5. EXERCISES

1. (a) Let $X$, $Y$ be compact metric spaces. Prove that a mapping $f: X \to Y$ is continuous if and only if its graph is closed in $X \times Y$.
   (b) Let $Y$ be a complete metric space, and let $X$ be a metric space. Let $A$ be a subset of $X$. Let $f: A \to Y$ be a mapping that is uniformly continuous.

Let $\bar{A}$ be the closure of $A$ in $X$. Show that there exists a unique extension of $f$ to a continuous map $\bar{f}: \bar{A} \to Y$, and that $\bar{f}$ is uniformly continuous. You may assume that $X$, $Y$ are subsets of a Banach space if you wish, in order to write the distance function in terms of the absolute value sign.

2. **Seminorms.** Let $E$ be a vector space. A function $\sigma: E \to \mathbf{R}$ is called a **seminorm** if it satisfies the same conditions as a norm except that we allow $\sigma(x) = 0$ without necessarily having $x = 0$. In other words, $\sigma$ satisfies the following conditions:

**SN 1.** *We have $\sigma(x) \geqq 0$ for all $x \in E$.*

**SN 2.** *If $x \in E$ and $a$ is a number, then $\sigma(ax) = |a|\sigma(x)$.*

**SN 3.** *We have $\sigma(x + y) \leqq \sigma(x) + \sigma(y)$ for all $x, y \in E$.*

We also denote a seminorm by the symbols $| \;|$.

(a) If $| \;|$ is a seminorm on $E$, show that the set $E_0$ of elements $x \in E$ with $|x| = 0$ is a subspace.

(b) Define open balls with respect to a seminorm as with a norm. Show that the topology whose base is the family of open balls is Hausdorff if and only if the seminorm is a norm.

(c) Let $\{\sigma_n\}$ be a sequence of seminorms on $E$ such that the values $\sigma_n(x)$ are bounded. Let $\{a_n\}$ be a sequence of positive numbers such that $\sum a_n$ converges. Show that $\sum a_n\sigma_n$ is a seminorm.

(d) Let $\{\sigma_i\}_{i \in I}$ be a family of seminorms on a vector space $E$. Let $x_0 \in E$ and let $i_1, \ldots, i_n$ be a finite number of indices. Let $r > 0$. We call the set of all $x \in E$ such that

$$\sigma_{i_k}(x - x_0) < r, \qquad k = 1, \ldots, n,$$

a basic open set. Show that the family of basic open sets is a base for a topology on $E$, which is said to be determined by the family of seminorms.

3. (a) Let $l^1$ be the set of all sequences $\alpha = \{a_n\}$ of numbers (say, real) such that $\sum |a_n|$ converges. Define

$$|\alpha| = \sum |a_n|.$$

Show that this is a norm on $l^1$, and that $l^1$ is complete under this norm.

(b) Let $\beta = \{b_n\}$ be a fixed sequence in $l^1$. Show that the set of all $\alpha \in l^1$ such that $|a_n| \leqq |b_n|$ is compact. Show that the unit sphere in $l^1$ is not compact.

4. Let $\alpha$ be a real number, $0 < \alpha \leqq 1$. A real valued function $f$ on $[0, 1]$ is said to satisfy a **Hölder condition of order** $\alpha$ if there is a constant $C$ such that for all $x, y$ we have

$$|f(x) - f(y)| \leqq C|x - y|^\alpha.$$

For such a function, define

$$\|f\|_\alpha = \sup_x |f(x)| + \sup_{\substack{x,y \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

(a) Show that the set of functions satisfying such a Hölder condition is a vector space, and that $\| \ \|_\alpha$ is a norm on this space.

(b) Show that the set of functions $f$ with $\|f\|_\alpha \leqq 1$ is a compact subset of $C([0, 1])$.

5. **Metric spaces.** (a) Let $X$ be a metric space with distance function $d$. Define $d'(x, y) = \min\{1, d(x, y)\}$. Show that $d'$ is a distance function, and that the notion of convergence and limit with respect to $d'$ is the same as with respect to $d$.

(b) As in normed vector spaces, one can define Cauchy sequences, i.e. sequences $\{x_n\}$ such that given $\varepsilon$, there exists $N$ such that for all $m, n \geqq N$ we have $d(x_n, x_m) < \varepsilon$. A metric space is called **complete** if every Cauchy sequence converges. Show that if a metric space $X$ as in part (a) is complete with respect to $d$, then it is complete with respect to $d'$.

(c) For each $x \in X$ define the function $f_x$ on $X$ by

$$f_x(y) = d(x, y).$$

Let $\| \ \|$ be the sup norm. Show that

$$d(x, y) = \|f_x - f_y\|.$$

Let $a$ be a fixed element of $X$ and let $g_x = f_x - f_a$. Show that the map $x \mapsto g_x$ is a distance-preserving embedding of $X$ into the normed vector space of bounded functions on $X$. (If the metric is bounded, you can use $f_x$ instead of $g_x$). Thus one need not fuss too much with abstract metric spaces. Besides, almost all metric spaces which occur naturally are in fact given as subsets of normed vector spaces.

A topological space is said to be **metrizable** if there exists a metric such that the open balls form a basis for the topology. Such a metric is said to be **compatible** with the topology.

6. Let $A$ be a subset of a metric space $X$. For each $x \in X$, let

$$d(x, A) = \inf d(x, y)$$

for all $y \in A$. Show that the map

$$x \mapsto d(x, A)$$

is a continuous function on $X$, and that $d(x, A) = 0$ if and only if $x$ lies in the closure of $A$. We call $d(x, A)$ the **distance** from $x$ to $A$.

7. (a) Show that a metrizable space is normal. [*Hint*: Let $A$, $B$ be disjoint closed subsets. Let $U$ be the set of $x$ such that $d(x, A) < d(x, B)$ and let $V$ be the set of $x$ such that $d(x, B) < d(x, A)$.]

(b) If $A$, $B$ are disjoint closed subsets of a metric space, show that the function

$$x \mapsto d(x, A)/(d(x, A) + d(x, B))$$

can be used to prove Urysohn's lemma.

8. Let $X$ be a topological space and $E$ a normed vector space. Let $M(X, E)$ be the set of all maps of $X$ into $E$ and $C(X, E)$ the space of all continuous maps of $X$ into $E$. Let $B(X, E)$ be the space of all bounded maps, and $BC(X, E)$ the space of bounded continuous maps.
   (a) Show that $BC(X, E)$ is closed in $B(X, E)$.
   (b) Suppose that $E$ is complete, i.e. a Banach space. Show that $B(X, E)$ is complete, with the sup norm.
   (c) If $X$ is compact, show that $C(X, E) = BC(X, E)$.

9. **Uniform convergence on compact sets.** Let $X$ be a Hausdorff space. Let $M(X, E)$ be the space of maps of $X$ into a Banach space $E$. A sequence $\{f_n\}$ in this space is said to be **uniformly Cauchy on compact subsets** if given a compact set $K$ and $\varepsilon > 0$, there exists $N$ such that for $m, n \geqq N$, we have

$$\|f_n - f_m\|_K < \varepsilon,$$

where $\|\ \|_K$ is the sup norm on $K$. In other words, the sequence restricted to $K$ is uniformly Cauchy. The sequence is said to be **uniformly convergent on compact sets** if there is some map $f$ having the following property. Given a compact set $K$ and $\varepsilon$, there exists $N$ such that for $n \geqq N$, we have

$$\|f_n - f\|_K < \varepsilon.$$

In other words, the sequence restricted to $K$ is uniformly convergent. We shall now make $M(X, E)$ into a metric space for which the above convergence is the same as convergence with respect to this metric, in certain cases.

A sequence $\{K_i\}$ of compact subsets of $X$ said to be **exhaustive** if their union is equal to $X$, and if every compact subset of $X$ is contained in some $K_i$. We assume that there exists such a sequence $\{K_i\}$.

(a) Define

$$d(f) = \sum_{i=1}^{\infty} 2^{-i} \min(1, \|f\|_{K_i}).$$

If $f$ is unbounded on $K$, then we set $\|f\|_K = \infty$ and $\min(1, \|f\|_K) = 1$. Show that $d(f)$ satisfies two of the properties of a norm, namely:

$$d(f) = 0 \quad \text{if and only if} \quad f = 0;$$

$$d(f + g) \leqq d(f) + d(g).$$

(b) Define $d(f, g)$ by $d(f - g)$. Show that $d(f, g)$ is a metric on $M(X, E)$.

(c) Show that

$$2^{-i} \inf(1, \|f\|_{K_i}) \leqq d(f) \qquad \text{and} \qquad d(f) \leqq \|f\|_{K_i} + 2^{-i}.$$

(d) Show that a sequence $\{f_n\}$ converges uniformly on compact sets if and only if it converges in the above metric.

(e) Let $K$ be a compact set and $\varepsilon > 0$. Given $f$, let $V(f, K, \varepsilon)$ be the set of all maps $g$ such that $\|f - g\|_K < \varepsilon$. Show that $V(f, K, \varepsilon)$ is open in the topology defined by the metric. Show that the family of all such open

sets for all choices of $f$, $K$, $\varepsilon$ is base for the topology. This proves that the topology does not depend on the choice of exhaustive sequence $\{K_i\}$.

(f) If $E$ is complete, i.e. a Banach space, show that $M(X, E)$ is complete in the metric defined above.

(g) If $X$ is locally compact, show that the space of continuous maps $C(X, E)$ is closed in $M(X, E)$ for the metric.

10. Let $U$ be the open unit disc in the plane. Show that there is an exhaustive sequence of compact subsets of $U$.

11. Let $U$ be a connected open set in the plane (or in Euclidean space $\mathbf{R}^k$). Show that there is an exhaustive sequence of compact subsets of $U$.

12. Let $U$ be an open subset of a normed vector space. Show that $U$ is connected if and only if $U$ is arcwise connected.

13. The diagonal $\Delta$ in a product $X \times X$ is the set of all points $(x, x)$.
    (a) Show that a space $X$ is Hausdorff if and only if the diagonal is closed in $X \times X$.
    (b) Show that a product of Hausdorff spaces is Hausdorff.

14. If $A$ is a subspace of a space $X$, we define the **boundary** of $A$ (denoted by $\partial A$) to be the set of all $x$ such that any open neighborhood $U$ of $x$ contains a point of $A$ and a point not in $A$. In other words, $\partial A = \bar{A} \cap (\overline{\mathscr{C}A})$.
    (a) Show that $\partial(A \cup B) \subset \partial A \cup \partial B$.
    (b) Show that $\partial(A \cap B) \subset \partial A \cup \partial B$.
    (c) Let $X$, $Y$ be topological spaces, and let $A$ be a subset of $X$, $B$ a subset of $Y$. Show that

$$\partial(A \times B) = (\partial A \times \bar{B}) \cup (\bar{A} \times \partial B).$$

    (d) Let $A$ be a subset of a complete normed vector space $E$. Let $x \in A$ and let $y$ be in the complement of $A$. Show that there exists a point on the line segment between $x$ and $y$ which lies on the boundary of $A$. (The line segment consists of all points $x + t(y - x)$ with $0 \leqq t \leqq 1$.)

## Separable Spaces

15. A topological space having a countable base for its open sets is called **separable**. Show that a separable space has a countable dense subset.

16. (a) If $X$ is a metric space and has a countable dense subset, then $X$ is separable.
    (b) A compact metric space is separable.

17. (a) Every open covering of a separable space has a countable subcovering.
    (b) A disjoint collection of open sets in a separable space is countable.
    (c) A base for the open sets of a separable space contains a countable base.

18. A denumerable product of separable (resp. metric) spaces is separable (resp. metric).

19. Let $X$ be separable. Show that the following conditions are equivalent:
   (a) $X$ is compact.
   (b) Every sequence $\{x_n\}$ in $X$ has at least one point of accumulation, that is $X$ is sequentially compact.
   (c) Every decreasing sequence $\{A_n\}$ of non-empty closed sets has a nonempty intersection.

20. Prove that a normal separable space $X$ is metrizable (**Urysohn metrization theorem**). [*Hint*: Let $\{U_n\}$ be a countable base for the topology. Let $(U_{n_i}, U_{m_i})$ be an enumeration of all pairs of elements in this base such that $\overline{U}_{n_i} \subset U_{m_i}$. For each $i$ let $f_i$ be a continuous function satisfying $0 \leqq f_i \leqq 1$ and such that $f_i$ is 0 on $\overline{U}_{n_i}$ and 1 on the complement of $U_{m_i}$. Let

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i}|f_i(x) - f_i(y)|.]$$

Show that $d$ is a metric and that the identity mapping is continuous with respect to the given topology on $X$ and the topology obtained from the metric. You will use the fact that given $x \in X$ and some open set $U_m$ in the base containing $x$, there exists another set $U_n$ in the base such that

$$x \in U_n \subset \overline{U}_n \subset U_m.$$

21. **Regular spaces.** A topological space $X$ is called **regular** if it is Hausdorff, and if given a point $x$ and a closed set $A$ not containing $x$, there exist disjoint open sets $U$, $V$ such that $x \in U$ and $A \subset V$.
   (a) A subspace of a regular space is regular.
   (b) Let $X$ be a topological space. If every point has a closed neighborhood which is regular, then $X$ is regular.
   (c) Every locally compact Hausdorff space is regular.
   (d) If $X$ is separable regular, show that every point $x$ has a sequence of open neighborhoods such that:
      (i) $\overline{U}_{n+1} \subset U_n$;
      (ii) $\{x\} = \bigcap U_n$.

The following exercises are of somewhat less general interest than the preceding ones (but some are more amusing).

22. **Proper maps.** Let $X$, $Y$ be topological spaces and $f: X \to Y$ a map. We say that $f$ is **closed** if $f$ maps closed sets into closed sets. We say that $f$ is **proper** if $f$ is continuous and if for every topological space $Z$ the map

$$f \times I_Z = f_Z: X \times Z \to Y \times Z$$

given by $f_Z(x, z) = \big(f(x), z\big)$ is closed.
   (a) Show that a proper map is closed.
   (b) For each $i = 1, \ldots, n$ let $f_i: X_i \to Y_i$ be a continuous map. Assume that $X_i$ is not empty for each $i$. Let $f: \prod X_i \to \prod Y_i$ be the product map. Show that $f$ is proper if and only if all $f_i$ are proper.
   (c) If $f: X \to Y$ is proper and $A$ is closed in $X$, show that $f|A$ is proper.

23. Let $f: X \to X'$ and $g: X' \to X''$ be continuous maps. Prove:
(a) If $f$ and $g$ are proper, so is $g \circ f$.
(b) If $g \circ f$ is proper and $f$ is surjective, then $g$ is proper.
(c) If $g \circ f$ is proper and $g$ is injective, then $f$ is proper.
(d) If $g \circ f$ is proper and $X'$is Hausdorff, then $f$ is proper.

24. Let $X$ be a topological space, $\{p\}$ a set consisting of one element $p$. The map $f: X \to \{p\}$ is proper if and only if $X$ is compact. [*Hint*: Assume that $f$ is proper. To show that $X$ is compact, let $\{S_\alpha\}$ be a family of non-empty closed sets having the finite intersection property. Let $Y = X \cup \{p\}$, where $p$ is disjoint from $X$. Define a base for a topology of $Y$ by letting a set be in this base if it is of type $S_\alpha \cup \{p\}$, or if it is an arbitrary subset of $X$. Show this is a base. The projection $\pi: X \times Y \to Y$ is a closed map. Let $D$ be the subset of $X \times Y$ consisting of all pairs $(x, x)$ with $x \in X$. Then $\pi(\overline{D})$ is closed and therefore contains $p$. Hence there exists $x \in X$ such that $(x, p) \in \overline{D}$, whence give an open $U$ in $X$ containing $x$, and any $S_\alpha$, the set $U \times (S_\alpha \cup \{p\})$ intersects $D$, whence $U$ intersects $S_\alpha$, and $x$ lies in $\bigcap S_\alpha$.]

25. Let $f: X \to Y$ be a continuous map. Show that the following properties are equivalent:
(a) $f$ is proper.
(b) $f$ is closed and for each $y \in Y$ the set $f^{-1}(y)$ is compact.

26. Let $f: X \to Y$ be proper. If $B$ is a compact subset of $Y$, then $f^{-1}(B)$ is compact.

27. (**The marriage problem** *so baptized by Hermann Weyl.*) Let $B$ be a set of boys, and assume that each boy $b$ knows a finite set of girls $G_b$. The problem is to marry each boy to a girl of his acquaintance, injectively. A necessary condition is that each set of $n$ boys know collectively at least $n$ girls. Prove that this condition is sufficient. [*Hint*: First assume that $B$ is finite, and use induction. Let $n > 1$. If for all $1 \leqq k < n$ each set of $k$ boys knows $> k$ girls, marry off one boy and refer the others to the induction hypothesis. If for some $k$ with $1 \leqq k < n$ there exists a subset of $k$ boys knowing exactly $k$ girls, marry them by induction. The remaining $n - k$ boys satisfy the induction hypothesis with respect to the remaining girls (obvious!) and thus the case of finite $B$ is settled. For the infinite case, which is really the relevant problem here, take the Cartesian product $\prod G_b$ over all $b \in B$, each $G_b$ being finite, discrete, and use Tychonoff's theorem. For this elegant proof, cf. Halmos and Vaughn, *Amer. J. Math.* January 1950, pp. 214–215.]

28. **The Cantor set.** Let $K$ be the subset of $[0, 1]$ consisting of all numbers having a trecimal expansion

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

where $a_n = 0$ or $a_n = 2$. This set is called the Cantor set. Show that $K$ is compact. Show that the complement of $K$ consists of a denumerable union of intervals, and that the sum of the lengths of these intervals is 1. Show that the connected component of each point in $K$ is the point itself. (One says that $K$ is totally disconnected.)

[It can be shown that a compact metric space is always a continuous image of a Cantor set, and also that a totally disconnected compact metric space is homeomorphic to a Cantor set. Cf. books on general topology. The Cantor set has measure 0, is not countable, and is a rich source for counterexamples.]

29. **Peano curve.** Let $K$ be the Cantor set of the preceding exercise. Let $S = [0, 1] \times [0, 1]$ be the unit square. Let $f: K \to S$ be the map which to each element $\sum a_n/3^n$ of the Cantor set assigns the pair of numbers

$$\left(\sum \frac{b_{2n+1}}{2^n}, \sum \frac{b_{2n}}{2^n}\right),$$

where $b_m = a_m/2$. Show that $f$ is well defined. Show that $f$ is surjective and continuous. One can then extend $f$ to a continuous map of the interval onto the square. This is called a Peano curve. Note that the interval has dimension 1 whereas its image under the continuous map $f$ has dimension 2. This caused quite a sensation at the end of the nineteenth century when it was discovered by Peano.

30. **The semi parallelogram law** (Bruhat-Tits). Let $X$ be a complete metric space. We say that $X$ satisfies the semi parallelogram law, or is seminegative, if given two points $x_1, x_2 \in X$ there is a point $z$ such that for all $x \in X$ we have

$$d(x_1, x_2)^2 + 4d(x, z)^2 \leqq 2d(x, x_1)^2 + 2d(x, x_2)^2.$$

Prove that under this law, $d(z, x_1) = d(x_1, x_2)/2$, and $z$ is uniquely determined. We call $z$ the **midpoint** of $x_1, x_2$.

31. (Serre, after Bruhat-Tits) Let $X$ be a seminegative complete metric space. Let $S$ be a bounded subset of $X$. Show that there exists a unique closed ball $\bar{\mathbf{B}}_r(x_1)$ of minimal radius containing $S$. [Use the semiparallelogram law both for uniqueness and existence. For existence, show that if $\{\bar{\mathbf{B}}_{r_n}(x_n)\}$ is a sequence of closed balls containing $S$ with $\lim r_n = r$ (the inf of all radii of closed balls containing $S$), then $\{x_n\}$ is Cauchy.] The center of that closed ball is called the **circumcenter** of $S$.

32. (Bruhat-Tits fixed point theorem) Let $X$ be a complete seminegative metric space. Let $G$ be a group of isometries of $X$, i.e. bijective maps $f: X \to X$ which preserve distance. Denote the action of $G$ by $(g, x) \mapsto g.x$. Suppose $G$ has a bounded orbit (i.e. there is a point $x$ such that the set $S$ of all elements $g.x$, $g \in G$, is bounded). Then $G$ has a fixed point (the **circumcenter**) of the orbit.

For the above exercises, cf. Bruhat-Tits, *Groupes Réductifs sur un Corps Local* I, *Pub. IHES* **41** (1972) pp. 5–251; and K. Brown, *Buildings*, Springer Verlag, 1989, Chapter VI, Theorem 2 of §5.

CHAPTER III

# Continuous Functions on Compact Sets

## III, §1. THE STONE–WEIERSTRASS THEOREM

Let $E$ be a normed vector space (over the real or the complex numbers). We can define the notion of Cauchy sequence in $E$ as we did for real sequences, and also the notion of convergent sequence (having a limit). If every Cauchy sequence converges, then $E$ is said to be **complete**, and is also called a **Banach space**. A closed subspace of a Banach space is complete, hence it is also a Banach space.

**Examples.** Let $S$ be a non-empty set, and let $F$ be a normed vector space. We denote by $B(S, F)$ the space of bounded maps from $S$ into $F$. It is a normed vector space under the sup norm, and if $F$ is a Banach space, then $B(S, F)$ is complete, and thus is also a Banach space. The proof that $B(S, F)$ is complete if $F$ is complete should be carried out as an exercise. (The reader should have had a similar proof as part of a course in advanced calculus but, at any rate, has had it for functions which are real valued. The proof applies as well to Banach spaces.) If $S$ is a subset of a normed vector space (or a metric space) we denote by $C(S, F)$ the space of continuous maps of $S$ into $F$, and by $BC(S, F)$ the subspace of bounded continuous maps. Then $BC(S, F)$ is closed in $B(S, F)$, this being nothing else but a special case of the assertion that a uniform limit of continuous maps is continuous. Again, the reader should have seen a proof in the case of functions, and that same proof (a $3\varepsilon$-proof) applies to the case of maps into Banach spaces. (Do Exercise 0 if you have never done it before, or look up *Undergraduate Analysis.*)

Let $X$ be a set. By an **algebra** $A$ of functions on $X$ (say, real valued) we mean a subset of the ring of all functions having the properties that if

$f$, $g \in A$, then $f + g$ and $fg$ are in $A$, and if $c \in \mathbf{R}$, then $cf \in A$. Most of the algebras we deal with also contain the constant functions (identified with $\mathbf{R}$ itself). We make a similar definition of an algebra over $\mathbf{C}$.

For example, a polynomials in one variable form an algebra, and so do polynomials in several variables. If $\varphi$ is a function on some set $S$, then the set of all functions which can be written in the form

$$a_0 + a_1\varphi + \cdots + a_n\varphi^n$$

with $a_i \in \mathbf{R}$ form an algebra, said to be generated by $\varphi$. Similarly, we have the notion of an algebra generated by a finite number of functions $\varphi_1, \ldots, \varphi_r$, or by a family of functions. It is the algebra of polynomials in $\varphi_1, \ldots, \varphi_r$. If $X$ is a topological space, the set of all continuous functions is an algebra, denoted by $C(X)$. If we wish to specify the range of values (real or complex), we write $C(X, \mathbf{C})$ or $C(X, \mathbf{R})$. Recall that a **function** is a mapping with values in $\mathbf{R}$ or $\mathbf{C}$.

Let $S$ be a compact set. Let $A$ be an algebra of continuous functions on $S$. Every function in $A$ is bounded because $S$ is compact, and consequently we have the sup norm on $A$, namely for $f \in A$,

$$\|f\| = \sup_{x \in S} |f(x)|.$$

Thus $A$ is contained in the normed vector space of all bounded functions on $S$. We are interested in determining the closure of $A$. Since $C(S)$ is closed, the closure of $A$ will be contained in $C(S)$. We shall find conditions under which it is equal to $C(S)$. In other words, we shall find conditions under which every continuous function on $S$ can be uniformly approximated by elements of $A$.

We shall say that $A$ **separates points** of $S$ if given points $x$, $y \in S$, and $x \neq y$, there exists a function $f \in A$ such that $f(x) \neq f(y)$. The ordinary algebra of polynomial functions obviously separates points, since the function $f(x) = x$ already does so.

**Theorem 1.1 (Stone–Weierstrass Theorem).** *Let $S$ be a compact set, and let $A$ be an algebra of real valued continuous functions on $S$. Assume that $A$ separates points and contains the constant functions. Then the uniform closure of $A$ is equal to the algebra of all real continuous functions on $S$.*

We shall first prove the theorem under an extra assumption. We shall get rid of the extra assumption afterwards.

**Lemma 1.2.** *In addition to the hypotheses of the theorem, assume also that if $f$, $g \in A$ then $\max(f, g) \in A$, and $\min(f, g) \in A$. Then the conclusion of the theorem holds.*

*Proof.* We give the proof in three steps. First, we prove that given $x_1$, $x_2 \in S$ and $x_1 \neq x_2$, and given real numbers $\alpha$, $\beta$, there exists $h \in A$ such that $h(x_1) = \alpha$ and $h(x_2) = \beta$. By hypothesis, there exists $\varphi \in A$ such that $\varphi(x_1) \neq \varphi(x_2)$. Let

$$h(x) = \alpha + (\beta - \alpha)\frac{\varphi(x) - \varphi(x_1)}{\varphi(x_2) - \varphi(x_1)}.$$

Then $h$ satisfies our requirements.

Next we are given a continuous function $f$ on $S$ and also given $\varepsilon$. We wish to find a function $g \in A$ such that

$$f(y) - \varepsilon < g(y) < f(y) + \varepsilon$$

for all $y \in S$. This will prove what we want. We shall satisfy these inequalities one after the other. For each pair of points $x$, $y \in S$ there exists a function $h_{x,y} \in A$ such that

$$h_{x,y}(x) = f(x) \qquad \text{and} \qquad h_{x,y}(y) = f(y).$$

If $x = y$, this is trivial. If $x \neq y$, this is what we proved in the first step. We now fix $x$ for the moment. For each $y \in S$ there exists an open ball $U_y$ centered at $y$ such that for all $z \in U_y$ we have

$$h_{x,y}(z) < f(z) + \varepsilon.$$

This is simply the continuity of $f - h_{x,y}$ at $y$. The open sets $U_y$ cover $S$, and since $S$ is compact, there exists a finite number of points $y_1, \ldots, y_n$ such that $U_{y_1}, \ldots, U_{y_n}$ already cover $S$. Let

$$h_x = \min(h_{x,y_1}, \ldots, h_{x,y_n}).$$

Then $h_x$ lies in $A$ according to the additional hypothesis of the lemma (and induction). Furthermore, we have for all $z \in S$:

$$h_x(z) < f(z) + \varepsilon,$$

and $h_x(x) = f(x)$, that is $(h_x - f)(x) = 0$.

Now for each $x \in S$ we find an open ball $V_x$ centered at $x$ such that, by continuity, for all $z \in V_x$ we have $(h_x - f)(z) > -\varepsilon$, or in other words,

$$f(z) - \varepsilon < h_x(z).$$

By compactness, we can find a finite number of points $x_1, \ldots, x_m$ such

that $V_{x_1}, \dots, V_{x_m}$ cover $S$. Finally, let

$$g = \max(h_{x_1}, \dots, h_{x_m}).$$

Then $g$ lies in $A$, and we have for all $z \in S$

$$f(z) - \varepsilon < g(z) < f(z) + \varepsilon,$$

thereby proving the lemma.

The theorem is an easy consequence of the lemma, and will follow if we can prove that whenever $f, g \in A$ then $\max(f, g)$ and $\min(f, g)$ lie in the closure of $A$. To prove this, we note first that we can write

$$\max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2},$$

$$\min(f, g) = \frac{f + g}{2} - \frac{|f - g|}{2}.$$

Consequently it will suffice to prove that if $f \in A$ then $|f| \in A$.

Since $f$ is bounded, there exists a number $c > 0$ such that

$$-c \leqq f(x) \leqq c$$

for all $x \in S$. The absolute value function can be uniformly approximated by ordinary polynomials on the interval $[-c, c]$ by Exercises 6, 7, or 8, which are very simple ad hoc proof. Given $\varepsilon$, let $P$ be a polynomial such that

$$|P(t) - |t|| < \varepsilon$$

for $-c \leqq t \leqq c$. Then

$$|P(f(x)) - |f(x)|| < \varepsilon,$$

and hence $|f|$ can be approximated by $P \circ f$. Explicitly, if

$$P(t) = a_n t^n + \cdots + a_0,$$

then

$$P \circ f = a_n f^n + \cdots + a_0,$$

i.e.

$$P(f(x)) = a_n f(x)^n + \cdots + a_0.$$

This concludes the proof of the Stone–Weierstrass theorem.

**Corollary 1.3.** *Let $S$ be a compact set in $\mathbf{R}^k$. Any real continuous function on $S$ can be uniformly approximated by polynomial functions in $k$ variables.*

*Proof.* The set of polynomials contains the constants, and obviously separates points of $\mathbf{R}^k$ since the coordinate functions $x_1, \ldots, x_k$ already do this. So the theorem applies.

There is a complex version of the Weierstrass–Stone theorem. Let $A$ be an algebra of complex valued functions on the set $S$. If $f \in A$, we have its complex conjugate $\bar{f}$ defined by

$$\bar{f}(x) = \overline{f(x)}.$$

For instance, if $f(x) = e^{ix}$ then $\bar{f}(x) = e^{-ix}$. If $A$ is an algebra over $\mathbf{C}$ of complex valued functions, we say that $A$ is **self conjugate** if whenever $f \in A$ the conjugate function $\bar{f}$ is also in $A$.

**Theorem 1.4 (Complex S–W Theorem).** *Let $S$ be a compact set and $A$ an algebra (over* $\mathbf{C}$*) of complex valued continuous functions on $S$. Assume that $A$ separates points, contains the constants, and is self conjugate. Then the uniform closure of $A$ is equal to the algebra of all complex valued continuous functions on $S$.*

*Proof.* Let $A_{\mathbf{R}}$ be the set of all functions in $A$ which are real valued. We contend that $A_{\mathbf{R}}$ is an algebra over $\mathbf{R}$ which satisfies the hypotheses of the preceding theorem. It is obviously an algebra over $\mathbf{R}$. If $x_1 \neq x_2$ are points of $S$, there exists $f \in A$ such that $f(x_1) = 0$ and $f(x_2) = 1$. (The proof of the first step of Lemma 1.2 shows this.) Let $g = f + \bar{f}$. Then $g(x_1) = 0$ and $g(x_2) = 2$, and $g$ is real valued, so $A_{\mathbf{R}}$ separates points. It obviously contains the real constants, and so the real S–W theorem applies to it. Given a complex continuous function $\varphi$ on $S$, we write $\varphi = u + iv$, where $u$, $v$ are real valued. Then $u$, $v$ are continuous, and $u$, $v$ can be approximated uniformly by elements of $A_{\mathbf{R}}$, say $f, g \in A_{\mathbf{R}}$ such that $\|u - f\| < \varepsilon$ and $\|v - g\| < \varepsilon$. Then $f + ig$ approximates $u + iv = \varphi$, thereby concluding the proof.

**Remark.** The Stone–Weierstrass theorem has a useful application to locally compact spaces. For such corollaries, we refer the reader to Chapter IX, §6, and Chapter XVI, §3. For explicit approximations in concrete cases, see the Exercises and also Chapter VIII, §1.

## III, §2. IDEALS OF CONTINUOUS FUNCTIONS

The second theorem of this chapter deals with ideals of continuous functions. Let $S$ be a topological space, and $R$ a ring of continuous functions (real valued) on $S$. An **ideal** $J$ of $R$ is a subset of $R$ satisfying the following properties: The zero function 0 is in $J$. If $f, g \in J$, then $f + g$ and $-f$ are in $J$, and if $h \in R$, then $hf \in J$. The reader should really have

met the definition of an ideal in an algebra course, but we don't assume this here, although some motivation from algebra is useful.

Let $f$ be continuous on $S$. A **zero** of $f$ is a point $x \in S$ such that $f(x) = 0$. The set of zeros of $f$ is a closed set denoted by $Z_f$. Let $J$ be an ideal. Then the set

$$Z(J) = \bigcap_{f \in J} Z_f,$$

equal to the intersection of the sets of zeros of all $f \in J$, is closed, and is called the set of **zeros** of $J$. If $J$, $J'$ are two ideals, and $J \subset J'$, then $Z(J) \supset Z(J')$. We ask to what extent the set of zeros of an ideal determines this ideal, and answer this question in an important case.

**Theorem 2.1.** *Let $X$ be a compact space, and let $R$ be the ring of continuous functions on $X$, with the sup norm. Let $J$ be a closed ideal (i.e. an ideal, closed under the sup norm). If $f \in R$ is such that $f(x) = 0$ for all zeros $x$ of $J$ (i.e. if $f$ vanishes on the set of zeros of $J$), then $f$ lies in $J$.*

*Proof.* Given $\varepsilon$, let $U$ be the subset of $X$ consisting of all $x \in X$ such that $|f(x)| < \varepsilon$. Then $U$ is open, and the complement $S$ of $U$ is closed, and hence compact. Note that $U$ contains $Z_f$. For each $y \in S$, we can find a function $g_y$ in $J$ such that $g_y(x) \neq 0$ in some open neighborhood $V_y$ of $y$ (by continuity). There is some finite covering $\{V_{y_1}, \ldots, V_{y_m}\}$ of $S$ corresponding to functions $g_{y_1}, \ldots, g_{y_m}$. Let

$$g = g_{y_1}^2 + \cdots + g_{y_m}^2.$$

Then $g$ is in $J$, is continuous, is nowhere 0 on $S$, and $\geqq 0$. Since $g$ has a minimum on $S$, there is a number $a > 0$ such that $g(x) \geqq a$ for all $x \in S$. The function

$$\frac{ng}{1 + ng}$$

lies in $J$, because $1 + ng$ is nowhere 0 on $X$, its inverse is continuous on $X$, so in $R$, and hence $(1 + ng)^{-1} ng \in J$. For $n$ large, the function $ng/(1 + ng)$ tends uniformly to 1 on $S$, and hence the function

$$f \frac{ng}{1 + ng}$$

lies in $J$, and approximates $f$ within $\varepsilon$ on $S$. Since $0 \leqq ng/(1 + ng) \leqq 1$ it follows that on $U$ we have the estimate

$$0 \leqq |fng/(1 + ng)| < \varepsilon,$$

and so $fng/(1 + ng)$ lies within $2\varepsilon$ of $f$. Thus we have shown that $f$ lies in the closure of $J$. Since $J$ is assumed closed, we conclude that $f$ lies in $J$, thereby proving our theorem.

**Remark 1.** Situations analogous to that of Theorem 2.1 arise frequently in mathematics. For instance, let $R$ be the ring of polynomials in $n$ variables over the complex numbers, $R = \mathbf{C}[t_1, \ldots, t_n]$. Let $J$ be an ideal of $R$, and define zeros of $J$ to be $n$-tuples of complex numbers $x$ such that $f(x) = 0$ for all $f \in J$. It is shown in algebraic geometry courses that if $f$ is a polynomial in $R$ which vanishes on $Z(J)$, then $f^m \in J$ for some positive integer $m$. This is called **Hilbert's Nullstellensatz**.

**Remark 2.** Theorem 2.1 is but an example of a type of theorem which describes the topology of a space and describes properties of a space in terms of the ring of continuous functions on that space. (Cf. also Exercise 5.) This is one way in which one can algebraicize the study of certain topological spaces.

## III, §3. ASCOLI'S THEOREM

In the examples of Chapter XVIII, §4, we shall deal with compact subsets of function spaces, and we need a criterion for compactness, which is provided by Ascoli's theorem. It is also used in other places in analysis, for instance in a proof of the Riemann mapping theorem in complex analysis. Therefore, we give a proof here in the general discussion of compact spaces.

Let $X$ be a subset of a metric space, and let $F$ be a Banach space. Let $\Phi$ be a subset of the space of continuous maps $C(X, F)$. We shall say that $\Phi$ is (or its elements are) **equicontinuous** at a point $x_0 \in X$ if given $\varepsilon$, there exists $\delta$ such that whenever $x \in X$ and $d(x, x_0) < \delta$, then

$$|f(x) - f(x_0)| < \varepsilon$$

for all $f \in \Phi$. We say that $\Phi$ is **equicontinuous** on $X$ if it is equicontinuous at every point of $X$.

**Theorem 3.1 (Ascoli's Theorem).** *Let $X$ be a compact subset of a metric space, and let $F$ be a Banach space. Let $\Phi$ be a subset of the space of continuous maps $C(X, F)$ with sup norm. Then $\Phi$ is relatively compact in $C(X, F)$ if and only if the following two conditions are satisfied:*

**ASC 1.** *$\Phi$ is equicontinuous.*

**ASC 2.** *For each $x \in X$, the set $\Phi(x)$ consisting of all values $f(x)$ for $f \in \Phi$ is relatively compact.*

*Proof.* Assume that $\Phi$ satisfies the two conditions. We shall prove that $\Phi$ is relatively compact. For this it is sufficient to show that $\Phi$ can be covered by a finite number of balls of prescribed radius (Corollary 3.9 of Chapter II). Let $r > 0$. By equicontinuity, for each $x \in X$ we select an open neighborhood $V(x)$ such that if $y \in V(x)$, then $|f(y) - f(x)| < r$ for all $f \in \Phi$. Then a finite number $V(x_1), \ldots, V(x_n)$ cover $X$. Each set

$$\Phi(x_1), \ldots, \Phi(x_n)$$

is relatively compact, and hence so is their union

$$Y = \Phi(x_1) \cup \cdots \cup \Phi(x_n).$$

Let $B(a_1), \ldots, B(a_m)$ be open balls of radius $r$ centered at points $a_1, \ldots, a_m$ which cover $Y$. Then $f(x_1), \ldots, f(x_n)$ lie in these balls. In fact, for each $i = 1, \ldots, n$ we have

$$f(x_i) \in B(a_{\sigma i})$$

where $\sigma: \{1, \ldots, n\} \to \{1, \ldots, m\}$ is some mapping. For each such map $\sigma$ let $\Phi_\sigma$ be the set of $f \in \Phi$ such that for all $i$, we have

$$|f(x_i) - a_{\sigma i}| < r.$$

Then the finite number of $\Phi_\sigma$ cover $\Phi$. It suffices now to prove that each $\Phi_\sigma$ has diameter $< 4r$. But if $f$, $g \in \Phi_\sigma$ and $x \in X$, then $x$ lies in some $V(x_i)$, and then:

$$\begin{aligned} |f(x) - g(x)| &\leqq |f(x) - f(x_i)| + |f(x_i) - a_{\sigma i}| + |a_{\sigma i} - g(x_i)| + |g(x_i) - g(x)| \\ &< 4r. \end{aligned}$$

This proves our implication, and the part of Ascoli's theorem which is used in the applications. The converse is trivial and left to the reader.

Ascoli's theorem is used mostly when $F$ is the real or complex numbers, and in that case, we reformulate it as a corollary.

**Corollary 3.2.** *Let $X$ be a compact subset of a metric space, and let $\Phi$ be a subset of the space of continuous functions on $X$ with sup norm. Then $\Phi$ is relatively compact if and only if $\Phi$ is equicontinuous and bounded (for the sup norm, of course).*

*Proof.* For each $x \in X$, our hypothesis that $\Phi(x)$ is bounded implies that $\Phi(x)$ is relatively compact, since a closed bounded subset of a finite dimensional space is compact. So we can apply the theorem.

**Remark.** Since $\Phi$ has a metric defined by the sup norm, as a relatively compact set it has the property that any sequence has a convergent subsequence, converging in its closure. Sometimes one deals with a locally compact set $X$ which is a denumerable union of compact sets. In that case, one obtains the following version of Ascoli's theorem.

**Corollary 3.3.** *Let $X$ be a metric space whose topology has a countable base $\{U_i\}$ such that the closure $\overline{U}_i$ of each $U_i$ is compact. Let $\{f_n\}$ be a sequence of continuous maps of $X$ into a Banach space. Assume that $\{f_n\}$ is equicontinuous (as a family of maps), and is such that for each $x \in X$, the closure of the set $\{f_n(x)\}$ $(n = 1, 2, \ldots)$ is compact. Then there exists a subsequence which converges pointwise to a continuous function $f$, and such that the convergence is uniform on every compact subset.*

*Proof.* We can find a sequence $\{V_i\}$ of open sets such that $\overline{V}_i \subset V_{i+1}$, such that $\overline{V}_i$ is compact, and such that the union of the $V_i$ is $X$. For each $i$, by the previous version of Ascoli's theorem, there exists a subsequence which converges uniformly on $\overline{V}_i$. The diagonal sequence with respect to all $i$ converges uniformly on every compact set. This proves the corollary.

**Remark.** In light of Urysohn's metrization lemma, the hypotheses on $X$ in the corollary could be given as $X$ separable locally compact.

## III, §4. EXERCISES

0. Let $S$ be a subset of a normed vector space (or a metric space), and let $\{f_n\}$ be a sequence of continuous maps of $S$ into a Banach space $F$. Assume that $\{f_n\}$ is a Cauchy sequence (for the sup norm). Show that $\{f_n\}$ converges to a continuous function $f$ (for the sup norm). Show that $BC(S, F)$ is closed in $B(S, F)$.

1. Let $X$ be a compact set and let $R$ be the ring of continuous (real valued) functions on $X$. Let $J$, $J'$ be closed ideals of $R$. Show that $J \subset J'$ if and only if $Z(J) \supset Z(J')$.

2. Let $S$ be a closed subset of $X$. Let $J$ be the set of all $f \in R$ such that $f$ vanishes on $S$. Show that $J$ is a closed ideal. Assume that $X$ is Hausdorff. Establish a ring-isomorphism between the factor ring $R/J$ and the ring of continuous functions on $S$. (We assume that you have had the notion of a factor ring in an algebra course.)

3. Let $X$ be a compact space and let $J$ be an ideal of $C(X)$. If the set of zeros of $J$ is empty, show that $J = C(X)$. (This result is valid in both the real and the complex case.)

4. Let $X$ be a compact Hausdorff space. Show that a maximal ideal of $C(X)$ has only one zero, and is closed. (Recall that an ideal $M$ is said to be maximal if $M \neq C(X)$, and if there is no ideal $J$ such that $M \subset J \subset C(X)$ other than $M$ and $C(X)$ itself.) Thus if $M$ is maximal, then there exists $p \in X$ such that $M$ consists of all continuous functions $f$ vanishing at $p$.

5. Let $X$ be a normal space, and let $R$ be the ring of continuous functions on $X$. Show that the topology on $X$ is the one having the least amount of open sets making every function in $R$ continuous.

6. Give a Taylor formula type proof that the absolute value can be approximated uniformly by polynomials. First, reduce it to the interval $[-1, 1]$ by multiplying the variable by $c$ or $c^{-1}$ as the case may be. Then write $|t| = \sqrt{t^2}$. Select $\delta$ small, $0 < \delta < 1$. If we can approximate $(t^2 + \delta)^{1/2}$, then we can approximate $\sqrt{t^2}$. Now to get $(t^2 + \delta)^{1/2}$ either use the Taylor series approximation for the square root function, or if you don't like the binomial expansion, first approximate

$$\log(t^2 + \delta)^{1/2} = \tfrac{1}{2}\log(t^2 + \delta)$$

by a polynomial $P$. Then take a sufficiently large number of terms from the Taylor formula for the exponential function, say a polynomial $Q$, and use $Q \circ P$ to solve your problems.

7. Give another proof for the preceding fact, by using the sequence of polynomials $\{P_n\}$, starting with $P_0(t) = 0$ and letting

$$P_{n+1}(t) = P_n(t) + \tfrac{1}{2}(t - P_n(t)^2).$$

Show that $\{P_n\}$ tends to $\sqrt{t}$ uniformly on $[0, 1]$, showing by induction that

$$0 \leqq \sqrt{t} - P_n(t) \leqq \frac{2\sqrt{t}}{2 + n\sqrt{t}},$$

whence $0 \leqq \sqrt{t} - P_n(t) \leqq 2/n$.

8. Look at Example 1 of Chapter VIII, §3 to see another explicit way of proving Weierstrass' approximation theorem for a continuous function on a finite closed interval. Do Exercise 1 of that chapter.

9. Let $X$ be a compact set in a normed vector space, and let $\{f_n\}$ be a sequence of continuous functions converging pointwise to a continuous function $f$, and such that $\{f_n\}$ is a monotone increasing sequence. Show that the convergence is uniform (**Dini's theorem**; cf. Chapter IX, §1).

10. Let $X$ be a compact metric space (whence separable). Show that the Banach space $C(X, \mathbf{R})$ or $C(X, \mathbf{C})$ of continuous functions on $X$ is separable. [*Hint:* Let $\{x_n\}$ be a countable dense set in $X$ and let $g_n$ be the function on $X$ given by

$$g_n(x) = d(x, x_n),$$

where $d$ is the distance function. Use the Stone-Weierstrass theorem applied to the algebra generated by all functions $g_n$ to conclude that $C(X, \mathbf{R})$ is separable.] *Note:* Since a compact Hausdorff space is normal, and since a normal separable space is metrizable, one can adjust the statement of the theorem proved in the exercise as follows:

*Let $X$ be a compact Hausdorff separable space. Then $C(X, \mathbf{R})$ is separable.*

11. Let $X$, $Y$ be compact Hausdorff spaces. If $f$, $g$ are continuous functions on $X$ and $Y$ respectively, we denote by $f \otimes g$ the function such that

$$(f \otimes g)(x, y) = f(x)g(y).$$

Show that every continuous function on $X \times Y$ can be uniformly approximated by sums $\sum_{i=1}^{n} f_i \otimes g_i$ where $f_i$ is continuous on $X$ and $g_i$ is continuous on $Y$.

12. Let $X$ be compact Hausdorff. By an **algebra automorphism** of $C(X)$ we mean a map $\sigma: C(X) \to C(X)$ such that $\sigma$ leaves the constants fixed, and satisfies

$$\sigma(f + g) = \sigma(f) + \sigma(g), \qquad \sigma(fg) = \sigma(f)\sigma(g).$$

Show that an algebra automorphism is norm preserving, i.e. $\|\sigma f\| = \|f\|$.

13. Let $X$ be a compact Hausdorff space and let $A$ be a subalgebra of $C(X, \mathbf{R})$. Show that there exists a continuous map $\varphi: X \to Y$ of $X$ onto a compact space $Y$ such that every element of $A$ can be written in the form $g \circ \varphi$, where $g$ is a continuous function on $Y$.

14. Let $X$, $Y$ be compact Hausdorff spaces. Show that $X$ is homeomorphic to $Y$ if and only if $C(X, \mathbf{C})$ is algebra-isomorphic to $C(Y, \mathbf{C})$.

15. Let $X$ be a compact Hausdorff space. Let $\mathcal{M}$ be the set of all maximal ideals in $C(X, \mathbf{C})$. Define a closed set in $\mathcal{M}$ to consist of all maximal ideals containing a given ideal. Show that this defines a topology on $\mathcal{M}$. For each $x \in X$, let $M_x$ be the ideal of functions in $C(X, \mathbf{C})$ which vanish at $x$. Show that the map

$$x \mapsto M_x$$

is a homeomorphism between $X$ and $\mathcal{M}$.

16. For $a \in \mathbf{R}$ let $f_a(x) = e^{iax}e^{-x^2}$. Prove that any function $\varphi$ which is $C^\infty$ and has compact support on $\mathbf{R}$ can be uniformly approximated by elements of the space generated by the functions $f_a$ over $\mathbf{C}$. [*Hint:* If $\psi$ is a function vanishing outside a compact set, and $N$ is a large integer, let $\psi_N$ be the extension of $\psi$ on $[-N, N]$ to $\mathbf{R}$ by periodicity. Use the partial sums of a Fourier series to approximate such an extension of $\varphi(x)e^{x^2}$, and then multiply by $e^{-x^2}$.] *Remark.* Instead of $e^{-x^2}$ you could use any function $h(x) > 0$ which is $C^\infty$, and tends to 0 at infinity. This would not be the case in Exercises 19 and 20 below.

The next four exercises form a connected set.

17. Let $X$ be compact Hausdorff and let $p$ be a point of $X$. Let $A$ be a subalgebra of $C(X, \mathbf{R})$ consisting of functions $g$ such that $g(p) = 0$. Assume that there is no point $q \neq p$ such that $g(q) = 0$ for all $g \in A$, and that $A$ separates the points of $X - \{p\}$. The the uniform closure of $A$ is equal to the ideal of all functions vanishing at $p$.

18. Let $X$ be locally compact Hausdorff, but not compact. Let $C_\infty(X, \mathbf{R})$ be the algebra of continuous functions $f$ on $X$ such that $f$ **vanishes at infinity** (meaning, given $\varepsilon$ there exists a compact $K$ such that $|f(x)| < \varepsilon$ if $x \notin K$). Let $A$ be a subalgebra of $C_\infty(X, \mathbf{R})$ which separates points of $X$. Assume that there is no common zero to all functions in $A$. Show that $A$ is dense in $C_\infty(X, \mathbf{R})$.

19. Let $f$ be a real valued continuous function on $\mathbf{R}_{\geqq 0}$ (reals $\geqq 0$). Assume that $f$ vanishes at infinity. Show that $f$ can be uniformly approximated by functions of the form $e^{-x}p(x)$, where $p$ is a polynomial. [*Hint:* First show that you can approximate $e^{-2x}$ by $e^{-x}q(x)$ for some polynomial $q(x)$, by using Taylor's formula with remainder. If $p$ is a polynomial, approximate $e^{-nx}p(x)$ by $e^{-x}q(x)$ for some polynomial $q$.]

20. Let $f$ be a continuous function on $\mathbf{R}$, vanishing at infinity. Show that $f$ can be uniformly approximated by functions of the form $e^{-x^2}p(x)$, where $p$ is a polynomial.

**Remark.** By changing variables, one can use $e^{-cx}$ and $e^{-cx^2}$ with a fixed $c > 0$ instead of $e^{-x}$ and $e^{-x^2}$ in Exercises 19 and 20.

21. Let $X$ be a metric space and $E$ a normed vector space. Let $BC(X, E)$ be the space of bounded continuous maps of $X$ into $E$. Let $\Phi$ be a bounded subset of $BC(X, E)$. For $x \in X$, let $ev_x\colon \Phi \to E$ be the map such that $ev_x(\varphi) = \varphi(x)$. Show that $ev_x$ is a continuous bounded map. Show that $\Phi$ is equicontinuous at a point $a \in X$ if and only if the map $x \to ev_x$ of $X$ into $BC(\Phi, E)$ is continuous at $a$.

22. Let $X$ be a compact subset of a normed vector space, and $E$ a normed vector space. Show that any equicontinuous subset $\Phi$ of $C(X, E)$ is uniformly equicontinuous. [This means: Given $\varepsilon$, there exists $\delta$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all $f \in \Phi$.]

23. Let $X$ be a subset of a normed vector space and $\Phi$ an equicontinuous subset of $BC(X, \mathbf{R})$. Let $Y$ be the set of points $x \in X$ such that $\Phi(x)$ is bounded. Prove that $Y$ is open and closed in $X$. If $X$ is compact and connected, and if for some point $a \in X$ the set $\Phi(a)$ is bounded, show that $\Phi$ is relatively compact in $C(X, \mathbf{R})$.

# PART TWO

# Banach and Hilbert Spaces

The two chapters of this part are absolutely basic for everything else that follows, and introduce the most useful of all the spaces encountered in analysis, namely Banach and Hilbert spaces. The reader who wishes to study integration theory as soon as possible may continue these chapters with Chapter VI, which will make essential use of the basic properties of these spaces, especially the completion of a normed vector space and the linear extension theorem. Indeed, the integral of the absolute value of a function defines a seminorm on a suitable space of functions, whose completion will be the main object of study of the chapters on integration.

On the other hand, readers may look directly at the functional analysis, as a continuation of the linear theory of Banach and Hilbert spaces. At some point, of course, these come together when we study the spectral theorems and the existence of spectral measures.

As in the algebraic theory of vector spaces, we shall consider continuous linear maps $L: E \to F$ of a normed vector space into another. The kernel and image of $L$ are defined as in the algebraic theory, namely the kernel is the set of elements $x \in E$ such that $L(x) = 0$. The image is simply $L(E)$. Both Ker $L$ and Im $L$ are subspaces, of $E$ and $F$ respectively. However, now that we have the norm, we note that the kernel is a closed subspace (being the inverse image of the closed set $\{0\}$). Warning: the image if not necessarily closed. For conditions under which the image is closed, see Chapter XV.

For the integration theory, we do not need such considerations of subspace and factor space. However, we shall consider the dual space in the context of integration, showing that various spaces of functions are dual to each other. Thus we deal at somewhat greater length with the dual space in this chapter. An application of the duality theory in the context of Banach algebras will be given in Chapter XVI.

# CHAPTER IV

# Banach Spaces

## IV, §1. DEFINITIONS, THE DUAL SPACE, AND THE HAHN–BANACH THEOREM

Let $E$ be a Banach space, i.e. a complete normed vector space. One can deal with series $\sum x_n$ in Banach spaces just as with series of numbers, or of functions, and the most frequent test for convergence (in fact absolute convergence) is the standard one:

*Let $\{a_n\}$ be a sequence of numbers $\geqq 0$ such that $\sum a_n$ converges. If $|x_n| \leqq a_n$ for all $n$, then $\sum x_n$ converges.*

The proof is standard and trivial.

Let $E$, $F$ be normed vector spaces. We denote by $L(E, F)$ the space of continuous linear maps of $E$ into $F$. *It is easily verified that a linear map $\lambda: E \to F$ is continuous if and only if there exists $C > 0$ such that $|\lambda(x)| \leqq C|x|$ for all $x \in E$.* Indeed, if the $C$ exists, continuity is obvious (even uniform continuity). Conversely, if $\lambda$ is continuous at 0, then there exists $\delta$ such that if $|x| \leqq \delta$, then $|\lambda(x)| < 1$. Hence for any non-zero $x \in E$, we get

$$\left|\lambda\left(\frac{\delta x}{|x|}\right)\right| < 1,$$

whence we can take $C = 2/\delta$.

Such a number $C$ is called a **bound** for $\lambda$, and $\lambda$ is also said to be **bounded**. Let $S_1$ be the unit sphere in $E$ (centered at the origin), that is

the set of all $x \in E$ such that $|x| = 1$. Then a bound for $\lambda$ is immediately seen to be the same thing as a bound for the values of $\lambda$ on $S_1$. The least upper bound of all values $|\lambda(x)|$, for $x \in S_1$, is called the **norm** of $\lambda$, and the map

$$\lambda \mapsto |\lambda|$$

is a norm on $L(E, F)$. It is immediately seen that $|\lambda|$ is the greatest lower bound of all numbers $C > 0$ such that

$$|\lambda(x)| \leqq C|x|, \qquad \text{all } x \in E.$$

*Let $E$, $F$, $G$ be normed vector spaces, let $u \in L(E, F)$, and let $v \in L(F, G)$. Then $v \circ u$ is in $L(E, G)$ and we have*

$$|v \circ u| \leqq |v||u|.$$

*Proof.* A composite of continuous maps is continuous, and a composite of linear maps is linear, so our first assertion is clear. As to the second, we have

$$|v \circ u(x)| = |v(u(x))| \leqq |v||u(x)| \leqq |v||u||x|,$$

so the desired inequality follows by definition.

*If $F$ is complete, then $L(E, F)$ is complete.*

This is but an exercise. If $\{\lambda_n\}$ is a Cauchy sequence of elements in $L(E, F)$, then for each $x \in E$ one verifies that $\{\lambda_n(x)\}$ is a Cauchy sequence in $F$, and hence converges to an element which we define to be $\lambda(x)$. One then verifies that $\lambda$ is linear, and that if $C = \lim|\lambda_n|$, then $C$ is a bound for $\lambda$, so that $\lambda$ is continuous. Finally one verifies that $\{\lambda_n\}$ converges to $\lambda$ in $L(E, F)$. (Fill in the details as Exercise 1, or look them up in *Undergraduate Analysis*.)

We give some terminology concerning the space $L(E, F)$ which is used constantly in this book, and in analysis.

A continuous (bounded) linear map of a Banach space into itself is called an **endomorphism**, or an **operator**.

In the case of two spaces $E$, $F$, an element $u \in L(E, F)$ is said to be **invertible** if there exists $v \in L(F, E)$ such that

$$u \circ v = I_F \qquad \text{and} \qquad v \circ u = I_E$$

(where $I$ is the identity mapping). In mathematics, the word **isomorphism** refers to invertibility in various contexts, for instance a map having a

continuous inverse, a linear inverse, a differentiable inverse, etc. ad lib. Thus in each case, one should add an adjective to the word **isomorphism** to make precise the kind of invertibility which is meant. In our present case, we shall call invertible elements of $L(E, F)$ **toplinear** isomorphisms, the adjective toplinear referring to the topology and the linearity. The set of toplinear isomorphisms of $E$ onto $F$ is denoted by $\text{Lis}(E, F)$. If $E = F$, then we call toplinear isomorphisms of $E$ with itself toplinear **automorphisms** of $E$; the set of such automorphisms is denoted by $\text{Laut}(E)$. (For euphony, the reader may prefer the adjective topolinear instead of toplinear.)

A toplinear isomorphism $u$ between Banach spaces $E$, $F$ which also preserves the norm (that is $|u(x)| = |x|$ for all $x \in E$) will be called a **Banach** isomorphism, or an **isometry**.

We shall also be dealing with bilinear maps. Let $E$, $F$, $G$ be normed vector spaces. A map

$$\varphi: E \times F \to G$$

is said to be **bilinear** if for each $x \in E$ the map $y \mapsto \varphi(x, y)$ is linear, and if for each $y \in F$ the map $x \mapsto \varphi(x, y)$ is linear. Such bilinear maps form a vector space. It is easily verified (in a manner similar to the case of linear maps) that $\varphi$ is continuous if and only if there exists $C$ such that

$$|\varphi(x, y)| \leqq C|x||y|$$

for all $x \in E$, $y \in F$. The greatest lower bound of such $C$ then defines a norm on the space of continuous bilinear maps, denoted by $L(E, F; G)$, and this space is a Banach space if $G$ is complete. (Cf. Exercise 3.)

In the differential calculus, and other applications, we need an isomorphism between $L(E, L(F, G))$ and $L(E, F; G)$ as follows. Let $\lambda \in L(E, L(F, G))$ and define $\varphi_\lambda$ by

$$\varphi_\lambda(x, y) = \lambda(x)(y).$$

Then $\varphi_\lambda$ is obviously bilinear, and we have

$$|\varphi_\lambda(x, y)| \leqq |\lambda(x)||y| \leqq |\lambda||x||y|$$

so that

$$|\varphi_\lambda| \leqq |\lambda|.$$

On the other hand, given $\varphi \in L(E, F; G)$, we can define $\lambda_\varphi$ by

$$\lambda_\varphi(x)(y) = \varphi(x, y).$$

Then

$$|\lambda_\varphi(x)(y)| \leqq |\varphi||x||y|$$

so that by definition,

$$|\lambda_\varphi(x)| \leqq |\varphi||x|.$$

Hence

$$|\lambda_\varphi| \leqq |\varphi|.$$

Thus we get a Banach isomorphism

$$L(E, L(F, G)) \leftrightarrow L(E, F; G).$$

As one example of a bilinear map, we have

$$L(E, F) \times E \to F$$

such that $(\lambda, x) \mapsto \lambda(x)$. This bilinear map has norm 1.

Similarly, we can treat multilinear maps. If $E_1, \ldots, E_n$, $F$ are normed vector spaces, a **multilinear** map

$$\varphi\colon E_1 \times \cdots \times E_n \to F$$

is a map which is linear in each variable. Such a map is continuous if and only if there exists $C$ such that for all $x_i \in E_i$ we have

$$|\varphi(x_1, \ldots, x_n)| \leqq C|x_1||x_2|\cdots|x_n|.$$

We have a norm-preserving isomorphism

$$L(E_1, L(E_2, \ldots, L(E_n, F)\ldots)) \leftrightarrow L(E_1, \ldots, E_n; F)$$

from the space of repeated continuous linear maps to the space of continuous multilinear maps exactly as in the bilinear case. If $F$ is complete, then all these spaces are also complete.

We now consider a specially important space of linear maps.

The normed vector space $L(E, \mathbf{R})$ [or $L(E, \mathbf{C})$ in the complex case] is called the **dual space** of $E$, and is denoted by $E'$. Elements of $E'$ are called **functionals** on $E$. Functionals can be used as substitutes for coordinates. Indeed, suppose that $E = \mathbf{R}^k$, and let $\lambda_i$ be the $i$-th coordinate function, that is

$$\lambda_i(x_1, \ldots, x_n) = x_i.$$

Then it is easily verified that $\{\lambda_1, \ldots, \lambda_n\}$ is a basis for the dual space of $\mathbf{R}^k$. Furthermore, the values of $\lambda_1, \ldots, \lambda_n$ on an element $x \in \mathbf{R}^k$ characterize this element. Although we do not have such convenient bases in the infinite dimensional case, we still have such a characterization of elements of $E$ in terms of the values of functionals. This is based on the following theorem.

**Theorem 1.1.** *Let $E$ be a real normed vector space, and let $F$ be a subspace. Let $\lambda: F \to \mathbf{R}$ be a functional, bounded by a number $C > 0$. Then there exists an extension of $\lambda$ to a functional of $E$, having the same bound.*

*Proof.* Changing the norm on $E$ (multiplying it by a number) we see that it suffices to prove our theorem when $C = 1$. We first prove that if $v \in E$ and $v \notin F$, then we can extend $\lambda$ to $F + \mathbf{R}v$, and preserve the bound 1. Every element of $F + \mathbf{R}v$ has a unique expression as $x + tv$ with $x \in F$ and $t \in \mathbf{R}$. Let $a \in \mathbf{R}$. The map $\lambda^*$ on $F + \mathbf{R}v$ such that

$$\lambda^*(x + tv) = \lambda(x) + ta$$

is certainly linear. We must show that we can select $a$ such that $\lambda^*$ is bounded by 1. Dividing both sides by $t$ (if $t \neq 0$), we see that it suffices to find a number $a$ such that

$$|\lambda(y) + a| \leqq |y + v|$$

for all $y \in F$, or equivalently that for all $y \in F$,

$$\lambda(y) + a \leqq |y + v| \quad \text{and} \quad -\lambda(y) - a \leqq |y + v|.$$

This determines inequalities for $a$, namely

$$-\lambda(y) - |y + v| \leqq a \leqq -\lambda(y) + |y + v|,$$

and it suffices to show that the set of real $a$ satisfying such inequalities is not empty. But for all $y, z \in F$ we have

$$|\lambda(y) - \lambda(z)| = |\lambda(y - z)| \leqq |y - z|$$

so that

$$-\lambda(z) - |z + v| \leqq -\lambda(y) + |y + v|.$$

From this we conclude that there is a non-empty interval of values of $a$ which satisfy our requirements.

We now use Zorn's lemma. We consider the set of pairs $(G, \lambda^*)$ where $G$ is a subspace of $E$ containing $F$, and $\lambda^*$ is a functional on $G$ having the same bound as $\lambda$, and extending $\lambda$. We order such pairs

$$(G_1, \lambda_1) \leqq (G_2, \lambda_2)$$

if $G_1$ is a subspace of $G_2$ and $\lambda_2$ is an extension of $\lambda_1$. This is an ordering, and our set of pairs is inductively ordered. The proof of this is the usual proof: Given a totally ordered set of pairs as above, say $\{(G_i, \lambda_i)\}$, we let $G$ be the union of all $G_i$. We can define a functional $\lambda^*$

on $G$ extending all $\lambda_i$: Any $x \in G$ is in some $G_i$, and we define $\lambda^*(x) = \lambda_i(x)$. This is independent of the choice of $i$ such that $x \in G_i$, and the pair $(G, \lambda^*)$ is an upper bound for our family. By Zorn's lemma, let $(G, \lambda^*)$ be a maximal element. Then $G = E$, for otherwise, there is some $v \in E$, $v \notin G$, and we can use the first part of the proof to get a bigger pair. This proves our theorem.

**Corollary 1.2.** *Let $E$ be a normed vector space, and $v \in E$, $v \neq 0$. Then there exists a functional $\lambda$ on $E$ such that $\lambda(v) \neq 0$.*

*Proof.* Let $F$ be the one-dimensional space generated by $v$. We define $\lambda$ on $F$ taking any non-zero value on $v$, and extend $\lambda$ to $E$ using Theorem 1.1.

Theorem 1.1, or its Corollary, is referred to as the **Hahn–Banach theorem**. We have formulated it over the reals, but it is also valid for complex Banach spaces, and the complex case is easily reduced to the real case. Indeed, given a complex functional $\lambda$ on a complex subspace $F$, let $\varphi$ be its real part. Let $\varphi'$ be a real extension of $\varphi$ to $E$, and define

$$\lambda'(v) = \varphi'(v) - i\varphi'(iv) \qquad \text{for } v \in E.$$

You can verify as Exercise 2 that $\lambda'$ is a desired complex extension of $\lambda$.

The dual space $E'$ is a special case of the space of linear maps $L(E, F)$ when $F$ is the space of scalars. As such, we have seen that it is a Banach space with its natural norm. Furthermore, we can form the double dual $E''$ in a similar fashion, and $E''$ is also a Banach space. Note that each element $x \in E$ gives rise to a functional $f_x \in E''$, given by

$$f_x\colon E' \to \text{scalars } \mathbf{R} \text{ or } \mathbf{C} \qquad \text{such that} \qquad f_x(\lambda) = \lambda(x),$$

continuous for the topology defined by the norm on $E'$.

**Proposition 1.3.** *The map $x \mapsto f_x$ is an injective linear map of $E$ into $E''$, which is norm preserving,* i.e. $|x| = |f_x|$.

*Proof.* Suppose $x, y \in E$ and $x \neq y$. Then $x - y \neq 0$. By the Hahn–Banach theorem, there exists $\lambda \in E'$ such that $\lambda(x - y) \neq 0$, so $\lambda(x) \neq \lambda(y)$. This proves that $f_x \neq f_y$, whence the map $x \mapsto f_x$ is injective. The inequality

$$|\lambda(x)| \leqq |\lambda||x|$$

shows that $|f_x| \leqq |x|$. We leave to the reader the opposite inequality $|x| \leqq |f_x|$, which concludes the proof that we have an isometric embedding of $E$ in $E''$.

In Chapter II, §1 we defined the weak topology on a space, determined by a set of mappings into a topological space. We now apply this notion to the dual space. We let $\mathscr{F}$ be the family of functions on $E'$ given by

$$\mathscr{F} = \{f_x\}_{x \in E}$$

as above. The weak topology on $E'$ determined by this family $\mathscr{F}$ is called simply the **weak topology** on $E'$. The next theorem gives one of its most important properties.

**Theorem 1.4 (Alaoglu's Theorem).** *Let $E$ be a Banach space, and let $E'_1$ be the unit ball in the dual space $E'$. Then $E'_1$ is compact for the weak topology.*

*Proof.* For each $x \in E$, let $K_x$ be the closed disc of radius 1 in $\mathbf{C}$. Let

$$K = \prod_{\substack{x \in E \\ |x| \leqq 1}} K_x$$

be the Cartesian product of all closed discs of radius 1, taken over all $x \in E$ satisfying $|x| \leqq 1$. We give $K$ the product topology, so that by Tychonoff's theorem, $K$ is compact. We map $E'_1$ into $K$ by the map

$$f\colon E'_1 \to K \qquad \text{such that} \qquad \lambda \mapsto \prod_{|x| \leqq 1} \lambda(x) = \prod_{|x| \leqq 1} f_x(\lambda)$$

Immediately from the definition, one sees that the map $f$ is injective, and thus gives an embedding of $E'_1$ into the product space. Furthermore, also from the definition of the weak topology defined in Chapter II, §1, we observe that the weak topology determined by the family $\mathscr{F}$ is the same as the weak topology determined by the family $\mathscr{F}_1$ of functionals $f_x$ with $x \in E_1$ (the closed unit ball in $E$), because any $x \in E$, $x \neq 0$ is a scalar multiple of a unit vector. More precisely, we also have an imbedding

$$f\colon E' \hookrightarrow \prod_{x \in E} \mathbf{C}_x \qquad \text{given by} \qquad \lambda \mapsto \prod_{x \in E} \lambda(x),$$

and the following diagram is commutative:

$$\begin{array}{ccc} E' & \hookrightarrow & \prod_{x \in E} \mathbf{C}_x \\ \uparrow & & \uparrow \\ E'_1 & \hookrightarrow & \prod_{x \in E_1} K_x \end{array}$$

The product topology induced on $\prod K_x$ is the same as the topology induced by viewing this product as a subspace of $\prod \mathbf{C}_x$. Therefore, it follows that the weak topology on $E'_1$ is the topology induced by viewing $E'_1$ as a subspace of $K$ via the embedding $f$, or also as a subspace of $\prod \mathbf{C}_x$ $(x \in E)$, via the embedding of $E'$ in $\prod \mathbf{C}_x$. To show that $E'_1$ is compact, it suffices therefore to show that $f(E'_1)$ is closed in $K$.

To do this, we first prove that $E'$ is closed in $\prod \mathbf{C}_x$ $(x \in E)$. Let $\prod \gamma(x)$ $(x \in E)$ be an element of the product which lies in the closure of $f(E')$. Given elements $x,\ y \in E$, we have to show that $x \mapsto \gamma(x)$ is a bounded functional. By definition of the weak topology, given there exists $\lambda \in E'$ such that

$$|\lambda(x) - \gamma(x)| < \varepsilon,$$

$$|\lambda(y) - \gamma(y)| < \varepsilon,$$

$$|\lambda(x + y) - \gamma(x + y)| < \varepsilon.$$

But $\lambda(x + y) = \lambda(x) + \lambda(y)$, whence $|\gamma(x + y) - \gamma(x) - \gamma(y)| < 3\varepsilon$, so

$$\gamma(x + y) = \gamma(x) + \gamma(y).$$

Similarly, one sees that $\gamma(cx) = c\gamma(x)$ for $c \in \mathbf{C}$, whence $\gamma$ is linear. Also similarly, one sees that $\gamma$ is bounded. Furthermore, if $\prod \gamma(x)$ lies in the closure of $E'_1$, then the above $\lambda$ can be chosen such that $|\lambda| \leqq 1$, that is $|\lambda(x)| \leqq |x|$. Then by a similar epsilson argument, one sees that $|\gamma(x)| \leqq |x|$, which proves that $f(E'_1)$ is closed, whence compact, thus concluding the proof of Theorem 1.4.

**Remark.** In the case of Hilbert space, to be defined in the next chapter, the Banach space $E$ is self dual, and so in this case, one may state that the unit ball in Hilbert space is compact in the weak topology.

## IV, §2. BANACH ALGEBRAS

An **algebra** (say over $\mathbf{R}$) is a vector space $A$, together with a mapping $A \times A \to A$ (called a multiplication) which is bilinear. This means that for all $u,\ v,\ w \in A$ and $c \in \mathbf{R}$ we have

$$u(v + w) = uv + uw, \qquad (u + v)w = uw + vw,$$

$$c(uv) = (cu)v = u(cv).$$

If in addition we have $uv = vu$, we say that the algebra is **commutative**. If $u(vw) = (uv)w$, we say that the algebra is **associative**. If there exists an element $e \in A$ such that $eu = ue = u$, we say that the algebra has a **unit**

**element** $e$, which is then uniquely determined, because if $e'$ is another unit element, then

$$e = ee' = e'.$$

A **normed algebra** is an associative algebra whose vector space is normed, and whose norm satisfies the condition $|uv| \leqq |u||v|$. A normed algebra which is complete is called a **Banach** algebra.

For convenience when there is a unit element, *we shall also assume that* $|e| = 1$. See Exercise 5 which shows that this condition can always be achieved by a simple redefinition of the norm.

**Example 1.** Let $A$ be the vector space of bounded functions on a set, multiplication being ordinary multiplication of functions. Then $A$ is a Banach algebra. So is the set of bounded continuous functions.

**Example 2.** Let $A = \mathbf{R}^3$ and let the product be the cross product. Then $A$ is neither commutative nor associative, but otherwise satisfies the other axioms of a normed algebra. Since non-associative algebras occur so rarely in what we do, we have taken associativity into the definition of a normed algebra, so that the present example is not that of a normed algebra in our sense.

**Example 3.** Let $E$ be a normed vector space. Then $L(E, E)$ is an algebra, if we define the multiplication to be composition of mappings. In other words, if $u, v \in L(E, E)$, then the product $u \circ v$ is again a continuous linear map of $E$ into itself, and we have associativity and bilinearity, which follow at once from the definition of the sum of two linear maps. Furthermore, $L(E, E)$ has a unit element $I$ which is the identity mapping. We often write $uv$ instead of $u \circ v$. Elements of $L(E, E)$ are also called **endomorphisms** of $E$, or **operators** on $E$, and we abbreviate $L(E, E)$ by $\text{End}(E)$. If $E$ is complete, i.e. a Banach space, then from remarks made in §1, we conclude that $\text{End}(E)$ is a Banach algebra. Of course, $\text{End}(E)$ is not necessarily commutative. It is the most important algebra studied in this book. If $E$ is finite dimensional, this algebra is essentially the algebra of $n \times n$ matrices, where $n = \dim E$.

**Example 4.** Let $E$ be the vector space of continuous functions on $\mathbf{R}$, periodic of period $2\pi$, with the sup norm. Then $E$ is a Banach space. If $f, g \in E$, we define a product called the **convolution product** by

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(x - t)\, dt.$$

It follows easily from elementary integrations that $E$ is then a commuta-

tive, associative Banach algebra. Note that $E$ does not have a unit element. In this direction, see Chapter III, §1.

We observe that an algebra with a unit element contains a replica of the scalars, under the map

$$c \mapsto ce,$$

which is injective, and preserves addition and multiplication. In the case of $L(E, E)$, an element $cI$ ($I$ = Identity) is simply "multiplication by $c$."

Let $A$ be an associative algebra with unit element $e$. An element $u$ of $A$ is said to be **invertible** if there exists $v \in A$ such that $uv = vu = e$. The element $v$ is uniquely determined by $u$, because if $uw = wu = e$, then multiplying on the left by $v$ shows that $w = vuw = v$. We call this element the **inverse** of $u$ and denote it by $u^{-1}$. An invertible element is also called a **unit**. If $u$, $v$ are invertible, then so is $uv$, because

$$(uv)^{-1} = v^{-1}u^{-1}.$$

**Theorem 2.1.** *Let $A$ be a Banach algebra with unit element $e$. Then the set of invertible elements is open in $A$. If $v \in A$ and $|v| < 1$, then $e + v$ is invertible.*

*Proof.* Let $|v| < 1$. Then the series $e + v + v^2 + \cdots$ converges (absolutely) and since

$$(e - v)(e + v + \cdots + v^n) = e - v^{n+1},$$

it follows that $e - v$ is invertible, and that its inverse is the limit of $e + v + \cdots + v^n$ as $n \to \infty$. That we have $-v$ instead of $v$ makes no difference, since $|-v| = |v|$. Suppose now that $u$ is invertible, and let

$$|w - u| < 1/|u^{-1}|.$$

Then

$$|wu^{-1} - e| = |(w - u)u^{-1}| \leqq |w - u||u^{-1}| < 1.$$

Hence $wu^{-1}$ is invertible, whence $w$ is invertible, thus proving our theorem.

We observe that the map $u \mapsto u^{-1}$ is continuous (as a map defined on the set of invertible elements). The usual proof is valid.

**Corollary 2.2.** *Let $E$, $F$ be Banach spaces. Then the set of toplinear isomorphisms of $E$ onto $F$ is open in $L(E, F)$.*

*Proof.* Suppose that this set is not empty, and let $u: E \to F$ be a toplinear isomorphism. Then for $v \in L(E, F)$ we have

$$|u^{-1}v - I| = |u^{-1}(v - u)| \leqq |u^{-1}||v - u|.$$

If $v$ is close to $u$, then $u^{-1}v$ is close to $I$, and is invertible by Theorem 2.1, so there exists $w_1$ such that

$$w_1 u^{-1} v = I_E.$$

Similarly, there exists a toplinear automorphism $w_2$ of $F$ such that

$$vu^{-1}w_2 = I_F.$$

Thus $v$ has a right inverse and a left inverse, say $v_1, v_2$, such that

$$v_1 v = I_E \qquad \text{and} \qquad vv_2 = I_F.$$

Considering $v_1 v v_2$ and using associativity shows that $v_1 = v_2$, whence $v$ is invertible.

## IV, §3. THE LINEAR EXTENSION THEOREM

**Theorem 3.1.** *Let $E$ be a normed vector space, $F$ a subspace, and $G$ a complete normed vector space. Let*

$$\lambda: F \to G$$

*be a continuous linear map, with norm $C$. Then the closure $\bar{F}$ of $F$ in $E$ is a subspace of $E$. There exists a unique extension of $\lambda$ to a continuous linear map $\bar{\lambda}: \bar{F} \to G$, and $\bar{\lambda}$ has the same norm as $\lambda$.*

*Proof.* Elements in $\bar{F}$ are limits of sequences in $F$. Thus if

$$x = \lim x_n \qquad \text{and} \qquad y = \lim y_n,$$

then

$$x + y = \lim(x_n + y_n)$$

and for $c \in \mathbf{R}$,

$$cx = \lim(cx_n).$$

Hence $\bar{F}$ is a subspace of $E$.

The uniqueness of $\bar{\lambda}$ is clear from continuity. We show its existence.

Let $x \in \bar{F}$, and let $x = \lim x_n$ with $x_n \in F$. Then

$$|\lambda(x_n) - \lambda(x_m)| = |\lambda(x_n - x_m)| \leqq C|x_n - x_m|.$$

Hence $\{\lambda x_n\}$ is a Cauchy sequence in $G$, and since $G$ is assumed to be complete, $\{\lambda x_n\}$ has a limit in $G$ which we denote by $\bar{\lambda}x$. This value is independent of the sequence $x_n \to x$, for if $x = \lim x'_n$ with $x'_n \in F$, then $\lim \lambda x_n = \lim \lambda x'_n$. If

$$y \in \bar{F} \qquad \text{and} \qquad y = \lim y_n$$

with $y_n \in F$, then for $c \in \mathbf{R}$,

$$x + y = \lim(x_n + y_n) \qquad \text{and} \qquad cx = \lim(cx_n).$$

Hence

$$\begin{aligned}\bar{\lambda}(x + y) = \lim \lambda(x_n + y_n) = \lim(\lambda x_n + \lambda y_n) &= \lim \lambda x_n + \lim \lambda y_n \\ &= \bar{\lambda}x + \bar{\lambda}y.\end{aligned}$$

Similarly, $\bar{\lambda}(cx) = c\bar{\lambda}(x)$. Hence $\bar{\lambda}$ is linear, and since for $x \in F$ we have $x = \lim x$, it follows that $\bar{\lambda}x = \lambda x$ if $x \in F$. Thus $\bar{\lambda}$ is an extension of $\lambda$.

Finally, we have

$$|\bar{\lambda}x| = \lim|\lambda x_n|$$

because the norm is a continuous function. Since

$$|\lambda x_n| \leqq C|x_n|,$$

it follows that

$$\lim|\lambda x_n| \leqq C|\lim x_n| = C|x|,$$

because limits preserve inequalities. This proves that a bound for $\lambda$ is also a bound for $\bar{\lambda}$ and hence that $|\lambda| = |\bar{\lambda}|$. This also concludes the proof of Theorem 3.1.

We shall see examples of Theorem 3.1 very frequently in the sequel, notably in the existence proof for the completion of a normed vector space, in integration, Chapter VI, §3 and Chapter XIII, §1; and in the spectral theorem of Chapter XVIII.

## IV, §4. COMPLETION OF A NORMED VECTOR SPACE

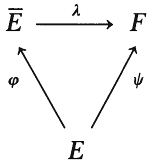
Let $E$ be a normed vector space. We wish to associate with $E$ a complete normed vector space in a manner analogous to that which associates the real numbers to the rational numbers. We shall follow

the method of Cauchy sequences. For another method, cf. Exercise 25.

We define a **completion** of $E$ to be a pair $(\bar{E}, \varphi)$ consisting of a Banach space $\bar{E}$ and a continuous linear map

$$\varphi: E \to \bar{E}$$

which is injective, such that $\varphi(E)$ is dense in $\bar{E}$, and such that $\varphi$ preserves the norm, i.e. $|\varphi x| = |x|$ for all $x \in E$. *We shall now prove that such a pair is essentially uniquely determined. In fact, if $(F, \psi)$ is another completion, then there exists a unique invertible element $\lambda \in L(\bar{E}, F)$ such that the following diagram is commutative, in other words $\psi = \lambda \circ \varphi$.*




The proof is in fact very easy. The map

$$\psi \circ \varphi^{-1}: \varphi(E) \to \psi(E) \subset F$$

is continuous and linear (it even preserves the norm) and consequently, by the linear extension theorem, it has a unique continuous linear extension of $\bar{E}$ into $F$, which we denote by $\lambda$. Similarly, the continuous linear map

$$\varphi \circ \psi^{-1}: \psi(E) \to \varphi(E) \subset \bar{E}$$

has a continuous linear extension of $F$ into $\bar{E}$, which we denote by $\mu$. Then $\mu \circ \lambda: \bar{E} \to \bar{E}$ gives the identity when restricted to $\varphi(E)$, and hence is equal to the identity on $\bar{E}$ itself by continuity (or by the uniqueness part of the linear extension theorem). Similarly, $\lambda \circ \mu: F \to F$ is the identity. This proves the uniqueness of the completion.

We observe that our toplinear isomorphism $\lambda$ preserves norms, that is

$$|\lambda x| = |x|$$

for all $x \in \bar{E}$. This again follows by continuity.

We shall now give two proofs of the existence of a completion. So let $E$ be a normed vector space and let $E'$ be its dual. As we saw in Proposition 1.3, we have a natural norm-preserving injection $E \to E''$. But $E''$ is complete because $E'' = L(E', F)$ with complete $F$ ($F$ = scalars). So the completion of $E$ is simply the closure $\bar{E}$ in $E''$. (Do Exercise 15.)

Next we give another proof, based on the same construction as the

real numbers from the rational numbers. This construction will be used in the integration theory. See the examples after the construction.

The Cauchy sequences of elements of $E$ form a vector space, which we denote by $S$, As usual, we have the notion of null sequences, that is sequences $\{x_n\}$ in $E$ such that given $\varepsilon$, there exists $N$ such that for all $n > N$ we have $|x_n| < \varepsilon$. The null sequences form a subspace. We define two Cauchy sequences $\xi = \{x_n\}$ and $\eta = \{y_n\}$ to be **equivalent** if there exists a null sequence $\alpha = \{a_n\}$ such that $\xi = \eta + \alpha$ (in other words $x_n = y_n + a_n$ for all $n$). This is an equivalence relation, and we denote the equivalence class of $\xi$ by $\bar{\xi}$. Then the equivalence classes of Cauchy sequences form a vector space in a natural way, and we have (for $c \in \mathbf{R}$):

$$\overline{\xi + \eta} = \bar{\xi} + \bar{\eta} \qquad \text{and} \qquad \overline{c\xi} = c\bar{\xi}.$$

We denote the vector space of equivalence classes of Cauchy sequences by $\bar{E}$. (It is nothing but the factor space of Cauchy sequences modulo the subspace of null sequences.)

If $\xi = \{x_n\}$ is a Cauchy sequence and $\eta = \{y_n\}$ is equivalent to $\xi$, then

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n|.$$

Then we define

$$|\bar{\xi}| = \lim_{n\to\infty} |x_n|.$$

It is verified at once that this is a norm of $\bar{E}$, which is thus a normed vector space.

We let

$$\varphi\colon E \to \bar{E}$$

be the map such that $\varphi(x)$ is the class of the Cauchy sequence $\{x, x, \dots\}$. Then it is clear that $\varphi$ is linear, and preserves norms. Furthermore, one sees at once that if $\bar{\xi}$ is the class of a Cauchy sequence $\xi$, and $x = \{x_n\}$, then

$$\bar{\xi} = \lim_{n\to\infty} \varphi(x_n).$$

Hence $\varphi(E)$ is dense in $\bar{E}$.

All that remains to prove is that $\bar{E}$ is complete. To do this, let $\{\bar{\xi}_n\}$ be a Cauchy sequence in $\bar{E}$. For each $n$ there exists an element $x_n \in E$ such that

$$|\bar{\xi}_n - \varphi x_n| < 1/n,$$

because $\varphi(E)$ is dense in $\bar{E}$. The sequence $\{x_n\}$ is then Cauchy (in $E$).

Indeed, we have

$$\begin{aligned} |x_n - x_m| &= |\varphi x_n - \varphi x_m| \\ &\leqq |\varphi x_n - \bar{\xi}_n| + |\bar{\xi}_n - \bar{\xi}_m| + |\bar{\xi}_m - \varphi x_m|, \end{aligned}$$

which gives a $3\varepsilon$-proof of the fact that $\{x_n\}$ is a Cauchy sequence. Let $\xi = \{x_n\}$. Then $\{\bar{\xi}_n\}$ converges to $\bar{\xi}$, because given $\varepsilon$,

$$|\bar{\xi}_n - \bar{\xi}| \leqq |\bar{\xi}_n - \varphi x_n| + |\varphi x_n - \bar{\xi}| < 2\varepsilon$$

for $n$ sufficiently large. This proves that $\bar{E}$ is complete, and concludes the proof for the existence of a completion of $E$.

**Example 1.** In integration theory, covered later in this book, one starts with the vector space of continuous functions, say on [0, 1], with the $L^1$-norm

$$\|f\|_1 = \int_0^1 |f(t)|\, dt.$$

One can also take the vector space of continuous functions on **R**, vanishing outside some bounded interval, and define the $L^1$-norm similarly. Then this space is not complete, and its completion is called $L^1$. It then becomes a problem to identify elements of $L^1$ with certain functions, and this is what we shall do.

Example 1 points to the need of a slight generalization of our normed vector spaces. Indeed, even in elementary integration theory, one deals with step functions, or piecewise continuous functions, which are such that if $\|f\|_1 = 0$, then $f$ may not be the zero function. For instance, if $f$ is 0 except at a finite number of points, then we do have $\|f\|_1 = 0$. In view of this, one defines a **seminorm** on a vector space $E$ to be a function satisfying all properties of a norm, except that we require

$$|x| \geqq 0$$

for all $x \in E$, but we allow $|x| = 0$ without having necessarily $x = 0$. Then it is clear that the set of all $x \in E$ such that $|x| = 0$ is a subspace $E_0$. The terminology of open and closed sets applies in the present context, and the topology defined by a seminorm is simply not Hausdorff. In fact, the closure of 0 is obviously the space $E_0$ itself.

In defining the completion, we can just as well define the completion of a space with a seminorm. We form Cauchy sequences and null

sequences, and we still get a map

$$j\colon E \to \bar{E},$$

the only difference being that $j$ has a kernel, which the reader will verify to be precisely $E_0$. In fact, we have a norm on the factor space $E/E_0$ if we define the norm of a coset $|x + E_0|$ to be $|x|$ (independent of the coset representative $x$ since we have

$$|x + y| = |x|$$

for all $y \in E_0$). Thus we can say that if $E$ has a seminorm, the completion $\bar{E}$ is simply the completion of $E/E_0$ as discussed in this section.

A vector space $E$ with a seminorm $|\ |$ can be called a **seminormed space**. We can define Cauchy sequences using the same definition as in the normed case. We shall say that $E$ is **complete** if every Cauchy sequence in $E$ converges—in other words, if given a Cauchy sequence $\{x_n\}$ in $E$, there exists $x \in E$ such that given $\varepsilon$, there exists $N$ such that for all $n \geqq N$ we have

$$|x_n - x| < \varepsilon.$$

Of course, the element $x$ to which our sequence $\{x_n\}$ converges is not uniquely determined, only up to an element of $E_0$. However, examples of this situation arise in practice, in integration theory. One must then distinguish between *a* complete seminormed space, and *the* completion of $E/E_0$ mentioned above.

**Example 2.** Let $E$ be the vector space of $C^\infty$ functions (say, real valued) on $\mathbf{R}$, vanishing outside a compact set (i.e. infinitely differentiable functions $f$ such that $f(t) = 0$ if $t$ is outside some bounded interval). We define the $H^0$-norm on $E$ by

$$\|f\|_{H^0} = \langle f, f\rangle^{1/2},$$

where

$$\langle f, f\rangle = \int_{-\infty}^{\infty} f(t)^2\, dt.$$

We define the **$H^p$-norm** by

$$\|f\|_{H^p}^2 = \sum_{k=0}^{p} \|D^k f\|_{H^0}^2,$$

where $D$ is the derivative. The completion of $E$ under the $H^p$-norm is called an **$H^p$ space**. This kind of space is used very frequently in analysis. For $p = 0$, the norm is also called the **$L^2$-norm**.

**Example 3.** On the interval [0, 1], we let $C^p$ be the space of functions having $p$ continuous derivatives. For $f \in C^p$ we define

$$\|f\|_{C^p} = \sup_{k \leqq p} \|D^k f\|.$$

Then this is a norm. It is an exercise to show that $C^p$ is already complete under this norm.

## IV, §5. SPACES WITH OPERATORS

Except for enumerating basic properties, it is rather rare in analysis that one meets merely a normed vector space, or a Banach space, just by itself. It is usually accompanied by a set of operators, and thus we make here some general comments on this situation.

Let $E$ be a normed vector space. Elements of $L(E, E)$ are also called **operators** on $E$. Let $S$ be a set of operators on $E$. By an $S$-**invariant subspace** $F$ we mean a subspace such that for every $A \in S$ we have $AF \subset F$, i.e. if $x \in F$ and $A \in S$, then $Ax \in F$. It is clear that if $F$ is an $S$-invariant subspace, then its closure is also $S$-invariant because if $x_n \in F$ and $x_n \to x$, then $Ax_n \to Ax$, so $Ax$ lies in the closure of $F$.

An operator $B$ is said to **commute** with $S$ if $AB = BA$ for all $A \in S$. *If $B$ commutes with $S$, then both the kernel of $B$ and its image are $S$-invariant subspaces.*

*Proof.* If $x \in E$ and $Bx = 0$, then $ABx = BAx = 0$ for all $A \in S$, so the kernel of $B$ is $S$-invariant. Similarly, also from the relation $ABx = BAx$, we see that the image of $B$ is $S$-invariant.

If $A$ is an operator on $E$, and $c_0, \ldots, c_n$ are numbers, we may form the operator

$$p(A) = c_n A^n + \cdots + c_0 I,$$

where

$$p(t) = c_n t^n + \cdots + c_0$$

is the polynomial having the numbers as coefficients. If $p$, $q$ are polynomials and $pq$ denotes the ordinary product of polynomials, then we have

$$(p + q)(A) = p(A) + q(A) \qquad \text{and} \qquad (pq)(A) = p(A)q(A).$$

Indeed, if $q(t) = b_m t^m + \cdots + b_0$, then

$$p(t)q(t) = \sum d_k t^k,$$

where

$$d_k = \sum_{r+s=k} c_r b_s.$$

But

$$p(A)q(A) = \sum d_k A^k$$

since associativity, commutativity, and distributivity hold in multiplying powers of $A$. The statement concerning the sum $p + q$ is even more trivial to see. Also, if $c$ is a number, then

$$(cp)(A) = cp(A).$$

All these rules are useful when considering the evaluation of polynomials on operators. In algebraic terminology, they express the fact that the map

$$p \mapsto p(A)$$

is a ring-homomorphism from the ring of polynomials into the ring of operators.

If $F$ is an $A$-invariant subspace, then it is clear that $F$ is also $p(A)$-invariant for all polynomials $p$. Thus if $F$ is in fact a subspace of $E$ which is invariant for an operator $A$, then it is also invariant for the set of all polynomials in $A$, called also the ring of operators generated by $A$. The same holds for any set of operators $S$, letting the ring of operators generated by $S$ be the set of all operators expressed as finite sums

$$\sum c_{i_1 \cdots i_n} A_1^{i_1} \cdots A_n^{i_n},$$

where $A_1, \ldots, A_n$ are elements of $S$, and the coefficients are numbers. Indeed, if $F$ is $A$- and $B$-invariant, then it is also $(A + B)$-invariant and $AB$-invariant.

If an operator $B$ commutes with all elements of $S$, then it is clear that $B$ also commutes with all elements in the ring of operators generated by $S$, because if $B$ commutes with $A_1$ and $A_2$, then $B$ commutes with $A_1 + A_2$ and also with $A_1 A_2$. Furthermore, if $F$ is a closed subspace and is $S$-invariant, then it is also $\bar{S}$-invariant, where $\bar{S}$ is the closure of $S$. Indeed, if $\{B_n\}$ is a sequence of operators in $S$ converging to some operator $B$, and if $x \in F$, then the sequence $\{B_n x\}$ is Cauchy, and hence converges to $Bx$ which lies in $F$.

In Chapters XVII and XVIII we study a pair $(E, A)$ consisting of a space $E$ and an operator $A$, and analyze this pair, describing its structure completely in important cases. The idea is to apply in the present context an all-pervasive point of view in mathematics, which is to decompose an object into a direct sum of simpler objects. In the present context, let us make some general definitions.

Let $E$ be a Banach space, and $F$, $G$ closed subspaces. We know that the product $F \times G$ consisting of all pairs $(y, z)$ with $y \in F$ and $z \in G$ is also a Banach space, say under the sup norm. If the map

$$F \times G \to E$$

given by

$$(y, z) \mapsto y + z$$

is a toplinear isomorphism, then we say that $E$ is the **direct sum** of the subspaces $F$ and $G$. Observe that our requirements involve both an algebraic and a topological condition. It follows from our conditions that

$$E = F + G \qquad \text{and} \qquad F \cap G = \{0\}.$$

It will be proved later that, in fact, these two conditions are sufficient; in other words, if they are satisfied, then the map

$$(y, z) \mapsto y + z$$

not only has an algebraic inverse, but this inverse is continuous (corollary of the open mapping theorem). When $E$ is a direct sum of $F$ and $G$, we write

$$E = F \oplus G.$$

If $A$ is an operator on $E$, then we are interested in expressing $E$ as a direct sum of $A$-invariant subspaces. Subsequent chapters give examples of this situation.

## APPENDIX: CONVEX SETS

## APP., §1. THE KREIN–MILMAN THEOREM

Although we shall not use the theorem of this section later in the book (except for some exercises), it is worthwhile giving it since it is used at the beginning of more advanced and specialized courses, in a wide variety of contexts. The exposition follows that of Artin (cf. *Collected Works*).

*Throughout this section, we let $E$ be a vector space over the reals (not normed). We let $E^*$ be a vector space of linear maps of $E$ into* $\mathbf{R}$ *(not necessarily the space of all such linear maps), and assume that $E^*$ separates $E$, that is given $x \in E$, $x \neq 0$ there exists $\lambda \in E^*$ such that $\lambda(x) \neq 0$. We*

*give E the topology having the smallest amount of open sets making all* $\lambda \in E^*$ *continuous.* A base for this topology is therefore given by the following sets: We take $x \in E$, and $\lambda_1, \ldots, \lambda_n \in E^*$, and $\varepsilon > 0$. We let $B$ be the set of all $y \in E$ such that

$$|\lambda_i(y) - \lambda_i(x)| < \varepsilon.$$

The set of all such $B$ is a base for the $E^*$-topology.

A subset $S$ of $E$ is said to be **convex** if given $x, y \in S$, the line segment

$$(1 - t)x + ty, \qquad 0 \leqq t \leqq 1,$$

joining $x$ to $y$ is contained in $S$.

*We observe that an arbitrary intersection of convex sets is convex.*

**Lemma 1.1.** *Let* $x_1, \ldots, x_n \in S$. *Any convex set containing* $x_1, \ldots, x_n$ *also contains all linear combinations*

$$t_1 x_1 + \cdots + t_n x_n$$

*with* $0 \leqq t_i \leqq 1$ *for all i, and* $t_1 + \cdots + t_n = 1$. *Conversely, the set of all such linear combinations is convex.*

*Proof.* If $t_n \neq 1$, then the above linear combination is equal to

$$(1 - t_n)\left(\frac{t_1}{1 - t_n} x_1 + \cdots + \frac{t_{n-1}}{1 - t_n} x_{n-1}\right) + t_n x_n.$$

The first assertion follows at once by induction. The converse is also an immediate consequence of the definitions.

The following properties of convex sets also follow at once from the definitions.

*Let* $\lambda: E \to F$ *be a linear map. If S is convex in E, then* $\lambda(S)$ *is convex in F. If T is convex in F, then* $\lambda^{-1}(T)$ *is convex in E. In other words, the image and inverse image of a convex set under a linear map are convex.*

Let $\lambda \in E^*$, $\lambda \neq 0$, and let $H_0$ be the kernel of $\lambda$ (i.e. the set of all $x \in E$ such that $\lambda(x) = 0$). Then $H_0$ is a closed subspace, and if $v \in E$ is such that $\lambda(v) \neq 0$, then

$$E = H_0 + \mathbf{R}v.$$

If $\lambda_1$, $\lambda_2$ are non-zero functionals with the same kernel $H$, then there exists $c \in \mathbf{R}$, $c \neq 0$ such that $\lambda_1 = c\lambda_2$. Indeed, one sees at once that $c = \lambda_1(v)/\lambda_2(v)$.

Let $\lambda \neq 0$ be an element of $E^*$, and let $c \in \mathbf{R}$. By the **hyperplane** $H_c$ we mean the set of all $x \in E$ such that $\lambda(x) = c$. In other words, $H_c = \lambda^{-1}(c)$. If $H_0$ is the kernel of $\lambda$, then $H_c$ consists of all elements $y + y_0$ with $y \in H_0$ and $y_0$ any fixed element of $E$ such that $\lambda(y_0) = c$.

The set of $x \in E$ such that $\lambda(x) \geqq c$ will be called a **closed half space** determined by the hyperplane, and so will the set of all $x$ such that $\lambda(x) \leqq c$. Similarly, we have the open half spaces, determined by the inequalities $\lambda(x) > c$ and $\lambda(x) < c$ respectively.

If $S$ is a closed subset of $E$ and $x_0$ a point, we say that a hyperplane $H$ **separates** $S$ and $x_0$ if $S$ is contained in one of the closed half spaces determined by $H$, and $x_0$ is not contained in this half space.

**Theorem 1.2.** *Let $S$ be a closed convex set in $E$, and let $x_0 \notin S$. Then there exists a separating hyperplane for $S$ and $x_0$, such that $S$ is contained in a closed half space determined by $H$.*

*Proof.* We begin by proving our statement in the finite dimensional case.

Let $T$ be a closed convex subset of $\mathbf{R}^n$, and let $P$ be a point of $\mathbf{R}^n$ such that $P \notin T$. The function $f(X) = |X - P|$ (euclidean norm) has a minimum on $T$, say at $Q \in T$. Let $N = Q - P$. Since $P \notin T$, we have $N \neq O$. We contend that the hyperplane passing through $Q$, perpendicular to $N$, will satisfy our requirements. The equation of this hyperplane is $X \cdot N = Q \cdot N$. Let $Q'$ be any point of $T$, and $Q' \neq Q$. For every $t$ with $0 < t \leqq 1$, we have

$$|Q - P| \leqq |Q + t(Q' - Q) - P| = |(Q - P) + t(Q' - Q)|.$$

Squaring gives

$$(Q - P)^2 \leqq (Q - P)^2 + 2t(Q - P)\cdot(Q' - Q) + t^2(Q' - Q)^2.$$

Canceling and dividing by $t$, we obtain

$$0 \leqq 2(Q - P)\cdot(Q' - Q) + t(Q' - Q)^2.$$

Letting $t$ tend to 0 yields

$$Q' \cdot N \geqq Q \cdot N \geqq P \cdot N + N \cdot N.$$

This proves that $T$ is contained in the closed half space defined by

$$X \cdot N \geqq c,$$

where $c = P \cdot N + N \cdot N$, thus proving our contention, and the fact that our hyperplane separates $T$ and $P$.

We return to the general case of the space $E$. There exists a neighborhood of $x_0$ which does not intersect $S$. In other words, there exists $\varepsilon$ and $\lambda_1, \ldots, \lambda_n \in E^*$ such that all $y \in E$ satisfying

$$|\lambda_i(y) - \lambda_i(x_0)| < \varepsilon \qquad (i = 1, \ldots, n)$$

do not lie in $S$. Consider the linear map

$$\varphi \colon E \to \mathbf{R}^n$$

given by

$$x \mapsto (\lambda_1(x), \ldots, \lambda_n(x)).$$

The image of $S$ is a convex set $\varphi(S)$ in $\mathbf{R}^n$, which does not intersect the neighborhood of $\varphi(x_0)$ determined by the inequality

$$\|Q - \varphi(x_0)\| < \varepsilon \qquad \text{(sup norm)}.$$

Its closure does not contain $\varphi(x_0)$. By our result in the finite dimensional case, there exists a non-zero vector

$$N = (c_1, \ldots, c_n) \in \mathbf{R}^n$$

such that $\varphi(S)$ lies in the closed half spaces determined by $N$ and a suitable constant $c$. We let

$$\lambda = c_1\lambda_1 + \cdots + c_n\lambda_n.$$

Then $\lambda \in E^*$ and $S$ is contained in a closed half space $\lambda \geqq c$, which does not contain $x_0$, thus proving Theorem 1.2.

**Remark.** All that we need in the sequel is that, the assumptions being as in the theorem, there exists a functional $\lambda \in E^*$ such that $\lambda(x_0)$ is not contained in $\lambda(S)$.

We define an **extreme point** of a convex set $S$ to be a point $x \in S$ having the following property: Whenever $y_1$, $y_2$ are points of $S$ such that we can write

$$x = ty_1 + (1 - t)y_2$$

with $0 < t < 1$, then $y_1 = y_2$.

**Theorem 1.3.** *Let $S$ be a non-empty, convex, compact subset of $E$. Then there exists an extreme point of $S$.*

*Proof.* Let $\mathscr{F}$ be the family of non-empty, convex, compact subsets of $E$ contained in $S$, and having the following additional property:

*If $K \in \mathscr{F}$ and $x \in K$, and if $y_1, y_2 \in S$ are such that*

$$x = ty_1 + (1 - t)y_2$$

*with $0 < t < 1$, then $y_1, y_2 \in K$.*

Then the set $S$ itself is in $\mathscr{F}$. We can order elements of $\mathscr{F}$ by descending inclusion, and if $\{K_i\}_{i \in I}$ is a totally ordered subfamily, then the intersection

$$\bigcap_{i \in I} K_i$$

is not empty, and clearly is again in $\mathscr{F}$. Hence by Zorn's lemma, there exists a minimal element $S_0$ in $\mathscr{F}$. We contend that $S_0$ consists of one point. (This will prove our theorem.) Since elements of $E^*$ separate points, it will suffice to prove that for each $\lambda \in E^*$, the set $\lambda(S_0)$ consists of one point. But $\lambda(S_0)$ is convex and compact, whence a closed bounded interval. Let $c$ be a right end point of this interval. Then the set $\lambda^{-1}(c) \cap S_0$ is non-empty, convex, compact. We contend that it lies in $\mathscr{F}$. Let $x$ be an element in $\lambda^{-1}(c) \cap S_0$, and suppose that we can write

$$x = ty_1 + (1 - t)y_2$$

with $y_1, y_2 \in S$ and $0 < t < 1$. Since $S_0 \in \mathscr{F}$, we get $y_1, y_2 \in S_0$. Applying $\lambda$, we find that

$$\lambda(x) = c = t\lambda(y_1) + (1 - t)\lambda(y_2).$$

Since $c$ is an end point of the interval $\lambda(S_0)$, it follows that

$$\lambda(y_1) = \lambda(y_2) = c.$$

Hence $y_1, y_2$ also lie in $\lambda^{-1}(c)$, and this shows that $\lambda^{-1}(c) \cap S_0$ is in $\mathscr{F}$. Since we took $S_0$ minimal, we conclude that $S_0$ is contained in $\lambda^{-1}(c)$, thereby proving our theorem.

**Corollary 1.4.** *Let $S$ be as in Theorem 1.3, and let $\lambda \in E^*$. Let $c$ be an end point of the interval $\lambda(S)$. Then $\lambda^{-1}(c) \cap S$ contains an extreme point of $S$.*

*Proof.* The intersection of the hyperplane $\lambda^{-1}(c)$ with $S$ is non-empty, convex, compact, and thus has an extreme point $x$, with respect to

$\lambda^{-1}(c) \cap S$. However, if $y_1, y_2 \in S$ and

$$x = ty_1 + (1 - t)y_2$$

with $0 < t < 1$, then $\lambda(x) = c = t\lambda(y_1) + (1 - t)\lambda(y_2)$, and hence $\lambda(y_1) = \lambda(y_2) = c$, so that $y_1, y_2 \in \lambda^{-1}(c) \cap S$. From this we conclude that $y_1 = y_2$, and hence that $x$ is also an extreme point of $S$ itself.

**Theorem 1.5 (Krein–Milman Theorem).** *Let $K$ be a convex, compact subset of $E$. Let $S$ be the set of extreme points of $K$. Then $K$ is the smallest closed convex set containing all elements of $S$ (i.e. the intersection of all closed convex sets containing $S$).*

*Proof.* Let $S'$ be the intersection of all closed convex sets containing $S$. Then $S' \subset K$, and since $K$ is compact, it follows that $S'$ is compact. Suppose that there exists $x_0 \in K$ but $x_0 \notin S'$. By Theorem 1.2, there exists $\lambda \in E^*$ such that $\lambda(x_0)$ is not contained in the interval $\lambda(S')$, say

$$\lambda(S') < \lambda(x_0).$$

Let $c$ be the right end point of the interval $\lambda(K)$. By Corollary 1.4, the set $\lambda^{-1}(c) \cap K$ contains an extreme point of $K$, contradicting the fact that $\lambda(S) < c$, and proving our theorem.

## APP., §2. MAZUR'S THEOREM

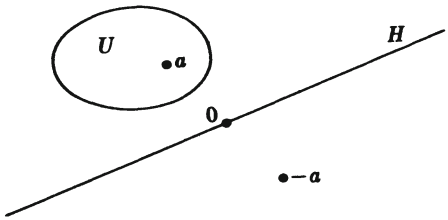
In the applications of Theorem 1.2, one starts frequently with a convex set in a Banach space, closed in the norm topology (i.e. the topology defined by the norm). In Theorem 1.2, we needed a convex set closed for the weak topology defined by a family of functionals. An example of such a family is simply the totality of all functionals, continuous for the norm topology. Of course, if a set $S$ is compact for the norm topology, it is also compact for the weak topology. One can then raise the question whether a closed convex set for the norm topology is also closed for the weak topology. The answer is yes:

**Theorem 2.1 (Mazur's Theorem).** *Let $E$ be a Banach space and let $A$ be a convex subset, closed for the norm topology. Then $A$ is also closed for the weak topology (that topology having the smallest amount of open sets making all functionals continuous). In fact, $A$ is the intersection of all closed half spaces containing $A$.*

The proof is self contained, and is based on the following lemma.

**Lemma 2.2.** *Let $U$ be an open non-empty convex set in $E$ which does not contain the origin. Then there exists a functional $\lambda$ on $E$ whose kernel does not intersect $U$.*

*Proof.* Let $a \in U$. Then $-a \notin U$, otherwise $0 \in U$ because $U$ is convex, and this is impossible. By a **cone** we shall mean a subset $C$ of $E$ such that if $x \in C$, then $tx \in C$ for all real $t \geqq 0$. Let $\Gamma$ be the set of all convex cones containing $U$ but not $-a$. Then $\Gamma$ is not empty because the set of all points $tx$ with $t \geqq 0$ and $x \in U$, is verified to be a convex cone directly from the definitions, and belongs to $\Gamma$. It is clear that $\Gamma$ is inductively ordered by ascending inclusion. Let $C$ be a maximal element of $\Gamma$. We contend that $C \cap (-C)$ is a closed hyperplane $H$ which does not intersect $U$. Picture:




First we prove that the maximal cone $C$ is closed. Suppose $C$ is not closed. Then we must have $-a \in \bar{C}$, for otherwise we have $C \subset \bar{C} \in \Gamma$ and $C \neq \bar{C}$, contradicting the maximality of $C$. On the other hand, we have $a \in U \subset C$. Since $U$ is open, there is a ball $B \subset U$ centered at $a$ and of radius $r > 0$. But $C$ is convex. Therefore $C$ contains the set $A$ of elements $(-a + x)/2$ with $x \in B$. It is easy to see that $A$ contains the ball centered at the origin and of radius $r/2$. This and the fact that $C$ is a cone imply that $C = E$, a contradiction. It follows that $H = C \cap (-C)$ is closed, is a cone, is convex, and $H = -H$. Therefore $H$ is immediately seen to be a closed subspace. We have $H \neq E$ because $-a \notin C$, so $-a \notin H$.

We have $E = C \cup (-C)$. To see this, let $x \in E$ and suppose $x \notin C$, $x \notin -C$. Since $C$ is maximal, the cone consisting of all elements $c + tx$ with $c \in C$, $t \geqq 0$ contains $-a$, and so does the cone of all elements $c + t(-x)$, $c \in C$, $t \geqq 0$. Hence we can write

$$-a = c_1 + t_1 x = c_2 - t_2 x$$

with $c_1, c_2 \in C$ and $t_1, t_2 \geqq 0$. Consequently

$$c_1 + (t_1 + t_2)x \in C.$$

However, $c_1 + t_1x = -a$ is on the line segment between $c_1$ and $c_1 + (t_1 + t_2)x$, and thus lies in $C$, a contradiction which proves that $E = C \cup (-C)$.

Now suppose that $x \in C$. Then the line segment between $x$ and $-a$ contains a point of $H$. For instance, on the segment $x + t(-a - x)$ with $0 \leqq t \leqq 1$, let $\tau$ be the sup of all $t$ such that $x + t(-a - x)$ lies in $C$. Then $x + \tau(-a - x)$ lies in $H$, and $\tau \neq 1$, otherwise $-a \in H$, which is impossible. We therefore have

$$(1 - \tau)x - \tau a = h \in H,$$

whence

$$x = \frac{\tau}{1 - \tau}a + \frac{1}{1 - \tau}h.$$

Working also with $-x$ instead of $x$, we conclude that $E$ is generated by $H$ and $a$, so that the factor space $E/H$ has dimension 1 and hence $H$ is a closed hyperplane.

Finally, $H$ does not intersect $U$, for otherwise let $h \in H \cap U$. Since $U$ is open, for small $s > 0$ we have $h - sa \in U$ so $h - sa \in C$. But $-h \in C$, whence $-sa \in C$ and $-a \in C$, which is impossible. This proves our lemma.

We now prove: *Let $b$ be a point of a Banach space $E$, which does not belong to the norm-closed non-empty convex set $A$. Then there exists a functional $\lambda$ and a number $\alpha$ such that $\lambda(x) > \alpha$ for all $x \in A$ and $\lambda(b) < \alpha$.*

*Proof.* Let $B$ be an open ball centered at $b$ and not intersecting $A$. Then the set $U = A - B$, consisting of all points $x - y$ with $x \in A$ and $y \in B$, is open, convex, non-empty, and does not contain the origin. ($U$ is open because it is a union of open sets $a - B$ with $a \in A$, and it is immediately verified to be convex because the sum of two convex sets is convex.) We apply our lemma to $U$ and find a functional $\lambda$ as in the lemma, so that $\lambda z \geqq 0$ for all $z \in A - B$, and therefore $\lambda x \geqq \lambda y$ for all $x \in A$, $y \in B$. Let $\beta = \inf \lambda x$ for $x \in A$. The map $\lambda$ is an open mapping, for instance because $\lambda$ gives an isomorphism of a one-dimensional subspace of $E$ onto $\mathbf{R}$. Therefore $\lambda y < \beta$ for all $y \in B$, so that in particular, $\lambda b < \beta$. We let $\alpha = \frac{1}{2}(\lambda b + \beta)$ to conclude the proof of our assertion.

Mazur's theorem follows at once, since we have proved that a non-empty closed convex set is the intersection of all closed half spaces containing it.

## IV, §6. EXERCISES

1. Fill in the details that if $F$ is complete, then $L(E, F)$ is complete.

2. Show that the Hahn–Banach theorem for the complex case follows easily from the real case of this theorem. In other words, finish the details of the argument given after Corollary 1.2.

3. Let $E$, $F$, $G$ be normed vector spaces. A **bilinear** map $\lambda: E \times F \to G$ is a map which is linear in each variable, i.e. for each $x \in E$ the map $y \mapsto \lambda(x, y)$ is linear, and for each $y \in F$, the map $x \mapsto \lambda(x, y)$ is linear. Show that a bilinear map $\lambda$ is continuous if and only if there exists $C > 0$ such that

$$|\lambda(x, y)| \leqq C|x||y|$$

for all $x \in E$, $y \in F$. Let $L(E, F; G)$ be the set of continuous bilinear maps of $E \times F$ into $G$. Show that $L(E, F; G)$ is a normed vector space, if the norm of $\lambda$ is defined to be the inf of all numbers $C$ as above. Show that if $G$ is complete, then $L(E, F; G)$ is complete.

4. Let $E$ be a Banach space and $F$ a closed subspace. For each coset $x + F$ of $F$, define $|x + F| = \inf|x + y|$ for $y \in F$. Show that this defines a norm on the factor space $E/F$, and that the natural map $E \to E/F$ is continuous linear. (Cf. Chapter XV, §1.)

5. Let $A$ be a Banach algebra. Suppose that there is a unit element $e \neq 0$, but that we do not necessarily have $|e| = 1$.
   (a) Show that $|e| \geqq 1$.
   (b) Define a new norm $\| \;\|$ on $A$ by putting

$$\|x\| = \sup_{y \neq 0} \frac{|xy|}{|y|}.$$

   Show that $\| \;\|$ is in fact a norm and that $\|e\| = 1$.
   (c) Show that $A$ is a Banach algebra under this new norm.

6. (a) Show that a finite dimensional subspace of a normed vector space is closed.
   (b) Let $E$ be a Banach space and $F$ a finite dimensional subspace. Show that there exists a closed subspace $G$ such that $F + G = E$ and

$$F \cap G = \{0\}.$$

   You will have to use the Hahn–Banach theorem.

7. Let $F$ be a closed subspace of a normed vector space $E$, and let $v \in E$, $v \notin F$. Show that $F + \mathbf{R}v$ is closed. If $E = F + \mathbf{R}v$, show that $E$ is the direct sum of $F$ and $\mathbf{R}v$. (You can give a simple ad hoc proof for this. A more general result will be proved later as a consequence of the open mapping theorem.)

8. Let $E$, $F$, $G$ be normed vector spaces and assume that $G$ is complete. Let $\lambda: E \times F \to G$ be a continuous bilinear map. Show that $\lambda$ can be extended to a continuous bilinear map of the completions $\bar{E} \times \bar{F} \to G$, which has the same norm as $\lambda$. (Identify $E$, $F$ as subspaces of their completions.)

9. Let $A$ be a Banach algebra, commutative, and with unit element. Let $J$ be an ideal. Show that the closure of $J$ is also an ideal. (The definition of an ideal is the same as in the case of rings of continuous functions. If the algebra is not commutative, then the same result is valid if we replace ideals by left ideals.)

10. Let $A$ be a commutative Banach algebra and let $M$ be a maximal ideal. Show that $M$ is closed.

11. Give the proof of the inequality left to the reader in Proposition 1.3.

12. Let $E$ be an infinite dimensional Banach space, and let $\{x_n\}$ be a sequence of linearly independent elements of norm 1. Show that there exists an element in the closure of the space generated by all $x_n$ which does not lie in any subspace generated by a finite number of $x_n$. [*Hint*: Construct this element as an absolutely convergent sum $\sum c_n x_n$.]

13. Let $\{E_n\}$ be a sequence of Banach spaces. Let $E$ be the set of all sequences $\xi = \{x_n\}$ with $x_n \in E_n$ such that $\sum |x_n|$ converges. Show that $E$ is a vector space, and that if we define

$$|\xi| = \sum |x_n|$$

then this is a norm, and $E$ is complete.

14. Let $E$ be a Banach space, and $P$, $Q$ two operators on $E$ such that $P + Q = I$, and $PQ = QP = O$. Show that

$$E = \operatorname{Ker} P + \operatorname{Ker} Q,$$

and that $\operatorname{Ker} P = \operatorname{Im} Q$. Show that $\operatorname{Ker} P \cap \operatorname{Ker} Q = \{O\}$, and that $\operatorname{Ker} P$ and $\operatorname{Im} P$ are closed subspaces.

15. Let $E$ be a Banach space and let $F$ be a vector subspace. Let $\bar{F}$ be the closure of $F$. Prove that $\bar{F}$ is a subspace, and is complete.

16. Let $A$ be a subset of a Banach space. By $c(A)$ we denote the convex closure of $A$, i.e. the intersection of all convex sets containing $A$. We let $\bar{c}(A)$ denote the closure of $c(A)$. Then $\bar{c}(A)$ is convex. Prove: If $K$ is compact, then $\bar{c}(K)$ is also compact. [*Hint*: Show $c(K)$ is totally bounded as follows. First find a finite number of points $x_1, \ldots, x_n$ such that $K$ is contained in the union of the balls of radius $\varepsilon$ around these points. Let $C$ be the convex closure of the set $\{x_1, \ldots, x_n\}$. Show that $C$ is compact, expressing $C$ as a continuous image of a compact set. Let $y_1, \ldots, y_m$ be points of $C$ such that $C$ is contained in the union of balls of radius $\varepsilon$ around these points. Then get the desired result.]

17. Let $F$ be the complete normed vector space of continuous periodic functions on $[-\pi, \pi]$ of period $2\pi$, with the sup norm. Let $E$ be the vector space of all real sequences $\alpha = \{a_n\}$ such that $\sum |a_n|$ converges. Define

$$|\alpha| = \sum_{n=1}^{\infty} |a_n|.$$

Show that this is a norm on $E$. Let

$$L\alpha(x) = \sum a_n \cos nx,$$

so that $L: E \to F$ is a linear map. Show that $L$ has norm 1. Let $B$ the closed unit ball of radius 1 centered at the origin in $E$. Show that $L(B)$ is closed in $F$. [*Hint:* Let $\{f_k\}$ $(k = 1, 2, \ldots)$ be a sequence of elements in $L(B)$ which converges uniformly to a function $f$ in $F$. Let $b_n$ be the Fourier coefficient of $f$ with respect to $\cos nx$. Let $\beta = \{b_n\}$. Show that $\beta$ is in $E$ and that $L(\beta) = f$.]

18. Let $K$ be a continuous function of two variables defined for $(x, y)$ in the square $[a, b] \times [a, b]$. Assume that $\|K\| \leqq C$ for some constant $C > 0$, where $\| \; \|$ is the sup norm. Let $E$ be the Banach space of continuous functions on $[a, b]$, and let $T: E \to E$ be the linear map such that

$$Tg(x) = \int_a^b K(t, x)g(t)\,dt.$$

Show that $T$ is bounded and $\|T\| < C(b - a)$. For more on $T$, see Chapter XIV, Exercise 5.

19. Let $A$ be a commutative Banach algebra with unit element $e$, over the reals, and define the exponential and logarithm maps by

$$\exp u = 1 + u + \frac{u^2}{2!} + \cdots$$

and

$$\log u = (u - e) - \frac{(u - e)^2}{2} + \frac{(u - e)^3}{3} - \cdots$$

Show that exp converges absolutely for all $u \in A$, and that log converges absolutely for all $u$ with $|u - e| < 1$. Show that the exp and log give inverse continuous mappings from a neighborhood of 0 onto a neighborhood of $e$ in $A$. Show that they satisfy the usual function equations

$$\exp(u + v) = (\exp u)(\exp v),$$

$$\log(uv) = \log u + \log v,$$

in these domains of definition. Show that every element of $A$ sufficiently close to $e$ is an $n$-th power for every positive integer $n$.

20. Let $X$ be a compact Hausdorff space and let $C(X)$ be the Banach space of real continuous functions on $X$. If $\lambda$ is a functional on $C(X)$ (sup norm) such that $\lambda(1) = |\lambda|$, show that $\lambda$ is positive, in the sense that if $f \in C(X)$, $f \geqq 0$, then $\lambda(f) \geqq 0$.

# CHAPTER V

# Hilbert Space

## V, §1. HERMITIAN FORMS

Essentially all of this chapter goes through over the real or the complex numbers with no change. Since the theory over the complex does introduce the extra conjugation, we use the complex language, and point out explicitly in one or two instances those results which are valid only over the complex.

Let $E$, $F$ be vector spaces over $\mathbf{C}$ and let $L: E \to F$ be a map. We say that $L$ is **antilinear**, or **semi-linear**, if $L$ is $\mathbf{R}$-linear, and $L(\alpha x) = \bar{\alpha} L(x)$ for all $x \in E$ and $\alpha \in \mathbf{C}$.

Let $E$ be a vector space over the complex numbers. A **sesquilinear form** or **scalar product** on $E$ is a map

$$E \times E \to \mathbf{C}$$

denoted by

$$(x, y) \mapsto \langle x, y \rangle$$

which is linear in its first variable, and **semi-linear** or **antilinear** in its second variable, meaning that for $x, y, y_1, y_2 \in E$, $\alpha \in \mathbf{C}$, we have

$$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle \qquad \text{and} \qquad \langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle.$$

If in addition we have for all $x, y \in E$

$$\langle x, y \rangle = \overline{\langle y, x \rangle},$$

we say that the form is **hermitian**. If furthermore we have $\langle x, x \rangle \geqq 0$ for

all $x \in E$, we say that the form is **positive**. We say the form is **positive definite** if it is positive, and $\langle x, x\rangle > 0$ if $x \neq 0$. *We shall assume throughout that our form* $\langle\ ,\ \rangle$ *is positive, but not necessarily definite.* We observe that a sesquilinear form is always **R**-bilinear.

We define $v$ to be **perpendicular** or **orthogonal** to $w$ if $\langle v, w\rangle = 0$. *Let $S$ be a subset of $E$. The set of elements $v \in E$ such that $\langle v, w\rangle = 0$ for all $w \in S$ is a subspace of $E$.* This is easily seen and will be left as an exercise. We denote this set by $S^\perp$. Let $E_0$ consist of all elements $v \in E$ such that $v \in E^\perp$, that is $\langle v, w\rangle = 0$ for all $w \in E$. Then $E_0$ is a subspace, which will be called the **null space** of the hermitian product.

**Theorem 1.1.** *If $w \in E$ is such that $\langle w, w\rangle = 0$, then $w \in E_0$, that is $\langle w, v\rangle = 0$ for all $v \in E$.*

*Proof.* Let $t$ be real, and consider

$$\begin{aligned} 0 \leqq \langle v + tw, v + tw\rangle &= \langle v, v\rangle + 2t\,\mathrm{Re}\langle v, w\rangle + t^2\langle w, w\rangle \\ &= \langle v, v\rangle + 2t\,\mathrm{Re}\langle v, w\rangle. \end{aligned}$$

If $\mathrm{Re}\langle v, w\rangle \neq 0$ then we take $t$ very large of opposite sign to $\mathrm{Re}\langle v, w\rangle$. Then $\langle v, v\rangle + 2t\,\mathrm{Re}\langle v, w\rangle$ is negative, a contradiction. Hence

$$\mathrm{Re}\langle v, w\rangle = 0.$$

This is true for all $v \in E$. Hence $\mathrm{Re}\langle iv, w\rangle = 0$ for all $v \in E$, whence $\mathrm{Im}\langle v, w\rangle = 0$. Hence $\langle v, w\rangle = 0$, as was to be shown.

We define $|v| = \sqrt{\langle v, v\rangle}$, and call it the **length** or **norm** of $v$. By definition and Theorem 1.1, we have $|v| = 0$ if and only if $v \in E_0$.

**Theorem 1.2 (Schwarz Inequality).** *For all $v, w \in E$ we have*

$$|\langle v, w\rangle| \leqq |v||w|.$$

*Proof.* Let $\alpha = \langle w, w\rangle$ and $\beta = -\langle v, w\rangle$. We have

$$\begin{aligned} 0 &\leqq \langle \alpha v + \beta w, \alpha v + \beta w\rangle \\ &= \langle \alpha v, \alpha v\rangle + \langle \beta w, \alpha v\rangle + \langle \alpha v, \beta w\rangle + \langle \beta w, \beta w\rangle \\ &= \alpha\bar{\alpha}\langle v, v\rangle + \beta\bar{\alpha}\langle w, v\rangle + \alpha\bar{\beta}\langle v, w\rangle + \beta\bar{\beta}\langle w, w\rangle. \end{aligned}$$

Note that $\alpha = |w|^2$. Substituting the values for $\alpha$, $\beta$, we obtain

$$0 \leqq |w|^4|v|^2 - 2|w|^2\langle v, w\rangle\overline{\langle v, w\rangle} + |w|^2\langle v, w\rangle\overline{\langle v, w\rangle}.$$

But

$$\langle v, w\rangle\overline{\langle v, w\rangle} = |\langle v, w\rangle|^2.$$

Hence

$$|w|^2|\langle v, w\rangle|^2 \leqq |w|^4|v|^2.$$

If $|w| = 0$, then $w \in E_0$ by Theorem 1.1 and the Schwarz inequality is obvious. If $|w| \neq 0$, then we can divide this last relation by $|w|^2$, and taking the square roots yields the proof of the theorem.

**Theorem 1.3.** *The function $v \mapsto |v|$ is a seminorm on E, that is:*

*We have $|v| \geqq 0$, and $|v| = 0$ if and only if $v \in E_0$.*

*For every complex $\alpha$, we have $|\alpha v| = |\alpha||v|$.*

*For $v, w \in E$ we have $|v + w| \leqq |v| + |w|$.*

*Proof.* The first assertion follows from Theorem 1.1. The second is left to the reader. The third is proved with the Schwarz inequality. It suffices to prove that

$$|v + w|^2 \leqq (|v| + |w|)^2.$$

To do this, we have

$$|v + w|^2 = \langle v + w, v + w\rangle = \langle v, v\rangle + \langle w, v\rangle + \langle v, w\rangle + \langle w, w\rangle.$$

But $\langle w, v\rangle + \langle v, w\rangle = 2\,\mathrm{Re}\langle v, w\rangle \leqq 2|\langle v, w\rangle|$. Hence by Schwarz,

$$\begin{aligned} |v + w|^2 &\leqq |v|^2 + 2|\langle v, w\rangle| + |w|^2 \\ &\leqq |v|^2 + 2|v||w| + |w|^2 = (|v| + |w|)^2. \end{aligned}$$

Taking the square root of each side yields what we want.

We call $|\ |$ the $L^2$**-norm** (or we should really say the $L^2$**-seminorm**).

An element of $E$ is said to be a **unit vector** if $|v| = 1$. If $|v| \neq 0$, then $v/|v|$ is a unit vector.

Let $w \in E$ be an element such that $|w| \neq 0$, and let $v \in E$. *There exists a unique number c such that $v - cw$ is perpendicular to w.* Indeed, for $v - cw$ to be perpendicular to $w$ we must have

$$\langle v - cw, w\rangle = 0,$$

whence $\langle v, w\rangle - \langle cw, w\rangle = 0$ and $\langle v, w\rangle = c\langle w, w\rangle$. Thus

$$c = \frac{\langle v, w\rangle}{\langle w, w\rangle}.$$

Conversely, letting $c$ have this value shows that $v - cw$ is perpendicular to $w$. We call $c$ the **Fourier coefficient of** $v$ with respect to $w$.

Let $v_1, \ldots, v_n$ be elements of $E$ which are not in $E_0$, and which are mutually perpendicular, that is $\langle v_i, v_j \rangle = 0$ if $i \neq j$. Let $c_i$ be the Fourier coefficient of $v$ with respect to $v_i$. Then

$$v - c_1 v_1 - c_2 v_2 - \cdots - c_n v_n$$

is perpendicular to $v_1, \ldots, v_n$. Indeed, all we have to do is to take the product of $v$ with $v_j$. All the terms involving $\langle v_i, v_j \rangle$ will give 0, and we shall have two terms

$$\langle v, v_j \rangle - c_j \langle v_j, v_j \rangle$$

which cancel. Thus subtracting linear combinations as above orthogonalizes $v$ with respect to $v_1, \ldots, v_n$.

We have two useful identities, namely:

**The Pythagoras Theorem.** *If $u$, $w \in E$ are perpendicular, then*

$$|u + w|^2 = |u|^2 + |w|^2.$$

**The Parallelogram Law.** *For $u$, $w \in E$, we have*

$$|u + w|^2 + |u - w|^2 = 2|u|^2 + 2|w|^2.$$

The proofs come immediately from expanding out the norm according to the definitions.

Let $\{v_i\}_{i \in I}$ be a family of elements of $E$ such that $|v_i| \neq 0$ for all $i$. For each finite subfamily, we can take the space generated by this subfamily, i.e. linear combinations

$$c_1 v_{i_1} + \cdots + c_n v_{i_n}$$

with complex coefficients $c_i$. The union of all such spaces is called the space generated by the family $\{v_i\}_{i \in I}$. Let us denote this space by $F$. We say that the family $\{v_i\}$ is **total** in $E$ if the closure of $F$ is equal to all of $E$.

As a matter of notation, we shall omit the double indices and write $v_1, \ldots, v_n$ instead of $v_{i_1}, \ldots, v_{i_n}$.

We say that the family $\{v_i\}$ is an **orthogonal** family if its elements are mutually perpendicular, that is $\langle v_i, v_j \rangle = 0$ if $i \neq j$, and if in addition $|v_i| \neq 0$ for all $i$. We say that it is an **orthonormal** family if it is orthogonal and if $|v_i| = 1$ for all $i$. One can always obtain an orthonormal family from an orthogonal family by dividing each vector by its norm. A total orthonormal family is called a **Hilbert basis**, or also an **orthonormal basis**. (**Warning:** It is not necessarily a "basis" in the sense of

abstract algebra, i.e. not every element of the space is a linear combination of a finite number of elements in a Hilbert basis.)

**Theorem 1.4.** *Let $\{v_i\}$ be an orthogonal family in $E$. Let $x \in E$ and let $c_i$ be the Fourier coefficient of $x$ with respect to $v_i$. Let $\{a_i\}$ be a family of numbers. Then*

$$\left| x - \sum_{k=1}^{n} c_k v_k \right| \leqq \left| x - \sum_{k=1}^{n} a_k v_k \right|.$$

*Proof.* We know that

$$x - \sum_{k=1}^{n} c_k v_k$$

is orthogonal to each $v_i$, $i = 1, \ldots, n$. Hence we get from Pythagoras:

$$\begin{aligned} \left| x - \sum_{k=1}^{n} a_k v_k \right|^2 &= \left| x - \sum_{k=1}^{n} c_k v_k + \sum_{k=1}^{n} (c_k - a_k) v_k \right|^2 \\ &= |x - \sum c_k v_k|^2 + |\sum (c_k - a_k) v_k|^2. \end{aligned}$$

This proves the desired inequality.

A **pre-Hilbert space** is a vector space with a positive definite hermitian form. If we start with a space with a form which is only positive (not definite), we can obtain a pre-Hilbert space by taking the factor space $E/E_0$ (i.e. equivalence classes of elements of $E$ modulo $E_0$). Similarly, we can form the completion of $E$. Viewing $E$ as a space over the reals, we can extend the **R**-bilinear form $\langle \; , \; \rangle$ to the completion. If $E$ is a pre-Hilbert space, then the extended form is hermitian positive definite. (That it is hermitian positive follows by continuity. For the definiteness, if $\{x_n\}$ is a sequence converging to $x$, and $x \neq 0$, we may assume that $x_n \neq 0$, and then that $\{x_n/|x_n|\}$ converges to $x/|x|$. Thus we may deal with unit vectors, whence the definiteness follows immediately.)

A **Hilbert space** is a vector space with a positive definite hermitian form, which is complete under the corresponding $L^2$-norm. Thus we see that the completion of a pre-Hilbert space is a Hilbert space.

**Lemma 1.5.** *Let $E$ be a Hilbert space, and $F$ a closed subspace. Let $x \in E$ and let*

$$a = \inf_{y \in F} |x - y|.$$

*Then there exists an element $y_0 \in F$ such that*

$$a = |x - y_0|.$$

*Proof.* Let $\{y_n\}$ be a sequence in $F$ such that $|y_n - x|$ approaches $a$. We show that $\{y_n\}$ is Cauchy. By the parallelogram law, we have

$$\begin{aligned} |y_n - y_m|^2 &= 2|y_n - x|^2 + 2|y_m - x|^2 - 4|\tfrac{1}{2}(y_n + y_m) - x|^2 \\ &\leqq 2|y_n - x|^2 + 2|y_m - x|^2 - 4a^2 \end{aligned}$$

because of the definition of $a$. This shows that $\{y_n\}$ is Cauchy, and thus converges to some vector $y_0$. The lemma follows by continuity.

**Theorem 1.6.** *Let $F$ be a closed subspace of the Hilbert space $E$, and assume that $F \neq E$. Then there exists an element $z \in E$, $z \neq 0$, such that $z$ is perpendicular to $F$.*

*Proof.* Let $x \in E$ and $x \notin F$. Let $y_0 \in F$ be at minimal distance from $x$ (by the lemma), and let $a$ be this distance. Let $z = x - y_0$. Then $z \neq 0$ since $F$ is closed. For all $y \in F$, $y \neq 0$ and complex $\alpha$, we have

$$|x - y_0|^2 \leqq |x - y_0 + \alpha y|^2$$

whence, expanding out, we obtain

$$0 \leqq \alpha\langle y, z\rangle + \bar{\alpha}\langle z, y\rangle + \alpha\bar{\alpha}\langle y, y\rangle.$$

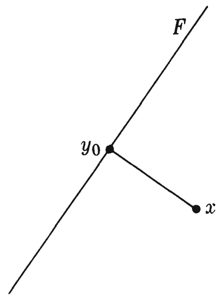



Figure 5.1

We let $\alpha = t\langle z, y\rangle$, with $t$ real $\neq 0$. We can then cancel $t$ and get a contradiction for small $t$, if $\langle y, z\rangle \neq 0$. This proves the theorem.

**Corollary 1.7.** *Let $E$ be a Hilbert space, $E \neq \{0\}$. Then there exists a total orthogonal basis for $E$.*

*Proof.* Let $S$ be the set of non-empty orthogonal families. If $\mathscr{F}_1$, $\mathscr{F}_2$ are orthogonal families, we define $\mathscr{F}_1 \leqq \mathscr{F}_2$ if $\mathscr{F}_1 \subset \mathscr{F}_2$. This gives an inductive ordering. Let $\mathscr{B}$ be a maximal element, and let $F$ be the subspace generated by $\mathscr{B}$. We contend that $F$ is dense in $E$. Otherwise,

$\bar{F} \neq E$, and by the theorem there exists $z \in E$, $z \neq 0$ and $z$ perpendicular to $\bar{F}$. We can then obtain a bigger orthogonal family than $\mathscr{B}$, a contradiction which proves our corollary.

**Corollary 1.8.** *Let $E$ be a Hilbert space, and $F$ a closed subspace. Then $E = F + F^{\perp}$.*

*Proof.* If $y_n \in F$ and $z_n \in F^{\perp}$, then the sequence $\{y_n + z_n\}$ is Cauchy if and only if $\{y_n\}$ is Cauchy and $\{z_n\}$ is Cauchy (by the Pythagoras theorem). Hence $F + F^{\perp}$ is closed. If $F + F^{\perp} \neq E$, then there exists $w \in E$, $w \neq 0$, which is perpendicular to $F + F^{\perp}$, whence perpendicular to $F$, so that $w \in F^{\perp}$, a contradiction which proves the corollary.

We observe that if $F$ is a closed subspace, then $F^{\perp\perp} = F$. For any $x \in E$, we can write uniquely

$$x = y + z$$

with $y \in F$ and $z \in F^{\perp}$. The map $P: E \to E$ such that

$$Px = y$$

is called the **orthogonal projection** on $F$. It is obviously a continuous linear map, and we study such maps in greater detail in Chapter XVIII, §5.

**Corollary 1.9.** *Let $E$ be a Hilbert space. Let $\{F_i\}$ $(i = 1, 2, \ldots)$ be a sequence of closed subspaces which are mutually perpendicular, that is $F_i \perp F_j$ if $i \neq j$. Let $\bar{F}$ be the closure of the space $F$ generated by all $F_i$. (In other words, $\bar{F}$ is the closure of the space $F$ consisting of all sums*

$$x_1 + \cdots + x_n, \qquad x_i \in F_i.)$$

*Then every element $x$ of $\bar{F}$ has a unique expression as a convergent series*

$$x = \sum_{i=1}^{\infty} x_i, \qquad x_i \in F_i.$$

*Let $P_i$ be the orthogonal projection on $F_i$. Then $x_i = P_i x$, and for any choice of elements $y_i \in F_i$ we have*

$$\left| x - \sum_{i=1}^{n} P_i x \right| \leqq \left| x - \sum_{i=1}^{n} y_i \right|.$$

*Proof.* Since

$$x - \sum_{i=1}^{n} P_i x$$

is orthogonal to $F_1, \dots, F_n$ we can use exactly the same argument as in Theorem 1.4, and the Pythagoras theorem to show the last inequality, writing

$$\left| x - \sum_{i=1}^{n} y_i \right|^2 = \left| x - \sum_{i=1}^{n} P_i x \right|^2 + \left| \sum_{i=1}^{n} (P_i x - y_i) \right|^2 .$$

There exists a sequence from $F$ which approaches $x$. It therefore follows that the partial sums

$$\sum_{i=1}^{n} P_i x$$

must approach $x$ also. If

$$x = \sum_{i=1}^{\infty} x_i$$

with $x_i \in F_i$, then we apply the projection $P_n$ (which is continuous!) to conclude that $P_n x = x_n$, thus proving the uniqueness.

It is convenient to call the family $\{F_i\}$ an **orthogonal decomposition of** $\bar{F}$ in the preceding theorem. If $\bar{F} = E$, then we call it an orthogonal decomposition of $E$, of course.

Suppose that the Hilbert space $E$ has a denumerable total family $\{v_n\}$, which we assume to be orthonormal. Then every element can be written as a convergent series

$$x = \sum_{n=1}^{\infty} a_n v_n$$

where $a_n$ is the Fourier coefficient of $x$ with respect to $v_n$, and the convergence is of course with respect to the $L^2$-norm. Namely, we take the spaces $F_n$ in the previous discussion to be the 1-dimensional spaces generated by $v_n$. In particular, we see that $\sum |a_n|^2$ converges, and that

$$|x|^2 = \sum_{n=1}^{\infty} |a_n|^2.$$

If $\{v_n\}$ is merely an orthonormal system, not necessarily a Hilbert basis, then of course we don't get the equality, merely the inequality

$$\sum_{n=1}^{\infty} |a_n|^2 \leqq |x|^2.$$

This is called the **Bessel inequality**, and it is essentially obvious from

previous discussions. For instance, for each $n$ we can write

$$v = v - \sum_{k=1}^{n} a_k v_k + \sum_{k=1}^{n} a_k v_k$$

and apply Pythagoras' theorem.

Conversely, we can define directly a set $l^2$ consisting of all sequences $\{a_n\}$ such that $\sum |a_n|^2$ converges. If $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ are two sequences in this space, then using the Schwarz inequality, on finite partial sums, one sees that

$$\sum |a_n b_n|$$

converges, whence we can define a product

$$\langle \alpha, \beta \rangle = \sum a_n \overline{b_n}.$$

Again from the above convergence, we conclude that $l^2$ is in fact a vector space, because

$$\sum |a_n + b_n|^2 \leqq \sum |a_n|^2 + \sum 2|a_n b_n| + \sum |b_n|^2.$$

Furthermore, this product is a hermitian product on it. Finally, it is but an exercise to verify that $l^2$ is complete. Indeed, the family $\{v_n\}$ is total, orthonormal in the completion of $l^2$, and in this completion any element can be expressed as a convergent series, described above. Thus the elements of the completion are precisely those of $l^2$.

The space $l^2$ can also be interpreted as the completion of a space of functions, those periodic of period $2\pi$, say, a total orthogonal family then being constituted by the functions

$$\chi_n(t) = e^{int}$$

where $n$ ranges over *all* integers (positive, negative, or zero).

It is clear that any two Hilbert spaces having denumerable orthonormal total families are isomorphic under the map which sends one family on the other. Indeed, if $G$ is another Hilbert space with total orthonormal family $\{e_n\}$, then the map

$$\sum a_n v_n \mapsto \sum a_n e_n$$

is linear and preserves the norm. In this way, we get a map from our space of periodic functions into $l^2$, which is injective and preserves the norm. It extends therefore uniquely to the completion.

In general, if two Hilbert spaces have total orthonormal families with

the same cardinality, then any bijection between these families extends to a unique norm-preserving linear map of one space to the other.

## V, §2. FUNCTIONALS AND OPERATORS

**Theorem 2.1.** *For every $y$ in the Hilbert space $E$, the map $\lambda_y$ such that $\lambda_y(x) = \langle x, y\rangle$ is a functional. The association*

$$y \mapsto \lambda_y$$

*is a norm-preserving antilinear isomorphism between $E$ and its dual space $E'$.*

*Proof.* The Schwarz inequality shows that $|\lambda_y| \leqq |y|$, and evaluating $\lambda_y$ at $y$ shows that $|\lambda_y| = |y|$, so we get a norm-preserving semi-linear map of $E$ into $E'$, semi-linear because of the hermitian nature of the scalar product, namely for complex $\alpha$,

$$\lambda_{\alpha y} = \bar{\alpha}\lambda_y.$$

There remains to show that every functional comes from some $y \in E$. Let $\lambda$ be a functional, and let $F$ be its kernel (the closed subspace of all $x$ such that $\lambda(x) = 0$). If $F \neq E$, there exists $z \in E$, $z \neq 0$ such that $z$ is perpendicular to $F$ (by Theorem 1.6). We contend that some scalar multiple of $z$ achieves our purpose, say $\alpha z$. A necessary condition on $\alpha$ is that

$$\langle z, \alpha z\rangle = \lambda(z)$$

or in other words, $\bar{\alpha} = \lambda(z)/\langle z, z\rangle$. This is also sufficient. Indeed, for any $x \in E$, we can write

$$x = x - \frac{\lambda(x)}{\lambda(z)} z + \frac{\lambda(x)}{\lambda(z)} z$$

and

$$x - \frac{\lambda(x)}{\lambda(z)} z$$

lies in $F$. Taking the product with $\alpha z$, we obtain

$$\langle x, \alpha z\rangle = \lambda(x)$$

thus proving our theorem.

By an **operator** we shall mean a continuous linear map of $E$ into itself. As we know, the space of operators $\mathrm{End}(E)$ is a Banach space.

By Herm($E$) we denote the set of all continuous hermitian forms on $E$. By Sesqu($E$) we denote the set of all continuous sesquilinear forms on $E$. It is immediately verified that both these sets are in fact Banach spaces, and that Herm($E$) is a closed subspace of Sesqu($E$). We shall now relate continuous sesquilinear forms on $E$ and operators.

Let $A: E \to E$ be an operator. We define $\varphi_A$ by

$$\varphi_A(x, y) = \langle Ax, y \rangle.$$

Then $\varphi_A$ is obviously a continuous sesquilinear form on $E$. Conversely, let $\varphi$ be such a form. For each $y \in E$ the map

$$x \mapsto \varphi(x, y)$$

is a functional, and consequently there exists a unique $y^* \in E$ such that for all $x \in E$ we have

$$\varphi(x, y) = \langle x, y^* \rangle.$$

The map $y \mapsto y^*$ is immediately verified to be linear, using the uniqueness of the element $y^*$ representing $\varphi$. Furthermore, from the Schwarz inequality, we find that

$$|y^*| \leqq |\varphi||y|.$$

If we define $A^*: E \to E$ to be the map such that $A^*y = y^*$, then we conclude that $A^*$ is a continuous linear map of $E$ into itself, i.e. an operator.

On the other hand, if we define $\psi(y, x) = \overline{\varphi(x, y)}$, then $\psi$ is sesquilinear continuous, and by what we have just seen, there exists a unique operator $A$ such that $\psi(y, x) = \langle y, Ax \rangle$, or in other words

$$\varphi(x, y) = \langle Ax, y \rangle.$$

Thus $\varphi = \varphi_A$ for some $A$.

**Theorem 2.2.** *The association*

$$A \mapsto \varphi_A$$

*is a norm-preserving isomorphism between* End($E$) *and the space of continuous sesquilinear forms on E.*

*Proof.* All that remains to be proved is that $|A| = |\varphi_A|$. But

$$|\varphi_A(x, y)| \leqq |A||x||y|$$

so that $|\varphi_A| \leqq |A|$. Conversely, we know that $|Ax| = |\lambda_{Ax}|$ and

$$|\lambda_{Ax}(y)| \leqq |\varphi_A||x||y|.$$

Hence $|Ax| \leqq |\varphi_A||x|$. This proves that $|A| \leqq |\varphi_A|$, whence our theorem follows.

We have also shown that to each operator $A$ we can associate a unique operator $A^*$ satisfying the relations

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

for all $x$, $y \in E$. We call $A^*$ the **adjoint** of $A$ (**transpose** of $A$ if our Hilbert space is over the reals).

**Theorem 2.3.** *The map $A \mapsto A^*$ satisfies the following properties:*

$$(A + B)^* = A^* + B^*, \qquad A^{**} = A,$$
$$(\alpha A)^* = \bar{\alpha}A^*, \qquad (AB)^* = B^*A^*,$$

*and for the norm,*

$$|A^*| = |A|, \qquad |A^*A| = |A|^2.$$

*Proof.* The first four properties are immediate from the definitions. For instance,

$$\langle \alpha Ax, y\rangle = \langle Ax, \bar{\alpha}y\rangle = \langle x, A^*\bar{\alpha}y\rangle = \langle x, \bar{\alpha}A^*y\rangle.$$

From the uniqueness we conclude that $(\alpha A)^* = \bar{\alpha}A^*$. The others are equally easy, and are left to the reader. As for the norm properties, we have

$$|\langle A^*x, y\rangle| = |\langle x, Ay\rangle| \leqq |A||x||y|$$

so that

$$|\varphi_{A^*}| = |A^*| \leqq |A|.$$

Since $A^{**} = A$, it follows that $|A| \leqq |A^*|$ so $|A| = |A^*|$. Finally,

$$|A^*A| \leqq |A^*||A| = |A|^2,$$

and conversely,

$$|Ax|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle \leqq |A^*A||x|^2$$

so that $|A| \leqq |A^*A|^{1/2}$. This proves our theorem.

If $\varphi$ is a continuous sesquilinear form on $E$, we define the function

$$q(x) = \varphi(x, x)$$

to be its associated **quadratic form**. In the complex case, we can recover the sesquilinear form from the quadratic form. We phrase this in terms of operators.

**Theorem 2.4.** *For a complex Hilbert space, if $A$ is an operator and $\langle Ax, x\rangle = 0$ for all $x$, then $A = O$.*

*Proof.* This follows from what is called the **polarization identity**,

$$\boxed{\langle A(x + y), x + y\rangle - \langle A(x - y), x - y\rangle = 2[\langle Ax, y\rangle + \langle Ay, x\rangle].}$$

Under the assumption of Theorem 2.4, the left-hand side is equal to 0. Replacing $x$ by $ix$, we get

$$\langle Ax, y\rangle + \langle Ay, x\rangle = 0,$$
$$i\langle Ax, y\rangle - i\langle Ay, x\rangle = 0.$$

From this it follows that $\langle Ax, y\rangle = 0$ and hence that $A = O$.

Theorem 2.4 is of course false in the real case, since a rotation is not necessarily $O$, but may map every vector on a vector perpendicular to it. However, in Chapter XVIII we shall deal with the case when $A = A^*$, in which case the result remains true, obviously.

Operators $A$ such that $A = A^*$ are called **hermitian**, or **self adjoint**. We shall study these especially in Chapter XVIII.

## V, §3. EXERCISES

For the first two exercises, recall that a sequence $\{x_n\}$ in a Hilbert space $H$ **converges weakly** to 0 if for all $y \in H$ we have $\lim\langle x_n, y\rangle = 0$.

1. Let $\{v_n\}$ $(n = 1, 2, \ldots)$ be a denumerable Hilbert basis for the Hilbert space $H$. Show that the sequence $\{v_n\}$ converges weakly to 0, and hence that the unit sphere is not closed in the unit ball for the weak topology.

2. Suppose the Hilbert space $H$ has a countable basis. Let $x \in H$ be such that $|x| \leqq 1$. Show that there exists a sequence $\{u_n\}$ in $H$ with $|u_n| = 1$ for all $n$ such that $\{u_n\}$ converges weakly to $x$.

3. Let $X$ be a closed convex subset of a Hilbert space. Show that there exists a point in $X$ which is at smallest distance from the origin.

4. Let $E$ be a Hilbert space, and let $\{x_n\}$ be an orthonormal basis. Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n^2$ converges. Let $C$ be the subset of $E$ consisting of all sums $\sum a_n x_n$ where $|a_n| \leqq c_n$. Show that $C$ is compact.

5. Show that a Hilbert space is separable (has a countable base for the topology) if and only if it has a countable orthonormal basis.

6. Let $A$ be an operator on a Hilbert space. Show that

$$\operatorname{Ker} A = (\operatorname{Im} A^*)^{\perp}.$$

7. Let $E$ be the vector space of real valued continuous functions on an interval $[a, b]$. Let $K = K(x, y)$ be a continuous function of two variables, defined on the square $a \leqq x \leqq b$ and $a \leqq y \leqq b$. An element $f$ of $E$ is said to be an **eigenfunction** for $K$, with respect to a real number $r$, if

$$f(y) = r \int_a^b K(x, y) f(x)\, dx.$$

We take $E$ with the $L^2$-norm of the hermitian product given by

$$\langle f, g \rangle = \int_a^b fg.$$

Prove that if $f_1, \ldots, f_n$ are in $E$, mutually orthogonal, and of $L^2$-norm equal to 1, and if they are eigenfunctions with respect to the same number $r$, then $n$ is bounded by a number depending only on $K$ and $r$. [*Hint:* Apply Bessel's inequality.]

8. Let $E$ be a pre Hilbert space.
   (a) If $E$ is complex, then $\operatorname{Im}\langle x, y \rangle = \operatorname{Re}\langle x, iy \rangle$.
   (b) Let $x, y \in E$. If $E$ is real, then

$$\langle x, y \rangle = \tfrac{1}{2}(|x + y|^2 - |x|^2 - |y|^2).$$

   If $E$ is complex, then

$$\langle x, y \rangle = \tfrac{1}{2}(|x + y|^2 - |x|^2 - |y|^2) + \tfrac{i}{2}(|x + iy|^2 - |x|^2 - |y|^2).$$

   (c) Let $F$ be a normed vector space such that the parallelogram law holds for its norm. Define $\langle x, y \rangle$ by the formula in (b). Show that this is a positive definite scalar product.

# PART THREE

# Integration

This part deals with integration in multiple contexts. We start with the integral on arbitrary measured spaces, setting the basic framework in a context which makes its structure particularly clear. The main idea is that one starts the theory of the integral by defining the integral on a natural space of simple functions where one sees immediately what the integral means. The space of step functions is the one which covers all cases, from the most general to the most special. As we shall also see, if one wants integration on the reals, or in euclidean space, then the space generated by characteristic functions of intervals or cubes, or the $C^\infty$ functions with compact support, also form a natural starting space for integration.

It turns out that for the basic framework of integration, all one needs for the space of values is linearity and completeness, so a Banach space. I think it obscures matters to assume (as is often done) that values are first taken in the real numbers, and to make abusive use of the ordering properties of the reals and of positivity in setting up the integral. Further comments on this will be made in Chapter VI, especially the introductory comments.

However, doing general Banach valued integration on measured spaces does not mean that one eventually slights special properties of complex valued integration over the real numbers. This entire part will mix general considerations with particular situations and examples, especially on euclidean space and the real line. Readers can see how having the general machinery of integration on measured spaces, or locally compact spaces, is used to make easier the formulation of more concrete results. For instance, in Chapter VIII, we give specific results on approximations on $\mathbf{R}$ or $\mathbf{R}^n$ with Dirac sequences and families. In Chapter IX, two

sections on functions of bounded variations and the Stieltjes integral illustrate the general relationships between measures and functionals on $C^\infty$ functions with compact support. They also emphasize what is peculiar to the real numbers, as distinguished from what holds when the values are taken in an arbitrary Banach space.

Thus, throughout this part, we see general integration theory on measured spaces alternate with special features on euclidean spaces or on the real line.

CHAPTER VI

# The General Integral

In this chapter we develop integration theory. We want two things from an integral which are not provided by the standard Riemann integral of bounded functions:

(1) We want to integrate unbounded functions.
(2) We want to be able to take limits under the integral sign, of a fairly general nature, more general than uniform limits.

To achieve this, we proceed in a manner entirely similar to the manner used when extending the integral to the completion of a space of step functions, except that instead of the sup norm we use the $L^1$-norm. Simple and basic lemmas then allow us to identify elements of the completion with actual functions, and all properties of the integral then become just as easy to prove as in the earlier versions of integration. The lemmas are designed to show that if in addition to $L^1$-convergence we require pointwise convergence almost everywhere, then we still recover essentially the $L^1$-completion, up to functions which vanish almost everywhere.

The treatment here is a conglomerate of various treatments in the literature. Unlike most treatments, however, I have based the existence and definition of the integral on a very simple lemma, which I call the **fundamental lemma of integration** (Lemma 3.1). It can be proved ab ovo with a very short proof, and shows immediately how an $L^1$-Cauchy sequence of functions converges (almost uniformly!). From this convergence, one can immediately see how to extend the integral "by continuity" from step maps to the most general class of mappings which is desired. In the basic lemma, positivity plays no role whatsoever. A posteriori, one notices that the monotone convergence theorem and the

"Fatou lemma" of other treatments become immediate corollaries of the basic approximation lemmas derived from Lemma 3.1. Thus it turns out that it is *easier* to work immediately with complex valued functions than to go through the sequence of many other treatments, via positive functions, real functions, and only then complex functions decomposed into real and imaginary parts. The proofs become shorter, more direct, and to me much more natural. One also observes that with this approach nothing but linearity and completeness in the space of values is used. Thus one obtains at once integration with Banach valued functions. But readers may well omit considering this case if it makes them more comfortable to deal with **C**-valued functions only. Note, however, that vector space valued functions are useful in giving an especially simple proof for the Fubini theorem, which again I find more transparent than the proof used in many treatments, based on positivity. Historically, Bochner was the first to consider integration of Banach valued functions. From the point of view taken here, there is no difference between Banach or complex valued functions.

Actually, it is a reasonable question why one should want to identify elements of the completion with functions: why not just work formally with Cauchy sequences? One of the basic reasons is that certain properties of the formal completion which one wishes to use are obvious if elements of this completion are identifiable with functions. For example, consider the space $L$ of continuous functions on $[0, 1]$. Let $T: L \to L$ be the linear map given by $Tf(x) = xf(x)$. Then $T$ is continuous for the $L^1$-norm on this space, whence $T$ extends uniquely to a continuous linear map $\overline{T}$ on the completion. Now it is clear that $T$ is injective on $L$, and one can ask if $\overline{T}: \overline{L} \to \overline{L}$ is also injective. If we can identify an element of the completion with a function $f$ so that $T$ is again given as multiplication by $x$, then one sees at once that $\overline{T}$ is injective. Otherwise, one has to prove some lemma about $L^1$-Cauchy sequences which amounts to a special case of those proved to establish the representation of elements of the completion by functions, and which serve in a wide variety of context.

I would also like to draw the reader's attention to the approximation Theorem 6.3, which gives a key result in line with our general approach: to prove something in integration theory, first prove it for a subspace of functions for which the result is obvious, then extend by linearity and continuity to the largest possible space.

## VI, §1. MEASURED SPACES, MEASURABLE MAPS, AND POSITIVE MEASURES

Let $X$ be a set (non-empty). By a **σ-algebra** in $X$ we mean a collection of subsets $\mathscr{M}$ having the following properties:

**σ-ALG 1.** *The empty set is in* $\mathscr{M}$.

**σ-ALG 2.** *The collection $\mathscr{M}$ is closed under taking complements (in $X$) and denumerable unions. In other words, if $A \in \mathscr{M}$ then $\mathscr{C}_X A \in \mathscr{M}$, and if $\{A_n\}$ is a sequence of elements of $\mathscr{M}$, then*

$$\bigcup_{n=1}^{\infty} A_n$$

*is also an element of $\mathscr{M}$.*

We conclude at once from these conditions that the whole set $X$ is in $\mathscr{M}$, and that a denumerable intersection of elements of $\mathscr{M}$ is also in $\mathscr{M}$. Also, using empty sets, we see that finite unions or intersections of elements of $\mathscr{M}$ are also in $\mathscr{M}$, and we could just as well have assumed this by saying "countable" instead of "denumerable" in our second axiom.

A set $X$ together with a $\sigma$-algebra $\mathscr{M}$ is called a **measurable space**, and the elements of $\mathscr{M}$ are called its **measurable sets**. We note that if $A$, $B$ are measurable, and if we denote by $A - B$ the set

$$A - B = A \cap \mathscr{C}_X B$$

consisting of all elements of $A$ not in $B$, then $A - B$ is measurable.

To prove that a collection of subsets is a $\sigma$-algebra, we shall often use the following characterization:

*A collection $\mathscr{M}$ of subsets of $X$ is a $\sigma$-algebra if and only if it contains the empty set, is closed under taking complements, finite intersection, and such that, if $\{A_n\}$ is a sequence of disjoint elements of $\mathscr{M}$ then the union $\bigcup A_n$ is in $\mathscr{M}$.*

*Proof.* This is clear since we can write

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 - A_1) \cup \big(A_3 - (A_2 \cup A_1)\big) \cup \cdots.$$

We could also define the notion of an **algebra** of subsets of $X$. It is a collection $\mathscr{A}$ of subsets satisfying the following conditions:

**ALG 1.** *The empty set is in $\mathscr{A}$.*

**ALG 2.** *If $A$, $B \in \mathscr{A}$, then $A \cap B$, $A \cup B$, and $A - B$ are in $\mathscr{A}$.*

Thus we can say that a $\sigma$-algebra is an algebra which is closed under taking countable unions, and containing the set $X$ itself.

**Terminology.** In some texts, what we call an algebra is called a ring (of subsets). However, in the theory of algebraic structures (groups, rings,

fields, vector spaces, etc.) it has become more or less standard practice to assume that a ring has a unit element for multiplication, while an "algebra" is merely an additive group with a bilinear law of composition. Our definitions have therefore been made to fit these conventions, in the analogous situation of algebras of subsets. Here, of course, the "unit element" is the whole space.

*Let $\mathcal{S}$ be a collection of subsets of $X$. Then there exists a smallest $\sigma$-algebra $\mathcal{M}$ in $X$ which contains $\mathcal{S}$.*

*Proof.* We can take for $\mathcal{M}$ the intersection of all $\sigma$-algebras containing $\mathcal{S}$. The collection of all subsets of $X$ is such an algebra, and does contain $\mathcal{S}$, so that we are not faced with the empty set. It is immediate that the intersection $\mathcal{M}$ above is itself a $\sigma$-algebra, so we are done.

In the preceding result, the $\sigma$-algebra $\mathcal{M}$ is said to be **generated** by $\mathcal{S}$.

**Example 1.**
Let $X$ be a topological space, and let $\mathcal{S}$ be the collection of all open sets. The $\sigma$-algebra generated by these open sets is called the algebra of **Borel** sets. An element of this algebra is called **Borel measurable**. In particular, every denumerable intersection of open sets and every denumerable union of closed sets is Borel measurable.

**Example 2.**
*Let $(X, \mathcal{M})$ be a measurable space. Let $f\colon X \to Y$ be a mapping of $X$ into some set $Y$. Let $\mathcal{N}$ be the collection of subsets $S$ of $Y$ such that $f^{-1}(S)$ is measurable in $X$. Then $\mathcal{N}$ is a $\sigma$-algebra.* The proof for this is immediate from basic properties of inverse images of sets. We call $\mathcal{N}$ the **direct image** of $\mathcal{M}$ under $f$, and could denote it by $f_*(\mathcal{M})$. (Cf. Exercise 1.)

**Example 3.**
Let $X$ be a measurable space, and let $Y$ be a subset. If $\mathcal{M}$ is the collection of measurable sets of $X$, we let $\mathcal{M}_Y$ consist of all subsets $A \cap Y$, where $A \in \mathcal{M}$. Then it is clear that $\mathcal{M}_Y$ is a $\sigma$-algebra, which is said to be **induced** by $\mathcal{M}$ on $Y$. Then $(Y, \mathcal{M}_Y)$ is a measurable space.

## Measurable Maps

If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, and $f\colon X \to Y$ is a map, we define $f$ to be **measurable** if for every $B \in \mathcal{N}$ the set $f^{-1}(B)$ is in $\mathcal{M}$. By condition **M2** below, one sees at once that if $Y$ is a topological space, and $\mathcal{N}$ is the $\sigma$-algebra of Borel sets, then $f$ is measurable in this general sense if and only if it satisfies the seemingly weaker condition stated in

**M2**, namely that the inverse image of an open set is measurable. In practice, we deal only with maps into topological spaces, and in fact into normed vector spaces.

**M1.** *If $f: X \to Y$ is measurable, and $g: Y \to Z$ is measurable, then the composite $g \circ f$ is measurable.* This is clear.

**M2.** *Let $f: X \to Y$ be a map into a topological space, with the $\sigma$-algebra of Borel sets. Suppose that for every open $V$ in $Y$, the inverse image $f^{-1}(V)$ is measurable. Then $f$ is measurable.*

*Proof.* Let $\mathcal{N}$ be the collection of subsets $S$ of $Y$ such that $f^{-1}(S)$ is measurable in $X$. Then $\mathcal{N}$ is a $\sigma$-algebra and contains the open sets. Hence it contains all Borel sets in $Y$, thus proving the desired result.

From now on, our maps will have values in a topological space, with the Borel sets as measurable sets.

We note at once that taking complements, we could have defined measurability by the condition that the inverse image of a closed set is measurable. Furthermore, we see that the inverse image of a countable union of closed sets, and the inverse image of a countable intersection of open sets is measurable because if $\{U_n\}$ is a sequence of open sets, then

$$f^{-1}\left(\bigcap_{n=1}^{\infty} U_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(U_n)$$

and similarly for closed sets. *Example:* Let $J$ be a half-open interval $(a, b]$ and let $f: X \to \mathbf{R}$ be measurable. Then

$$f^{-1}((a, b])$$

is measurable because we can write $(a, b]$ as the union of closed intervals

$$\left[a + \frac{1}{n}, b\right] \qquad \text{for} \qquad n = 1, 2, \ldots.$$

We shall now give a large number of criteria for mappings and sets to be measurable, and we shall see that limit operations preserve measurability, and algebraic operations likewise, under extremely mild hypotheses on the image space $Y$. These hypotheses will always be satisfied in practice, and trivially so in the case when we deal with maps into the real or complex numbers, or into Euclidean $n$-space.

**M3.** *Let $f: X \to Y \times Z$ be a map of a measurable space $X$ into a product of topological spaces $Y$, $Z$. Write $f$ in terms of its coordi-*

*nate maps, $f = (g, h)$ where $g: X \to Y$ and $h: X \to Z$. If $f$ is measurable, then so are $g$ and $h$. Conversely, if $g$, $h$ are measurable, and every open set in $Y \times Z$ is a countable union of open sets $V \times W$, where $V$ is open in $Y$ and $W$ is open in $Z$, then $f$ is measurable.*

*Proof.* If $f$ is measurable, then composing $f$ with the projections of $Y \times Z$ on $Y$ or $Z$ shows that both $g$ and $h$ are measurable. Conversely, if $g$, $h$ are measurable, then for any open sets $V$, $W$ in $Y$, $Z$ respectively, we have

$$f^{-1}(V \times W) = g^{-1}(V) \cap h^{-1}(W).$$

Hence $f^{-1}(V \times W)$ is measurable. The measurability of $f^{-1}(U)$ for any open set $U$ now follows from the assumption made on the topology of $Y \times Z$.

**M4.** In particular, we conclude that *a complex function $f$ on $X$ is measurable if and only if its real part and imaginary part are measurable.*

Note that the condition expressed on the product space $Y \times Z$ in our criterion is satisfied if $Y$, $Z$ are metric spaces and have denumerable everywhere dense sets. Thus they are satisfied if $Y$, $Z$ are separable Banach spaces, and in particular for euclidean $n$-space. Actually, in most applications we integrate complex valued functions, so that there is no problem with this extra condition.

**M5.** *If $f$ is a measurable map of $X$ into a normed vector space, then the absolute value $|f|$ is measurable, being composed of $f$ and the continuous function $y \mapsto |y|$.*

We would like the sum of two measurable maps $f$, $g$ into a normed vector space $E$ to be measurable. Since the sum can be viewed as the composite of the map $x \mapsto (f(x), g(x))$ and the sum map $E \times E \to E$, which is continuous, what we want follows from our criterion concerning maps into a product space, provided the extra condition is satisfied. In particular, we obtain the following.

**M6.** *Measurable complex valued functions on $X$ form a vector space, and similarly if the values are in a finite dimensional space, or if we restrict ourselves to maps whose image is separable (i.e. contains a countable dense set). Similarly, if $f$, $g$ are measurable complex functions on $X$, then the product $fg$ is measurable.*

For this last assertion, we note that the product is composed of the map $(f, g)$ and the product $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, which is continuous.

**M7.** *Let $f\colon X \to Y$ be a mapping of $X$ into a metric space. Let $\{f_n\}$ be a sequence of measurable mappings of $X$ into $Y$ which converges pointwise to $f$. Then $f$ is measurable.*

*Proof.* Let $U$ be open in $Y$. If $x \in f^{-1}(U)$, then for all $k$ sufficiently large, we must have $x \in f_k^{-1}(U)$ because $f_k(x)$ converges to $f(x)$. Hence for each $m$,

$$f^{-1}(U) \subset \bigcup_{k=m}^{\infty} f_k^{-1}(U)$$

and consequently

$$f^{-1}(U) \subset \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(U).$$

On the other hand, let $A$ be a closed set. Suppose that $x$ lies in every union

$$\bigcup_{k=m}^{\infty} f_k^{-1}(A)$$

for all positive integers $m$. Then for arbitrarily large $k$, we see that $f_k(x)$ lies in $A$, and hence by assumption the limit $f(x)$ lies in $A$ because $A$ is closed. Hence we obtain the reverse inclusion

$$\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(A) \subset f^{-1}(A).$$

Let $V$ be a fixed open set. For each positive integer $n$ let $A_n$ be the closed set of all $y \in Y$ such that $d(y, \mathscr{C}V) \geqq 1/n$, and let $V_n$ be the open set of all $y \in Y$ such that $d(y, \mathscr{C}V) > 1/n$. Then

$$V_n \subset A_n$$

and

$$V = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} V_n.$$

Thus we have the inclusions

$$\begin{aligned} f^{-1}(V) = \bigcup_n f^{-1}(A_n) &\supset \bigcup_n \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(A_n) \\ &\supset \bigcup_n \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(V_n) \end{aligned}$$

and

$$f^{-1}(V) = \bigcup_n f^{-1}(V_n) \subset \bigcup_n \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(V_n).$$

This proves that the equality holds, and shows that $f^{-1}(V)$ is measurable.

This last result is really the main thing we were after. We need it immediately in the next section to know that if $f$ is a limit of measurable real valued functions, then for every real $a$, the set

$$f^{-1}(J)$$

is measurable when $J$ is equal to the interval of all $t > a$ or the interval of all $t \geqq a$.

In the definition and development of the first properties of the integral in the subsequent sections, the limit property we have just proved, combined with our definition, is the one which will be most useful. It turns out that there is a condition which is necessary and sufficient for a map to be measurable in all applications, but which we preferred to postpone and state as a criterion rather than take as definition. We now discuss this condition. It will be the useful one in dealing with further properties.

A map $f: X \to Z$ into any set $Z$ is said to be a **simple map** if it takes on only a finite number of values, and if, for each $v \in Z$ the inverse image $f^{-1}(v)$ is measurable. Thus $X$ can be written as a finite disjoint union,

$$X = \bigcup_{i=1}^{m} X_i$$

where each $X_i$ is measurable, and $f$ is constant on $X_i$.

It is clear that simple maps of $X$ into a Banach space $E$ form themselves a vector space.

If $\{\varphi_n\}$ is a sequence of simple maps of $X$ into a Banach space $E$, and $\{\varphi_n\}$ converges pointwise, then the limit is measurable, according to the criterion **M7**. The converse is almost true, and is indeed true when $E$ is finite dimensional (so in particular when $E$ represents the real or complex numbers). We have:

**M8.** *A map $f: X \to E$ of $X$ into a finite dimensional space is measurable if and only if it is a pointwise limit of simple maps.*

*Proof.* The result reduces immediately to the case when $E = \mathbf{R}$. We leave the reduction to the reader. Thus assume that $f$ is measurable real valued. For each integer $n \geqq 1$ cut up the interval $[-n, n]$ into intervals of equal length $1/n$ and denote these intervals by $J_1, \ldots, J_N$. We take each $J_k$ to be closed on the left and open on the right. We let $J_{N+1}$ consist of all $t$ such that $|t| \geqq n$. Let

$$A_k = f^{-1}(J_k) \qquad \text{for} \quad k = 1, \ldots, N+1$$

so that each $A_k$ is measurable, the sets $A_k$ $(k = 1, \ldots, N+1)$ are disjoint, and their union is $X$. On each $A_k$ we define a constant map $\psi_n$ by

$$\psi_n(A_k) = \inf_{A_k} f \qquad \text{if} \quad k = 1, \ldots, N.$$

We can write $A_{N+1} = B \cup B'$ where $B$ consists of those elements $x$ such that $f(x) \geqq n$ and $B'$ consists of those $x$ such that $f(x) < -n$. We define

$$\psi_n(B) = n \qquad \text{and} \qquad \psi_n(B') = -n.$$

Then the sequence $\{\psi_n\}$ converges pointwise to $f$, and each $\psi_n$ is a simple function. This proves that measurability implies the other condition. The converse is already known from **M7**, and thus our characterization of measurable maps is proved.

The construction of the case we just discussed yields a useful additional property in the positive case:

**M9.** *Let $f: X \to \mathbf{R}_{\geqq 0}$ be a positive real valued measurable map. Then $f$ is a pointwise limit of an increasing sequence of simple maps.*

*Proof.* The functions $\psi_n$ defined above are all $\leqq f$, and we let

$$\varphi_n = \max(\psi_1, \ldots, \psi_n).$$

Then $\{\varphi_n\}$ is increasing to $f$, as desired.

After discussing positive measures, we shall discuss a variant of condition **M8**, related to a given measure.

## Positive Measures

We shall now define positive measures. To do this, it is convenient to introduce the symbol $\infty$ in the context of positivity (after all, we want some sets to have infinite measure).

We let $\infty$ be a symbol unequal to any real number. By $[0, \infty]$ (which we call also an interval) we mean all $t$ which are real $\geqq 0$ or $\infty$. We introduce the obvious ordering in $[0, \infty]$, with $a < \infty$ for every real $a$. We define addition and multiplication in $[0, \infty]$ by the convention that

$$\infty \cdot a = a \cdot \infty = 0 \qquad \text{if } a = 0,$$

$$\infty \cdot a = a \cdot \infty = \infty \qquad \text{if } 0 < a \leqq \infty,$$

$$\infty + a = a + \infty = \infty \qquad \text{if } 0 \leqq a \leqq \infty.$$

Then associativity, distributivity, and commtativity hold in $[0, \infty]$. The sum of a sequence of elements in $[0, \infty]$ then can be viewed to converge to a number $\geqq 0$ or to $\infty$.

Let $X$ be a measurable space and let $\mathscr{M}$ be the collection of its measurable sets. A **positive measure** on $\mathscr{M}$ (or on $X$, by abuse of language) is a map

$$\mu: \mathscr{M} \to [0, \infty]$$

which is **countably additive**. In other words $\mu(\emptyset) = 0$, and if $\{A_n\}$ is a sequence of measurable sets which are mutually disjoint ($A_n \cap A_m$ is empty if $n \neq m$), then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

If $A$ is measurable, we call $\mu(A)$ its **measure**, or $\mu$-measure if the reference to $\mu$ is necessary to avoid confusion.

**Examples.** Let $X$ be a set and $x_0$ an element of $X$. If $A$ is a subset of $X$ containing $x_0$, we define $\mu(A) = 1$. If $A$ does not contain $x_0$ we define $\mu(A) = 0$. It is immediately verified that this defines a measure, called the **Dirac** measure at $x_0$.

As another example, if a subset is finite, we define its measure to be its number of elements, and if a subset is infinite, we define its measure to be $\infty$. Again it is immediately verified that this defines a measure, called the **counting measure**.

We shall identify measures with integrals later.

A measurable space together with a measure is called a **measured space**. When we want to specify all data in the notation, we write the full triple $(X, \mathscr{M}, \mu)$ for a measured space.

We derive some trivial consequences from the definition of a positive measure.

First we note that the additivity of $\mu$ holds for finite sequences since we can take all but a finite number of the $A_n$ to be empty.

Next, a measure satisfies properties of monotonicity, namely:

*If $A$, $B$ are measurable, $A \subset B$, then $\mu(A) \leqq \mu(B)$.*

This is obvious because we can write $B = A \cup (B - A)$.

**Proposition 1.1.** *If $\{A_n\}$ is a sequence of measurable sets and $A_n \subset A_{n+1}$ for all $n$, and if*

$$A = \bigcup_{n=1}^{\infty} A_n$$

*then*

$$\mu(A) = \lim_{n \to \infty} \mu(A_n).$$

(This is understood in the obvious sense if $\mu(A) = \infty$.) To prove this, we let $A_0$ be the empty set, write

$$A = A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup \cdots \cup (A_{n+1} - A_n) \cup \cdots,$$

and use the countable additivity. We get

$$\mu(A) = \lim_{N\to\infty} \sum_{n=0}^{N} \mu(A_{n+1} - A_n) = \lim_{N\to\infty} \mu(A_N),$$

as was to be shown.

It will occasionally be useful to have the following characterization of measures:

**Proposition 1.2.** *A map $\mu\colon \mathcal{M} \to [0, \infty]$ is a measure if and only if $\mu(\emptyset) = 0$, $\mu$ is finitely additive, and if $\{A_n\}$ is an increasing sequence of measurable sets whose union is $A$, then*

$$\lim \mu(A_n) = \mu(A).$$

Our assertion is obvious, taking into account our preceding arguments.

**Proposition 1.3.** *If $A_n$ is a decreasing sequence of measurable sets, i.e. $A_{n+1} \subset A_n$ for all $n$, if some $A_n$ has finite measure, and if*

$$A = \bigcap_{n=1}^{\infty} A_n,$$

*then*

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

To prove this, say $\mu(A_1) \neq \infty$. We write

$$A_1 = (A_1 - A_n) \cup A_n.$$

The sets $A_1 - A_n$ form an ascending sequence, whose union is $A_1 - A$. By our previous result, we conclude that

$$\begin{aligned}\mu(A_1) &= \lim_{n\to\infty} \mu(A_1 - A_n) + \lim_{n\to\infty} \mu(A_n) \\ &= \mu(A_1 - A) + \lim_{n\to\infty} \mu(A_n) \\ &= \mu(A_1) - \mu(A) + \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

Our assertion follows.

Note that if we do not assume that some $A_n$ has finite measure, then the conclusion may be false. Indeed if all $A_n$ have infinite measure, their intersection may be empty. Think of the real numbers $\geqq n$.

*If $A_n$ is an arbitrary sequence of measurable sets, then in general we have only*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} \mu(A_n).$$

This is again obvious.

Having the notion of (positive) measure on $\mathscr{M}$ we emphasize the role played by sets of measure 0, and we shall use the following terminology. A property of elements of $X$ is said to hold **almost everywhere**, or for **almost all** $x$, if there exists a set $S$ of measure 0 such that the property holds for all $x \notin S$. For instance, if $f: X \to \mathbf{R}$ is a map of $X$ into the reals, we say that $f \geqq 0$ almost everywhere if $f(x) \geqq 0$ for almost all $x$, i.e. for all $x$ outside a set of measure 0. Of course, we should really put the $\mu$ into the notation, and say $\mu$-almost everywhere or $\mu$-almost all, but since we deal with a fixed measure, we omit the prefix $\mu$- for simplicity.

In developing the theory of the integral, we follow the oldest idea, which is first to integrate step maps and then take limits. We shall now discuss the measure theoretic aspect of this procedure.

Let $A$ be a set of finite measure. By a **partition** of $A$ we mean a finite sequence $\{A_i\}$ $(i = 1, \ldots, r)$ of measurable sets which are disjoint and such that

$$A = \bigcup_{i=1}^{r} A_i.$$

Let $E$ be a Banach space. A map $f: X \to E$ is called a **step map with respect to such a partition** if $f$ is equal to 0 outside $A$ (that is $f(x) = 0$ if $x \notin A$), and $f(A_i)$ has one element for each $i$ (i.e. $f$ is constant on $A_i$). A map $f: X \to E$ is said to be a **step map** if it is step with respect to some partition of some set of **finite measure**. We denote the set of all step maps by $\mathrm{St}(\mu, E)$ or more briefly by $\mathrm{St}(\mu)$.

If $Y$ is a measurable subset of $X$, then the restriction to $Y$ of a step map on $X$ is a step map on $Y$. Conversely, a step map on $Y$ can be extended to a step map on $X$ by giving it value 0 outside $Y$. If $f$ is a map on $X$, we denote by $f_Y$ the map such that $f_Y(x) = 0$ if $x \in Y$ and $f_Y(x) = f(x)$ if $x \in Y$.

*The set of step maps $\mathrm{St}(\mu, E)$ is a vector space. If $f$ is a step map, then so is $|f|$. If $f: X \to E$ is a step map and $g: X \to \mathbf{C}$ is a step function, then $gf$ (also written $fg$) is a step map.*

*Proof.* This is proved trivially using a refinement of two partitions. Indeed, if $\{A_i\}$ and $\{B_j\}$ are two partitions of $A$, then

$$\{A_i \cap B_j\}$$

is also a partition. Also, if $f$ is 0 outside $A$, and $g$ is 0 outside $B$, and $A$, $B$ are measurable of finite measure, then $A \cup B$ has finite measure, and we can find a partition of $A \cup B$ with respect to which both $f$ and $g$ are step maps. From this our assertions are obvious.

*We shall not use the rest of this section until the corollaries of the dominated convergence theorem in* §5.

We shall define the integral on certain maps which are limits of step maps. The present discussion is devoted to such limits. We define a map to be **$\mu$-measurable** if it is a pointwise limit of a sequence of step maps almost everywhere. In other words, if there exists a set $Z$ of measure 0 and a sequence of step maps $\{\varphi_n\}$ such that $\{\varphi_n(x)\}$ converges to $f(x)$ for all $x \notin Z$. Let $f: X \to Y$ be $\mu$-measurable, and let $A \subset X$ and $B \subset Y$ be measurable subsets with $f(A) \subset B$. Then the induced map $f: A \to B$ is $\mu$-measurable. Instead of **M1**, we have: if $f: X \to E$ is $\mu$-measurable, and $g: E \to F$ is continuous, then $g \circ f$ is $\mu$-measurable.

**M10.** *The $\mu$-measurable maps of $X$ into $E$ form a vector space. If $f$, $g$ are $\mu$-measurable functions (complex), so is their product. In fact, if $f: X \to E$ and $g: X \to F$ are $\mu$-measurable maps into Banach spaces, and $E \times F \to G$ is a continuous bilinear map, then the product $fg$ (with respect to this map) is $\mu$-measurable. The absolute value $|f|$ is $\mu$-measurable. If $f$ is a $\mu$-measurable function such that $f(x) \neq 0$ for all $x$, then $1/f$ is $\mu$-measurable.*

*Proof.* All statements are clear, except possibly the last, for which we give the argument: If $\{\varphi_n\}$ is a sequence of step functions converging pointwise to $f$, then we let $\psi_n(x) = 1/\varphi_n(x)$ if $\varphi_n(x) \neq 0$ and $\psi_n(x) = 0$ if $\varphi_n(x) = 0$. Then $\psi_n$ is step, and the sequence $\{\psi_n\}$ converges pointwise to $1/f$.

The property of $\mu$-measurability builds in some very strong finiteness properties on both the set of departure and the set of arrival of the map. To begin with, it is clear that a *$\mu$-measurable map vanishes outside a countable union of sets of finite measure.* Such sets are important. We give a name to them, and say that a measurable subset $Y$ of $X$ is **$\sigma$-finite** if it is a countable union of sets of finite measure. More accurately, we should really say that $\mu$ is **$\sigma$-finite** on $Y$, and we should say that $\mu$ is **$\sigma$-finite** if it is $\sigma$-finite on $X$. However, we allow ourselves the other terminology when $\mu$ is fixed throughout a discussion.

*Secondly, there exists a set $Z$ of measure 0 such that the image $f(X - Z)$ of the complement of $Z$ contains a countable dense set* (i.e. is separable). This is clear since outside such $Z$ the map $f$ is a pointwise limit of step maps, and thus the image of $X - Z$ lies in the closure of a

set which is a countable union of finite sets. Thus we now have two necessary conditions for a measurable map to be $\mu$-measurable, namely countability conditions on its domain and range. It turns out that these are sufficient.

**M11.** *Let $f: X \to E$ be a map of $X$ into a Banach space. The following two conditions are equivalent:*

*(i) There exists a set $Z$ of measure 0 such that the restriction of $f$ to the complement of $Z$ is measurable, $f$ vanishes outside a $\sigma$-finite subset of $X$, and the image $f(X - Z)$ contains a countable dense set.*

*(ii) The map $f$ is a pointwise limit almost everywhere of a sequence of step maps (that is, $f$ is $\mu$-measurable).*

*In particular, if $\mu$ is $\sigma$-finite and if $f$ is a function (complex valued), then $f$ is $\mu$-measurable if and only if there exists a subset $Z$ of measure 0 such that $f$ is measurable on the complement of $Z$.*

*Proof.* We have already proved that (ii) implies (i), using our preceding remarks, and **M7**. Conversely, assume (i). We may assume that $X$ is a disjoint union of subsets $X_k$ $(k = 1, 2, \ldots)$ of finite measure. If we can prove that the restriction $f|X_k$ of $f$ to each $X_k$ is $\mu$-measurable, then for each $k$ there is a sequence $\{\varphi_j^{(k)}\}$ $(j = 1, 2, \ldots)$ of step maps on $X_k$ which converges almost everywhere to $f|X_k$. We define $\varphi_n$ by the following values:

$$\varphi_n \text{ is } \varphi_n^{(k)} \text{ on } X_k \qquad \text{for} \quad k = 1, \ldots, n,$$

$$\varphi_n(x) = 0 \qquad \text{if} \quad x \notin X_1 \cup \cdots \cup X_n.$$

Then each $\varphi_n$ is a step map, and the sequence $\{\varphi_n\}$ converges almost everywhere to $f$. This reduces the proof that $f$ is $\mu$-measurable to the case when $X$ has finite measure.

Suppose therefore that $X$ has finite measure. We may also assume that the image of $f$ contains a countable dense set $\{v_k\}$ $(k = 1, 2, \ldots)$. For each positive integer $n$, let $B_{1/n}(v_k)$ be the open ball of radius $1/n$ centered at $v_k$. The union of these balls for all $k = 1, 2, \ldots$ covers the image of $f$, whence the union of the inverse images under $f$ covers $X$ itself. If we take $k$ large, it follows that the finite union of inverse images

$$f^{-1}(B_{1/n}(v_1)) \cup \cdots \cup f^{-1}(B_{1/n}(v_k)) = X - Y_n$$

differs from $X$ by a set $Y_n$ such that $\mu(Y_n) < 1/2^n$. We let

$$Z_n = Y_n \cup Y_{n+1} \cup \cdots$$

so that $\mu(Z_n) \leqq 1/2^{n-1}$. Then $Z_n \supset Z_{n+1} \supset \cdots$ is a decreasing sequence. On $X - Y_n$ we can obviously find a step map $\varphi_n$ such that

$$|f(x) - \varphi_n(x)| < 1/n \qquad \text{for} \quad x \notin Y_n.$$

We simply define the map $\varphi_n$ inductively to have the value $v_1$ on the inverse image of $B_{1/n}(v_1)$, the value $v_2$ on the inverse image of

$$B_{1/n}(v_2) - B_{1/n}(v_1),$$

and so forth. We let $\psi_n$ be equal to $\varphi_n$ on $X - Z_n$ and give $\psi_n$ the value 0 on $Z_n$. Then $\psi_n$ is a step map, and the sequence $\{\psi_n\}$ converges pointwise to $f$, except possibly on the set $Z$ equal to the intersection of all $Z_n$, which has measure 0. This proves what we wanted.

**Remark 1.** The proof is substantially the same as that of **M8**, granting the necessary adjustment to the more general situation.

**Remark 2.** We get some uniformity of convergence from the proof, outside a set of arbitrarily small measure.

**Remark 3.** We took values of $f$ in a Banach space, but for purposes of **M11**, values in any complete metric space would have done just as well. The additive structure plays no role. However, in all subsequent applications, we deal with maps in vector spaces where the additive structure does play a role.

**Remark 4.** Let $\mathscr{M}$ be the $\sigma$-algebra of all subsets of the set $X$. Let $f: X \to E$ be an arbitrary map into a Banach space. Then $f$ is measurable, and $\mu$-measurable if $\mu$ is such that $\mu(Y) = 0$ for all subsets $Y$ of $X$. This shows that it is reasonable to exclude the behavior on a set of measure 0 in our definition of $\mu$-measurability.

**M12.** *Let $\{f_n\}$ be a sequence of $\mu$-measurable maps, converging almost everywhere to a map $f$. Then $f$ is $\mu$-measurable.*

*Proof.* This is clear by using (i) of **M11**, and the following facts: A denumerable union of sets of measure 0 has measure 0. A denumerable union of sets having countable dense subsets has a countable dense subset. [If $\{D_k\}$ is a sequence of denumerable sets in a metric space, then

$$\overline{\bigcup_{k=1}^{\infty} D_k} \supset \overline{D_n} \quad \text{for all } n, \qquad \text{whence} \quad \overline{\bigcup_{k=1}^{\infty} D_k} \supset \bigcup_{n=1}^{\infty} \overline{D_n},$$

so that

$$\overline{\bigcup_{k=1}^{\infty} D_k} = \overline{\bigcup_{n=1}^{\infty} \overline{D_n}},$$

and our second statement is clear also.]

Property **M12** concludes the list of properties which show that $\mu$-measurability is preserved under the standard operations of analysis, with the sole exception of composition of maps, contrary to **M1**.

**For the rest of this chapter, we let $(X, \mathcal{M}, \mu)$ be a measured space, i.e. $\mathcal{M}$ is a $\sigma$-algebra in $X$, and $\mu$ is a positive measure on $\mathcal{M}$. We let $E$ be a Banach space. At first reading, the reader may assume that all maps $f$ are complex or real valued, that is $E = \mathbf{C}$ or $\mathbf{R}$. No proof or notation would be made shorter by this assumption.**

## VI, §2. THE INTEGRAL OF STEP MAPS

If $A$ is a measurable set of finite measure, and $f$ is a step map with respect to a partition $\{A_i\}$ $(i = 1, \ldots, r)$ of $A$, then we define its **integral** to be

$$\int_X f\,d\mu = \sum_{i=1}^{r} \mu(A_i)f(A_i).$$

If $\{B_j\}$ $(j = 1, \ldots, s)$ is another partition of $A$, then $f$ is step with respect to the partition $\{A_i \cap B_j\}$ and we have

$$\sum_{j=1}^{s} \mu(A_i \cap B_j)f(A_i) = \mu(A_i)f(A_i).$$

Summing over $i$ shows that our integral does not depend on the partition of $A$. If $f$ is step with respect to a partition of a set $A$ and a set $B$, then it is also step with respect to a partition of $A \cup B$, and we see that our integral is therefore well defined.

If $A$ is an arbitrary measurable subset and $f$ is a step map on $X$, recall that $f_A$ is the map such that $f_A(x) = f(x)$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$. Then $f_A$ is a step map both on $A$ and on $X$, and we define

$$\int_A f\,d\mu = \int_X f_A\,d\mu.$$

If $\mu$ remains fixed throughout a discussion, we write

$$\int_X f \quad \text{instead of} \quad \int_X f\,d\mu,$$

and even omit the $X$ if the total space $X$ is fixed, so that we also write

$$\int f \quad \text{instead of} \quad \int_X f.$$

If we integrate over a subset of $X$, then we shall always specify this subset, however. We now have trivial properties of the integral.

First, the integral is obviously a linear map

$$\int : \mathrm{St}(\mu, E) \to E$$

which satisfies the following properties.

*If $A$, $B$ are disjoint, then*

(1) $$\int_{A \cup B} f = \int_A f + \int_B f.$$

This is clear from the linearity, and the fact that $f_{A \cup B} = f_A + f_B$.

*Over the reals, the integral is an increasing function of its variables. This means: If $E = \mathbf{R}$ and $f \leqq g$, then*

(2) $$\int f \leqq \int g.$$

*Furthermore, if $f \geqq 0$ and $A \subset B$, then*

(3) $$\int_A f \leqq \int_B f.$$

Property 2 can be obtained from its positive alternate, namely

(2P) *If $f \geqq 0$, then* $\int f \geqq 0$.

Indeed, we just use linearity on $g - f$.

*Finally, the integral satisfies the inequalities*

(4) $$\left| \int_A f \, d\mu \right| \leqq \int_A |f| \, d\mu \leqq \|f\| \mu(A),$$

where $\| \ \|$ is the sup norm. This is an obvious estimate on a finite sum expressing the integral.

We can define a seminorm on the space of step maps, by letting

$$\|f\|_1 = \int_X |f|\,d\mu = \int |f|.$$

That this is a seminorm is immediately verified. For instance, to show that

$$\|f + g\|_1 \leqq \|f\|_1 + \|g\|_1,$$

we take a partition of a set of finite measure such that both $f$ and $g$ are step maps with respect to this partition, and then we estimate using the triangle inequality. This seminorm will be called the **$L^1$-seminorm**.

**Note.** The results of this section are at the level of a first course in calculus. We don't take limits, and our results depend only on the presence of an algebra (not necessarily a $\sigma$-algebra) and a map $\mu$ of this algebra into the reals $\geqq 0$ which is additive, i.e.

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for $A$, $B$ disjoint in the algebra.

## VI, §3. THE $L^1$-COMPLETION

We wish to investigate the completion of our space of step maps with respect to the $L^1$-seminorm. We recall that the completion is defined to be the space of equivalence classes of $L^1$-Cauchy sequences, and that two Cauchy sequences are said to be equivalent if their difference is an $L^1$-null sequence. We denote the completion by $L^1(\mu)$. We recall that the $L^1$-seminorm extends by continuity to a norm on this completion. We have a linear map

$$\mathrm{St}(\mu) \to L^1(\mu)$$

whose kernel is the subspace of step maps whose $L^1$-norm is 0. We shall describe this kernel in a more general situation later.

We want to determine a certain space of functions corresponding as closely as possible to the elements of $L^1(\mu)$. If every $L^1$-Cauchy sequence were also pointwise convergent, there would be no problem. This is however not the case, but the situation is close enough to this so that we can almost think in these terms.

We define $\mathscr{L}^1(\mu)$ to be the set of mappings such that there exists an $L^1$-Cauchy sequence of step mappings converging almost everywhere to

$f$. If $\{f_n\}$ and $\{g_n\}$ are $L^1$-Cauchy sequences of step mappings converging almost everywhere to $f$ and $g$ respectively, then $\{f_n + g_n\}$ and $\{\alpha f_n\}$ (for any number $\alpha$) are $L^1$-Cauchy and converge almost everywhere to $f + g$ and $\alpha f$ respectively. Consequently $\mathscr{L}^1(\mu)$ is a vector space.

In this section and the next, we speak of Cauchy sequences instead of $L^1$-Cauchy sequences since this is the only seminorm which will enter into considerations. Since we have several notions of convergence, however, we still specify by an adjective the type of convergence meant in each case. Actually, it will be useful to say that a sequence $\{f_n\}$ **approximates** and element $f$ of $\mathscr{L}^1$ if $\{f_n\}$ is $L^1$-Cauchy and converges to $f$ almost everywhere.

We shall extend the integral to $\mathscr{L}^1$, and we need two lemmas, which show that our approximation technique is not far removed from uniform approximation. The first is the **fundamental lemma of integration.**

**Lemma 3.1.** *Let $\{f_n\}$ be a Cauchy sequence of step mappings. Then there exists a subsequence which converges pointwise almost everywhere, and satisfies the additional property: given $\varepsilon$ there exists a set $Z$ of measure $< \varepsilon$ such that this subsequence converges absolutely and uniformly outside $Z$.*

*Proof.* For each integer $k$ there exists $N_k$ such that if $m, n \geqq N_k$, then

$$\|f_m - f_n\|_1 < \frac{1}{2^{2k}}.$$

We let our subsequence be $g_k = f_{N_k}$, taking the $N_k$ inductively to be strictly increasing. Then we have for all $m, n$:

$$\|g_m - g_n\|_1 < \frac{1}{2^{2n}}, \qquad \text{if} \quad m \geqq n.$$

We shall show that the series

$$g_1(x) + \sum_{k=1}^{\infty} \big(g_{k+1}(x) - g_k(x)\big)$$

converges absolutely for almost all $x$ to an element of $E$, and in fact we shall prove that this convergence is uniform except on a set of arbitrarily small measure.

Let $Y_n$ be the set of $x \in X$ such that

$$|g_{n+1}(x) - g_n(x)| \geqq \frac{1}{2^n}.$$

Since $g_n$ and $g_{n+1}$ are step mappings, it follows that $Y_n$ has finite measure. On $Y_n$, we have the inequality

$$\frac{1}{2^n} \leqq |g_{n+1} - g_n|$$

whence

$$\frac{1}{2^n}\mu(Y_n) = \int_{Y_n} \frac{1}{2^n} \leqq \int_X |g_{n+1} - g_n| \leqq \frac{1}{2^{2n}}.$$

Hence

$$\mu(Y_n) \leqq \frac{1}{2^n}.$$

Let

$$Z_n = Y_n \cup Y_{n+1} \cup \cdots.$$

Then

$$\mu(Z_n) \leqq \frac{1}{2^{n-1}}.$$

If $x \notin Z_n$, then for $k \geqq n$ we have

$$|g_{k+1}(x) - g_k(x)| < \frac{1}{2^k},$$

and from this we conclude that our series

$$\sum_{k=n}^{\infty} (g_{k+1}(x) - g_k(x))$$

is absolutely and uniformly convergent, for $x \notin Z_n$. This proves the statement concerning the uniform convergence. If we let $Z$ be the intersection of all $Z_n$, then $Z$ has measure 0, and if $x \notin Z$, then $x \notin Z_n$ for some $n$, whence our series converges for this $x$. This proves the lemma.

**Lemma 3.2.** *Let $\{g_n\}$ and $\{h_n\}$ be Cauchy sequences of step mappings of $X$ into $E$, converging almost everywhere to the same map. Then the following limits exist and are equal:*

$$\lim \int_X g_n = \lim \int_X h_n.$$

*Furthermore, the Cauchy sequences $\{g_n\}$ and $\{h_n\}$ are equivalent, i.e. $\{g_n - h_n\}$ is an $L^1$-null sequence.*

*Proof.* The existence of the limit of each integral is of course a triviality. To see the argument once more, we have

$$\left| \int g_n - \int g_m \right| \leqq \int |g_n - g_m| = \|g_n - g_m\|_1,$$

so that $\{\int g_n\}$ is a Cauchy sequence, whence converges. Let $f_n = g_n - h_n$. Then $\{f_n\}$ is Cauchy, converges almost everywhere to 0, and we must prove that the integrals

$$\int_X f_n \quad \text{and} \quad \int_X |f_n|$$

converge to 0.

Given $\varepsilon$, there exists $N$ such that if $m, n \geqq N$ we have

$$\|f_n - f_m\|_1 < \varepsilon.$$

Let $A$ be a set of finite measure outside of which $f_N$ vanishes. Then for all $n \geqq N$ we have

$$\int_{\mathscr{C}A} |f_n| = \int_{\mathscr{C}A} |f_n - f_N| \leqq \int_X |f_n - f_N| < 2\varepsilon.$$

By Lemma 3.1, there exists a subset $Z$ of $A$ such that

$$\mu(Z) < \frac{\varepsilon}{1 + \|f_N\|}$$

and a subsequence of $n$ such that $\{f_n\}$ tends to 0 uniformly on $A - Z$. Then for $n$ large in this subsequence, we conclude that

$$\int_{A-Z} |f_n| < \varepsilon.$$

Finally for $n$ large in this subsequence we have

$$\begin{aligned} \int_Z |f_n| &\leqq \int_Z |f_n - f_N| + \int_Z |f_N| \\ &\leqq \|f_n - f_N\|_1 + \mu(Z)\|f_N\| < 2\varepsilon. \end{aligned}$$

Taking the sum of our integrals over $\mathscr{C}A$, $A - Z$, and $Z$ we find the

desired bound,

$$\|f_n\|_1 = \int_X |f_n| < 5\varepsilon.$$

This proves the lemma.

In view of Lemma 3.2, for every $f$ in $\mathscr{L}^1$ we can define the integral

$$\int_X f\,d\mu = \int_X f = \lim \int_X f_n\,d\mu$$

using any approximating sequence of step maps $\{f_n\}$ to $f$. Elements of $\mathscr{L}^1$ will therefore be called **integrable** maps. It is clear that the integral is a linear map of $\mathscr{L}^1$ into $E$.

We want to extend the seminorm $\|\ \|_1$ to $\mathscr{L}^1$. We need a lemma for this.

**Lemma 3.3.** *If $f$ is integrable and $\{f_n\}$ is an approximating sequence of step maps, then $|f|$ is integrable, and $\{|f_n|\}$ approximates $|f|$. In particular,*

$$\int_X |f| = \lim \int_X |f_n| = \lim \|f_n\|_1.$$

*Proof.* It is clear that $|f_n|$ converges to $|f|$ almost everywhere, so that $|f|$ is integrable. To see that $\{|f_n|\}$ is a Cauchy sequence, we note that

$$\big||f_n| - |f_m|\big| \leqq |f_n - f_m|$$

whence

$$\big\||f_n| - |f_m|\big\|_1 = \int_X \big||f_n| - |f_m|\big| \leqq \int_X |f_n - f_m| = \|f_n - f_m\|_1.$$

This proves the lemma.

Lemma 3.3 implies in particular that

$$\lim \|f_n\|_1$$

is independent of the choice of approximating sequence $\{f_n\}$ to $f$, and thus allows us to define

$$\|f\|_1 = \int_X |f| = \lim \|f_n\|_1.$$

By continuity, this is trivially verified to be a seminorm on $\mathscr{L}^1$.

Let us summarize what we have done. Our purpose was to construct a completion (essentially) of the space of step mappings, under the $L^1$-seminorm. In any case, we have constructed a space $\mathscr{L}^1$ on which we have extended the integral and the seminorm by continuity. We must still show that this space is complete. We could now either relate our $\mathscr{L}^1$ with the space of equivalence classes of Cauchy sequences, and use the result of Chapter 4, §4, that this latter space is complete, or reproduce independently the proof of that result in the present instance. For convenience, we do this.

**Theorem 3.4.** *The space $\mathscr{L}^1$ is complete, under the seminorm $\| \ \|_1$.*

*Proof.* Let $\{f_n\}$ *be a Cauchy sequence in* $\mathscr{L}^1$. For each $n$ there exists an element $g_n \in \mathrm{St}(\mu)$ such that

$$\|f_n - g_n\|_1 < 1/n.$$

The sequence $\{g_n\}$ is then Cauchy. Indeed, we have

$$\|g_n - g_m\|_1 \leqq \|g_n - f_n\|_1 + \|f_n - f_m\|_1 + \|f_m - g_m\|_1,$$

which gives a $3\varepsilon$-proof of the fact that $\{g_n\}$ is a Cauchy sequence. For a subsequence of $n$, we know by Lemma 3.1 that $\{g_n\}$ converges almost everywhere to a function $f$ in $\mathscr{L}^1$. For this subsequence, we then have

$$\|f_n - f\|_1 \leqq \|f_n - g_n\|_1 + \|g_n - f\|_1$$

and this is $< 2\varepsilon$ for $n$ sufficiently large in the subsequence. Hence the subsequence is $L^1$-convergent to $f$. It follows that the sequence $\{f_n\}$ itself is $L^1$-convergent to $f$, and concludes the proof.

Note that the statement of Theorem 3.4 is to be interpreted in the sense that given a Cauchy sequence $\{f_n\}$ of elements in $\mathscr{L}^1$, there exists some $f$ in $\mathscr{L}^1$ such that given $\varepsilon$, we have $\|f_n - f\|_1 < \varepsilon$ for $n$ sufficiently large. We still have the possibility that the seminorm $\| \ \|_1$ is not a norm, so that strictly speaking, "the" completion in the sense of Chapter IV, §4, would be the factor space of $\mathscr{L}^1$ by the subspace of all elements $f$ such that $\|f\|_1 = 0$.

Let us now take for granted the existence of a completion as the space of equivalence classes of Cauchy sequences of step maps, modulo null sequences. Denote this by $L^1(\mu)$. Then we can define a map

$$\gamma\colon \mathscr{L}^1(\mu) \to L^1(\mu)$$

which to each integrable $f \in \mathscr{L}^1$ associates the equivalence class of a

Cauchy sequence $\{f_n\}$ approximating $f$. Lemma 3.2 shows that this map is well defined, and it is obviously linear. The definition of the seminorm on $\mathscr{L}^1$ means that in this notation, we have

$$\|f\|_1 = \|\gamma(f)\|_1.$$

Similarly, the integral, which is a continuous linear map

$$\int_X d\mu\colon \mathrm{St}(\mu, E) \to E$$

for the $L^1$-seminorm of $\mathrm{St}(\mu)$, extends in a natural way to $L^1(\mu)$. What we have shown in Lemma 3.2 is that there is a way of lifting it to $\mathscr{L}^1$ in such a way that for $f \in \mathscr{L}^1$ we have

$$\int_X f = \int_X \gamma(f).$$

The continuity of the integral with respect to our $L^1$-seminorm is implied by the relation

$$\left|\int_X f\right| \leqq \int_X |f| = \|f\|_1.$$

This relation is true for step maps $f$, and consequently holds for the extension of our continuous linear map to the completion. Therefore, it holds also for elements of $\mathscr{L}^1$ by Lemma 3.3 and the definition of the seminorm $\| \; \|_1$ on $\mathscr{L}^1$. The preceding relation also shows that the integral has norm $\leqq 1$, as a linear map.

## VI, §4. PROPERTIES OF THE INTEGRAL: FIRST PART

We note that if $f \in \mathscr{L}^1$ and $g$ differs from $f$ only on a set of measure 0, then $g$ lies in $\mathscr{L}^1$, and the integrals of $f$, $g$ coincide, as well as their $L^1$-seminorms.

We also note that if $f \in \mathscr{L}^1$, we can always redefine $f$ on a set of measure 0, say by giving it constant value on such a set, so that our new map is measurable. Indeed, if $\{\varphi_n\}$ is a sequence of step maps converging to $f$ except on some set $Z$ of measure 0, we let $\psi_n$ be the same map as $\varphi_n$ outside $Z$, and define $\psi_n(x) = 0$, say, for $x \in Z$. Then $\psi_n$ is measurable, and the sequence $\{\psi_n\}$ converges everywhere to a map $g$ which is equal to $f$ except on $Z$. Furthermore, $g$ is measurable, by **M7**.

The properties of the integral which we obtained for step maps now extend to the integral of elements of $\mathscr{L}^1$. We shall go through these

properties systematically once more. We start by repeating that

$$\int_X d\mu \colon \mathscr{L}^1(\mu, E) \to E$$

is linear.

We observe that if $f$, $g$ are in $\mathscr{L}^1(\mu)$ then $|f|$, $|g|$ are in $\mathscr{L}^1(\mu, \mathbf{R})$, and consequently if $E = \mathbf{R}$, then

$$\sup(f, g) = \tfrac{1}{2}(f + g + |f - g|)$$

is in $\mathscr{L}^1$, and so is $\inf(f, g)$ for a similar reason, namely

$$\inf(f, g) = \tfrac{1}{2}(f + g - |f - g|).$$

The expression for the sup also shows that if $\{f_n\}$, $\{g_n\}$ *are sequences in* $\mathscr{L}^1(\mu, \mathbf{R})$ *which are* $L^1$*-convergent to functions* $f$, $g$ *respectively, then* $\sup(f_n, g_n)$ *is* $L^1$*-convergent to* $\sup(f, g)$.

If $f$ is a real function, then we can write

$$f = f^+ - f^-$$

where $f^+ = \sup(f, 0)$ and $f^- = -\inf(f, 0)$. It follows that $f$ is in $\mathscr{L}^1$ if and only if $f^+$ and $f^-$ are in $\mathscr{L}^1$. Such a decomposition is occasionally useful in dealing with real valued maps.

*For any measurable set* $A$ *and any* $f \in \mathscr{L}^1(\mu)$ *the map* $f_A$ *is also in* $\mathscr{L}^1$.

(Recall that $f_A$ is the same as $f$ on $A$, and zero outside $A$.) *Proof:* If $\{\varphi_n\}$ is a sequence of step maps approximating $f$, then $\{\varphi_{nA}\}$ converges almost everywhere to $f_A$, and is Cauchy because

$$\int_X |\varphi_{nA} - \varphi_{mA}| \leqq \int_X |\varphi_n - \varphi_m| = \|\varphi_n - \varphi_m\|_1.$$

Hence $\{\varphi_{nA}\}$ approximates $f_A$. From the linearity of the integral, we thus obtain:

*If* $A$, $B$ *are disjoint measurable sets, then*

$$\int_{A \cup B} f = \int_A f + \int_B f. \tag{1}$$

This follows from the fact that $f_{A \cup B} = f_A + f_B$.

*Over the reals, the integral is an increasing function of its variables. This means: if $E = \mathbf{R}$ and $f \leqq g$, then*

$$\int f \leqq \int g. \tag{2}$$

*Furthermore, if $f \geqq 0$ and $A \subset B$ are measurable, then*

$$\int_A f \leqq \int_B f. \tag{3}$$

Property 2 can be obtained from its positive alternate, namely

(2P) *If $f \geqq 0$, then* $\int f \geqq 0.$

This is clear since an approximating sequence of step functions $\{\varphi_n\}$ can always be taken such that $\varphi_n \geqq 0$, replacing $\varphi_n$ by $\sup(\varphi_n, 0)$ if necessary. Property 2 follows by linearity, and Property 3 is then obvious.

*Finally, the integral on $\mathscr{L}^1(\mu)$ satisfies the inequalities*

$$\left|\int_A f\,d\mu\right| \leqq \int_A |f|\,d\mu \leqq \|f\|\mu(A) \tag{4}$$

where $\|\ \|$ is the sup norm. (We recall that $0 \cdot \infty = 0$.) This is immediate, taking an approximating sequence $\{\varphi_n\}$ of step maps to $f$, using continuity for the first inequality, and (2) for the second. When $\|f\|$ or $\mu(A)$ is infinite, the inequality is clear, and when both are finite, we use (2).

The next properties are general properties, immediate from the continuity of the integral. We make the Banach space explicit here.

**Theorem 4.1.** *Let $\lambda: E \to F$ be a continuous linear map of Banach spaces. Then $\lambda$ induces a continuous linear map*

$$\mathscr{L}^1(\mu, E) \to \mathscr{L}^1(\mu, F)$$

by

$$f \mapsto \lambda \circ f,$$

*and we have*

$$\lambda \int_X f\,d\mu = \int_X \lambda \circ f\,d\mu.$$

This is obvious for step maps, and follows by continuity for $\mathscr{L}^1$.

**Theorem 4.2.** *Let E, F be Banach spaces. Then we have a toplinear isomorphism*

$$\mathscr{L}^1(\mu, E \times F) \to \mathscr{L}^1(\mu, E) \times \mathscr{L}^1(\mu, F).$$

*If $f: X \to E \times F$ is a map, with coordinate maps $f = (g, h)$ in E and F respectively, then $f \in \mathscr{L}^1$ if and only if g, h are in $\mathscr{L}^1$, and then*

$$\int f = \left( \int g, \int h \right).$$

The proof is a simple exercise which we leave to the reader. (The projection is a continuous linear map on each factor!) It applies in particular in $\mathbf{R}^n$, or in $\mathbf{C}$, and we see that a complex map is in $\mathscr{L}^1$ if and only if its real and imaginary parts are in $\mathscr{L}^1$. Actually, this particular case can be seen even more easily, for if we write a complex function

$$f = g + ih$$

where $g$, $h$ are real, we note that a sequence of complex step functions approximates $f$ if and only if its real part approximates $g$ and its imaginary part approximates $h$ (with our definition of approximation, that is $L^1$-Cauchy , and convergence almost everywhere). Thus

$$\int f = \int g + i \int h,$$

whenever $f$ is in $\mathscr{L}^1(\mu, \mathbf{C})$.

All the properties mentioned up to now are essentially routine, and are listed for the sake of completeness. It is natural to make such a list involving properties like linearity, monotonicity, sup, inf, behavior under linear maps, and product mappings, which are the standard finite operations on maps and spaces.

We now turn to the limiting operations, and list the properties of the integral under these operations, giving a large number of criteria for limit mappings to be in $\mathscr{L}^1$.

## VI, §5. PROPERTIES OF THE INTEGRAL: SECOND PART

We first generalize the basic and crucial Lemma 3.1 to arbitrary maps in $\mathscr{L}^1$. This will be formulated as Theorem 5.2. We need a minor lemma to use in the proof, which was automatically satisfied when we dealt with step maps. We define a measurable set to be **$\sigma$-finite** if it is a countable union of sets of finite measure.

**Lemma 5.1.** *Let $f \in \mathscr{L}^1(\mu)$ be measurable. Let $c > 0$. Let $S_c$ be the set of all $x \in X$ such that $|f(x)| \geqq c$. Then $S_c$ has finite measure. Furthermore, $f$ vanishes outside a $\sigma$-finite set.*

*Proof.* Let $\{\varphi_n\}$ be an approximating sequence of step functions to $f$. Taking a subsequence if necessary and using Lemma 3.1, we can assume that there exists a set $Z$ of measure $< \varepsilon$ such that the convergence of $\{\varphi_n\}$ is uniform on the complement of $Z$. Hence for all sufficiently large $n$, we have

$$|\varphi_n(x)| \geqq c/2 \qquad \text{if} \quad x \in S_c - Z.$$

This proves that $S_c$ has finite measure. Taking the values $c = 1/k$ for $k = 1, 2, \ldots$ shows that $f$ vanishes outside a $\sigma$-finite set. Actually we can see this even more easily, since each $\varphi_n$ vanishes outside a set of finite measure, and $f$ is the limit almost everywhere of $\{\varphi_n\}$, whence $f$ vanishes outside a countable union of sets of finite measure.

We see that Lemma 5.1 applies in particular to the characteristic function of a measurable set: if it is in $\mathscr{L}^1$, then the measure of this set is finite.

**Theorem 5.2.** *Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{L}^1$ which is $L^1$-convergent to an element $f$ in $\mathscr{L}^1$. Then there exists a subsequence which converges to $f$ almost everywhere, and also such that given $\varepsilon$, there exists a set $Z$ of measure $< \varepsilon$ such that the convergence is uniform on the complement of $Z$.*

*Proof.* Considering $f_n - f$ instead of $f_n$, we are reduced to proving our theorem in the case $f = 0$. Selecting a subsequence, we may assume without loss of generality that we have

$$\|f_n\|_1 < \frac{1}{2^{2n}}.$$

Also, changing the $f_n$ on a set of measure 0, we can assume that all $f_n$ are measurable. We proceed as in Lemma 3.1. Let $Y_n$ be the set of $x$ such that $|f_n(x)| \geqq 1/2^n$. Then

$$\frac{1}{2^n}\mu(Y_n) \leqq \int_{Y_n} |f_n| \leqq \int_X |f_n| \leqq \frac{1}{2^{2n}},$$

whence

$$\mu(Y_n) \leqq \frac{1}{2^n}.$$

Let $Z_n = Y_n \cup Y_{n+1} \cup \cdots$. Then $\mu(Z_n) \leqq 1/2^{n-1}$. If $x \notin Z_n$, then for $k \geqq n$ we have

$$|f_k(x)| \leqq \frac{1}{2^k},$$

whence $\{f_k\}$ converges uniformly to 0 on the complement of $Z_n$. We let $Z$ be the intersection of all $Z_n$. Then $Z$ has measure 0, and it is clear that $\{f_n\}$ converges pointwise to 0 on $Z$. This proves our theorem.

**Corollary 5.3.** *An element $f \in \mathscr{L}^1$ has seminorm $\|f\|_1 = 0$ if and only if $f$ is equal to 0 almost everywhere.*

*Proof.* Assume that $\|f\|_1 = 0$. Then the sequence $\{0, 0, \ldots\}$ converges in $\mathscr{L}^1$ to $f$, and by Theorem 5.2, it converges pointwise almost everywhere to $f$, so that $f$ is 0 almost everywhere. The converse is obvious.

Corollary 5.3 is a major result in our theory. We define two maps of $X$ into $E$ to be **equivalent** if they differ only on a set of measure 0. We see that the actual completion of the space of step maps under the $L^1$-seminorm is the space of equivalence classes of functions in $\mathscr{L}^1$, under the equivalence defined by the property of being equal almost everywhere. In other words, the kernel of the map

$$\gamma: \mathscr{L}^1(\mu) \to L^1(\mu)$$

is the space of maps $f$ which are 0 almost everywhere.

**Corollary 5.4.** *Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{L}^1$ which converges almost everywhere to a mapping $f$. Then $f$ is in $\mathscr{L}^1$, and is the $L^1$-limit of $\{f_n\}$.*

*Proof.* The sequence $\{f_n\}$ is $L^1$-convergent to some $g \in \mathscr{L}^1$, and by the theorem, some subsequence converges almost everywhere to $g$. Since this subsequence converges almost everywhere also to $f$, it follows that $f = g$ almost everywhere. This proves our corollary.

**Theorem 5.5 (Monotone Convergence Theorem).** *Let $\{f_n\}$ be an increasing (resp. decreasing) sequence of real valued functions in $\mathscr{L}^1$ such that the integrals*

$$\int_X f_n \, d\mu$$

*are bounded. Then $\{f_n\}$ is Cauchy, and is both $L^1$ and almost everywhere convergent to some function $f \in \mathscr{L}^1$.*

*Proof.* Suppose that we deal with the increasing case. Let

$$\alpha = \sup_k \int_X f_k .$$

Then for $n \geqq m$ we have

$$\|f_n - f_m\|_1 = \int (f_n - f_m) = \int f_n - \int f_m \leqq \alpha - \int f_m,$$

whence we see that the sequence of functions is Cauchy. By Theorem 5.2 a subsequence converges almost everywhere, and since the sequence $\{f_n\}$ is increasing, it follows that $\{f_n\}$ itself converges almost everywhere. That convergence is in $L^1$-seminorm by Corollary 5.4. This proves our assertion in the increasing case, and the decreasing case is similar, or follows by considering the sequence $\{-f_n\}$.

**Corollary 5.6.** *If $\{f_n\}$ is a sequence of real valued functions in $\mathscr{L}^1$, and if there exists a real-valued function $g \in \mathscr{L}^1$ such that $g \geqq 0$ and $|f_n| \leqq g$ for all $n$, then $\sup f_n$ and $\inf f_n$ are in $\mathscr{L}^1$, and*

$$\sup \int f_n \leqq \int \sup f_n \qquad \textit{and} \qquad \int \inf f_n \leqq \inf \int f_n .$$

*Proof.* The functions

$$\sup(f_1, \ldots, f_n)$$

are in $\mathscr{L}^1$, and form an increasing sequence bounded by $g$. Hence they converge almost everywhere and we can apply the theorem to conclude the proof for the sup. The inf is dealt with similarly.

For the next corollary, we recall a definition. Let $\{f_n\}$ be a sequence of real valued functions $\geqq 0$. If

$$\lim_{k\to\infty} \inf_{n \geqq k} f_n$$

exists, we call it the **lim inf** of the sequence $\{f_n\}$. It is clear that if $\{f_n\}$ converges pointwise, then its lim inf exists and is equal to the limit. Actually, in the next corollary,

$$\lim_{k\to\infty} \inf_{n \geqq k} f_n(x)$$

will exist for almost all $x$, and the resulting function, which we may

define arbitrarily on a set of measure zero, will be in $\mathscr{L}^1$. By abuse of language, we still denote it by $\liminf f_n$.

**Corollary 5.7 (Fatou's Lemma).** *Let $\{f_n\}$ be a sequence of real valued functions $\geqq 0$ in $\mathscr{L}^1$. Assume that*

$$\liminf \|f_n\|_1$$

*exists (so is a real number $\geqq 0$). Then* $\liminf f_n(x)$ *exists for almost all $x$, the function* $\liminf f_n$ *is in $\mathscr{L}^1$, and we have*

$$\int_X \liminf f_n \, d\mu \leqq \liminf \int_X f_n \, d\mu = \liminf \|f_n\|_1 .$$

*Proof.* We apply the monotone convergence theorem twice, first to the decreasing sequence $\{g_m\}$ given by

$$g_m = \inf(f_k, f_{k+1}, \ldots, f_{k+m}).$$

Since $\{g_m\}$ is a decreasing sequence, converging to $\inf_{n \geqq k} f_n$, and since

$$\int \inf(f_k, f_{k+1}, \ldots, f_{k+m}) \leqq \int f_{k+j} \qquad \text{for} \quad j = 1, \ldots, m$$

we conclude from the monotone convergence theorem that

$$\int \inf_{n \geqq k} f_n \leqq \inf_{n \geqq k} \int f_n \leqq \lim_{k \to \infty} \inf_{n \geqq k} \int f_n .$$

Let $h_k = \inf_{n \geqq k} f_n$. Then $\{h_k\}$ is an increasing sequence for $k = 1, 2, \ldots$, and we can apply the monotone convergence theorem to $h_k$. The limit $\lim_{k \to \infty} h_k$ is precisely $\liminf f_n$, and Fatou's lemma drops out as desired.

**Note.** Fatou's lemma is used most often in the simple case when $\{f_n\}$ is pointwise convergent almost everywhere, and when the $L^1$-seminorms $\|f_n\|_1$ are bounded, thus ensuring that the pointwise $\lim f_n$ is in $\mathscr{L}^1$.

**Theorem 5.8 (Dominated Convergence Theorem).** *Let $\{f_n\}$ be a sequence of mappings in $\mathscr{L}^1(\mu)$. Assume that there exists some function $g \in \mathscr{L}^1(\mu, \mathbf{R})$ such that $g \geqq 0$ and $|f_n| \leqq g$ for all $n$. Assume that $\{f_n\}$ converges almost everywhere to some map $f$. Then $f$ is in $\mathscr{L}^1$ and $\{f_n\}$ is $L^1$-convergent to $f$.*

*Proof.* For each positive integer $k$, let

$$g_k = \sup_{m,n \geqq k} |f_n - f_m|.$$

Then $\{g_k\}$ is a decreasing sequence of real valued functions, and since $|f_n - f_m| \leqq 2g$, it follows from Corollary 5.6 that each $g_k$ is in $\mathscr{L}^1$. By the monotone convergence theorem and the hypothesis, the sequence $\{g_k\}$ converges almost everywhere to 0. Hence $\{f_n\}$ is actually a Cauchy sequence, and we can apply Corollary 5.4 to conclude the proof.

We now refer for the first time since the definition of $\mathscr{L}^1$ to the notion of $\mu$-measurability. The point is that we want to give criteria for the limit of a sequence of maps to be in $\mathscr{L}^1$, and $\mu$-measurability is the natural hypothesis here. We refer the reader to **M11** and emphasize the countability implications arising from a map being in $\mathscr{L}^1$, and hence $\mu$-measurable (by definition).

**Corollary 5.9.** *Let $f$ be $\mu$-measurable. Then $f$ is in $\mathscr{L}^1(\mu)$ if and only if its absolute value $|f|$ is in $\mathscr{L}^1(\mu, \mathbf{R})$. More generally, assume that there exists an element $g \in \mathscr{L}^1(\mu, \mathbf{R})$ such that $g \geqq 0$ and such that $|f| \leqq g$. Then $f$ is in $\mathscr{L}^1(\mu)$.*

*Proof.* Let $\{\varphi_n\}$ be a sequence of step maps converging pointwise to $f$. Without loss of generality we can assume that $g$ is measurable. (We may have to change all $\varphi_n$, $f$, and the given $g$ on a set of measure 0.) Define a map $h_n$ by

$$h_n(x) = \varphi_n(x) \qquad \text{if} \quad |\varphi_n(x)| \leqq 2g(x),$$

$$h_n(x) = 0 \qquad \text{if} \quad |\varphi_n(x)| > 2g(x).$$

The set $S_n$ of all $x$ such that $2g(x) - |\varphi_n(x)| \geqq 0$ is measurable, and it follows that $h_n$ is in $\mathscr{L}^1(\mu)$ for each $n$. Furthermore $\{h_n\}$ converges pointwise to $f$, and $|h_n| \leqq 2g$. We can therefore apply the dominated convergence theorem to conclude the proof.

**Note.** Corollary 5.9 explains the role of positivity in integration theory.

**Corollary 5.10.** *Let $\{f_n\}$ be a sequence of maps in $\mathscr{L}^1(\mu)$ which converges pointwise almost everywhere to $f$. If there exists $C \geqq 0$ such that $\|f_n\|_1 \leqq C$ for all $n$, then $f$ is in $\mathscr{L}^1$ and $\|f\|_1 \leqq C$.*

*Proof.* All $f_n$ are $\mu$-measurable, and hence $f$ is $\mu$-measurable, by **M12** of §1. By Corollary 5.9, it suffices to prove that $|f|$ is in $\mathscr{L}^1(\mu, \mathbf{R})$. But $|f| = \lim |f_n|$, and Fatou's lemma applies to conclude the proof.

**Remark.** In Corollary 5.10, we don't assert of course that $\{f_n\}$ is $L^1$-convergent to $f$. This is in general not true since for instance we can find a sequence $\{f_n\}$ converging everywhere to 0 such that each $f_n$ has $\|f_n\|_1 = 1$. (Take very thin tall vertical strips moving towards the $y$-axis.) To get $L^1$-convergence, we must of course cut down such $f_n$ in a manner similar to that used in Corollary 5.9.

**Corollary 5.11.** *Let $f \in \mathscr{L}^1(\mu)$. Let $g$ be a bounded measurable function on $X$ (so real or complex). Then $gf$ is in $\mathscr{L}^1(\mu)$.*

*Proof.* Let $\{\varphi_n\}$ be a sequence of step maps converging both $L^1$ and almost everywhere to $f$. Using **M8** of §1, let $\{\psi_n\}$ be a sequence of simple functions converging pointwise to $g$. Then $\{\varphi_n\psi_n\}$ is a sequence of step maps, and as $n \to \infty$, this sequence converges almost everywhere to $fg$. Changing $f$ and $g$ on a set of measure 0 (e.g. giving them the value 0), we can assume that this convergence is pointwise everywhere. If $C$ is a bound for $g$, i.e. $|g(x)| \leqq C$ for all $x$, then $|fg| \leqq C|f|$. We can now apply Corollary 5.9 to conclude the proof that $fg$ is in $\mathscr{L}^1$. We can also reproduce the proof of Corollary 5.9, i.e. after suitable adjustment we may suppose that

$$|\varphi_n| \leqq 2|f|$$

for all $n$, whence $|\varphi_n\psi_n| \leqq 2C|f|$ for all $n$, and then apply the dominated convergence theorem directly.

**Corollary 5.12.** *Let $E \times F \to G$ be a continuous bilinear map of Banach spaces into another. Let $f \in \mathscr{L}^1(\mu, E)$ and let $g$ be a bounded $\mu$-measurable map of $X$ into $F$. Then $fg \in \mathscr{L}^1(\mu, G)$.*

*Proof.* There is nothing to change in the preceding proof.

**Corollary 5.13.** *Let $\{f_n\}$ be a sequence of maps in $\mathscr{L}^1$ such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu$$

*converges. Then the series*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

*converges almost everywhere, the map $f$ is in $\mathscr{L}^1$, and*

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

*Proof.* Immediate from the dominated convergence theorem, considering the partial sums, and using the function

$$g(x) = \lim_n \sum_{k=1}^{n} |f_k(x)|.$$

**Example.** It is often useful to consider sums as in Corollary 5.13 in the following context. Let $\{A_n\}$ be a sequence of disjoint measurable sets whose union is equal to $X$. For each $n$ let $f_n$ be integrable over $A_n$, and define $f_n$ to be 0 outside $A_n$, so that $f_n$ is then defined over all of $X$. Let

$$f = \sum f_n.$$

(Conversely, if $f$ is given on all of $X$, we could let $f_n = f_{A_n} = f\chi_{A_n}$ where $\chi_{A_n}$ is the characteristic function of $A_n$.) If

$$\sum_{n=1}^{\infty} \int_{A_n} |f_n|$$

converges, then it follows that $f$ is in $\mathscr{L}^1$ over all of $X$.

**Remark.** In our discussion of measurability, we have already pointed out that a pointwise limit of step maps takes its values in a separable set, i.e. having a countable dense subset. Actually, taking the space generated by the values of the step maps in a sequence converging to $f$ we see that this space, and its closure, have a countable dense subset. This applies when $f$ is in $\mathscr{L}^1$ since we can change $f$ on a set of measure 0, say giving $f$ the value 0 on such a set, so that $f$ is a pointwise limit of step maps on the complement of this set. Furthermore, we also recall that a limit of step maps vanishes outside a countable union of sets of finite measure, and this also applies to an element of $\mathscr{L}^1$.

**Corollary 5.14.** *Let $f$ be in $\mathscr{L}^1$. Given $\varepsilon$, there exists a set of finite measure $A$ such that*

$$\left| \int_X f\, d\mu - \int_A f\, d\mu \right| < \varepsilon.$$

*Proof.* As we have remarked, we can change $f$ on a set of measure 0 such that $f$ vanishes outside a countable union of sets of finite measure, say $\{A_n\}$. Let

$$B_n = A_1 \cup \cdots \cup A_n.$$

The sets $B_n$ are increasing, and without loss of generality we may assume

that $X = \bigcup B_n$. Then

$$\left| \int_X f\, d\mu - \int_{B_n} f\, d\mu \right| = \left| \int_{X-B_n} f\, d\mu \right| \leqq \int_{X-B_n} |f|\, d\mu$$
$$\leqq \int_X |f|(1 - \chi_{B_n})\, d\mu.$$

We let $A = B_n$ for $n$ large. Our corollary follows from the monotone convergence theorem.

The next theorem has a probabilistic interpretation as follows. If $A$ is a set of finite measure $\neq 0$, we may view

$$\frac{1}{\mu(A)} \int_A f\, d\mu$$

as the average of $f$ over $A$. The theorem will assert that if the average of $f$ over all such $A$ lies in some closed set $S$, then in fact the values $f(x)$ must lie in $S$ for almost all $x$. We call this the **averaging theorem**.

**Theorem 5.15.** *Let $f \in \mathscr{L}^1(\mu, E)$. Let $S$ be a closed subset of $E$ and assume that for all measurable sets $A$ of finite measure $\neq 0$ we have*

$$\frac{1}{\mu(A)} \int_A f\, d\mu \in S.$$

*Suppose $0 \in S$ or $X$ is $\sigma$-finite. Then $f(x) \in S$ for almost all $x$.*

*Proof.* Changing $f$ on a set of measure 0, we may assume without loss of generality that $f$ vanishes outside a set which is a countable union of sets of finite measure, and that $E$ has a countable dense subset. It is then clear that it will suffice to prove our theorem under the additional assumption that $\mu(X) < \infty$, which we now make. Let $v \in E$ and $v \notin S$. Let $B_r(v)$ be an open ball of radius $r$ centered at $v$ and not intersecting $S$. Let $A$ be the set of all $x \in X$ such that $f(x) \in B_r(v)$. We prove that $A$ has measure 0. Indeed, if $\mu(A) > 0$ we have

$$\left| \frac{1}{\mu(A)} \int_A f\, d\mu - v \right| = \left| \frac{1}{\mu(A)} \int_A f\, d\mu - \frac{1}{\mu(A)} \int_A v\, d\mu \right|$$
$$\leqq \frac{1}{\mu(A)} \int_A |f - v|\, d\mu < r,$$

which is a contradiction. Hence $\mu(A) = 0$. The lemma follows using the

countability assumption on $E$, and using a countable dense set in the complement of $S$, together with open balls of rational radii around the elements of this set, which form a base for the topology.

**Corollary 5.16.** *Let $f \in \mathscr{L}^1(\mu)$ and assume that*

$$\int_A f \, d\mu = 0$$

*for every measurable set $A$ of finite measure. Then $f$ is equal to 0 almost everywhere.*

*Proof.* We take $S$ to consist of 0 alone, and apply the theorem.

**Corollary 5.17.** *Let $f \in \mathscr{L}^1(\mu)$. For each step function $g$ the map $fg$ is in $\mathscr{L}^1(\mu)$, and if*

$$\int_X fg \, d\mu = 0$$

*for all step functions $g$, then $f(x) = 0$ for almost all $x$.*

*Proof.* Apply Corollary 5.16 to characteristic functions $\chi_A$.

**Corollary 5.18.** *Let $f \in \mathscr{L}^1(\mu)$. Let $b \geqq 0$. If*

$$\left| \int_A f \, d\mu \right| \leqq b\mu(A)$$

*for all sets $A$ of finite measure, then $|f(x)| \leqq b$ for almost all $x$.*

*Proof.* Let $S_n$ be the subset of $E$ consisting of those elements $v$ such that $|v| \geqq b + 1/n$ and apply the theorem. Then take the union for $n = 1, 2, \ldots$.

The next corollary is included for later applications. The reader interested only in the case of complex or real functions may omit it.

**Corollary 5.19.** *Let $E$ be a Hilbert space and $f \in \mathscr{L}^1(\mu, E)$. If*

$$\int_X \langle f, g \rangle \, d\mu = 0$$

*for all step maps $g$, then $f(x) = 0$ for almost all $x$.*

*Proof.* The proof is really just like that of Corollary 5.17. First we may assume that the image of $f$ is contained in a separable Hilbert subspace. Let $e$ be a unit vector. For any measurable set $A$ of finite measure, the step map $e\chi_A$ having value $e$ in $A$ and 0 outside $A$ is bounded measurable. Let us denote by $f_e$ the Fourier coefficient of $f$ along $e$ so that $f_e$ is a function. We have

$$0 = \int_X \langle f, e\chi_A \rangle \, d\mu = \int_A f_e \, d\mu.$$

This being true for all $A$ it follows that $f_e$ is equal to 0 almost everywhere. Since there is a countable Hilbert basis in our Hilbert space, it follows that $f$ is 0 almost everywhere.

**Corollary 5.20.** *Let $E$ be a Hilbert space and $f \in \mathscr{L}^1(\mu, E)$. For each unit vector $e \in E$, let $f_e$ be the component of $f$ along $e$. Let $b \geqq 0$. Assume that for each unit vector $e$ and each set of finite measure $A$ we have*

$$\left| \int_A f_e \, d\mu \right| \leqq b\mu(A).$$

*Then $|f(x)| \leqq b$ for almost all $x$.*

*Proof.* We may assume that $E$ is separable as in Theorem 5.15, and that $\mu(X) < \infty$. Let $v \in E$ and $|v| > b$. Let $B_r(v)$ be an open ball of radius $r$ centered at $v$ not intersecting $\bar{B}_b(0)$. If $A$ is the set of all $x \in X$ such that $f(x) \in B_r(v)$, we take $e$ to be a unit vector in the direction of $v$. Let $c = |v|$. If $x \in A$, then $|f_e(x) - c| < r$ so that $f_e(x) \in B_r(c)$. By Corollary 5.18 it follows that $A$ has measure 0. Our Corollary 5.20 follows at once.

## VI, §6. APPROXIMATIONS

We shall analyze Theorem 5.2 more closely, so as to fit certain situations which arise in practice. Let us look at a special case, the real line. The most natural definition of any integral on $\mathbf{R}$ is to start with step functions defined on bounded intervals (open, closed, or half open or closed), and define the integral for these. However, the sets which are finite unions of bounded intervals do not form a $\sigma$-algebra, only an algebra. Thus we are faced with two problems: extend the measure (length) function on bounded intervals to a measure on the $\sigma$-algebra generated by the finite intervals, and second, show that the step functions taken with respect to finite intervals are still $L^1$-dense in the $\mathscr{L}^1$-completion. The

problem of extending the measure to a $\sigma$-algebra is dealt with in §7. Here, we settle the other question, and a countability condition arises naturally.

Let $\mathscr{A}$ be a subalgebra of $\mathscr{M}$, and assume that $\mathscr{A}$ consists of sets of finite measure. We shall say that $X$ is **$\sigma$-finite with respect to** $\mathscr{A}$ if $X$ is a countable union of elements of $\mathscr{A}$. Taking the usual inductive complementation, we see that if $X$ is $\sigma$-finite with respect to $\mathscr{A}$, then in fact, there is a sequence $\{A_n\}$ of disjoint elements of $\mathscr{A}$ such that

$$X = \bigcup_{n=1}^{\infty} A_n.$$

We recall that a **step map** $f$ with respect to $\mathscr{A}$ is a map which is equal to 0 outside some element $A$ of $\mathscr{A}$, and such that there is a partition $\{A_1, \ldots, A_r\}$ of $A$ consisting of elements of $\mathscr{A}$, such that $f$ is step with respect to this partition. We shall denote the space of step maps with respect to $\mathscr{A}$ by $\mathrm{St}(\mathscr{A})$. We are interested in giving conditions under which the closure of $\mathrm{St}(\mathscr{A})$ in $\mathscr{L}^1(\mu)$ is equal to $\mathscr{L}^1$. The next two lemmas lead to the theorem giving such criterion. We first consider those measurable subsets contained in some element $A$ of $\mathscr{A}$. Thus we denote by $\mathscr{A}_A$ the algebra induced by $\mathscr{A}$ on $A$, i.e. the algebra of all elements of $\mathscr{A}$ contained in $A$. We let $\mathrm{St}(\mathscr{A}_A)$ be the vector space of step maps with respect to $\mathscr{A}_A$.

**Remark.** Let $Y$ be a measurable subset of an element $A$ of $\mathscr{A}$ and $\chi_Y$ its characteristic function. Let $\varphi$ be a step function such that

$$\|\chi_Y - \varphi\|_1 < \varepsilon.$$

If we let $\varphi_1 = \inf(\varphi, 1)$, then $|\chi_Y - \varphi_1| \leqq |\chi_Y - \varphi|$, and hence

$$\|\chi_Y - \varphi_1\|_1 < \varepsilon.$$

We have a similar situation taking $\sup(\varphi, 0)$. We are interested in those $Y$ such that $\chi_Y$ lies in the closure of $\mathrm{St}(\mathscr{A}_A, \mathbf{R})$. Our remark shows that in determining those $Y$, we may restrict our attention to those step functions $\varphi$ such that

$$0 \leqq \varphi \leqq 1.$$

For what follows, we also observe that $\mathrm{St}(\mathscr{A}_A, \mathbf{R})$ is closed under the operations of sup and inf.

**Lemma 6.1.** *Let $A$ be an element of $\mathscr{A}$. Let $\mathscr{N}_A$ be the collection of measurable subsets $Y$ of $A$ whose characteristic function $\chi_Y$ lies in the $L^1$-closure of* $\mathrm{St}(\mathscr{A}_A, \mathbf{R})$, *i.e. such that given $\varepsilon$, there exists a step*

*function $\varphi \in \mathrm{St}(\mathscr{A}_A, \mathbf{R})$ satisfying*

$$(*) \qquad \|\chi_Y - \varphi\|_1 < \varepsilon.$$

*Then $\mathscr{N}_A$ is a $\sigma$-algebra in $A$.*

*Proof.* First we show that $\mathscr{N}_A$ is an algebra. If $Y, Z \in \mathscr{N}_A$, then

$$\sup(\chi_Y, \chi_Z) = \chi_{Y \cup Z} \qquad \text{and} \qquad \inf(\chi_Y, \chi_Z) = \chi_{Y \cap Z}$$

are in $\mathscr{N}_A$. Also, $\chi_A - \chi_Y = \chi_{A-Y}$ is in $\mathscr{N}_A$. Hence $\mathscr{N}_A$ is an algebra. To show that it is a $\sigma$-algebra, it suffices to show that if $\{Y_n\}$ is a sequence in $\mathscr{N}_A$ of **disjoint** elements, then $\bigcup Y_n$ is $\mathscr{N}_A$. (If we have an arbitrary sequence in $\mathscr{N}_A$, we can always adjust it by taking relative complementations to yield another sequence of disjoint elements in $\mathscr{N}_A$, having the same union.) Thus let $\{Y_n\}$ be a disjoint sequence in $\mathscr{N}_A$, and let $\{\varphi_n\}$ be step functions in $\mathrm{St}(\mathscr{A}_A, \mathbf{R})$ such that

$$\|\chi_{Y_n} - \varphi_n\|_1 < \frac{\varepsilon}{2^n}.$$

Let

$$Y = \bigcup_{n=1}^{\infty} Y_n.$$

Then

$$\left\| \chi_Y - \sum_{k=1}^{n} \varphi_k \right\|_1 \leqq \|\chi_Y - \chi_{Y_1 \cup \cdots \cup Y_n}\|_1 + \left\| \chi_{Y_1 \cup \cdots \cup Y_n} - \sum_{k=1}^{n} \varphi_k \right\|_1.$$

We take $n$ so large that the first term on the right is $< \varepsilon$. The second term on the right is estimated by

$$\sum_{k=1}^{n} \|\chi_{Y_k} - \varphi_k\|_1 < \varepsilon.$$

This proves that $\mathscr{N}_A$ is a $\sigma$-algebra in $A$.

The next lemma pertains to a completely general situation.

**Lemma 6.2.** *Let $\{A_i\}_{i \in I}$ be a family of sets whose union is equal to $X$. For each $i$, let $\mathscr{N}_i$ be a $\sigma$-algebra of subsets of $A_i$. Let $\mathscr{N}$ be the collection of subsets $Y$ of $X$ such that $Y \cap A_i \in \mathscr{N}_i$ for all $i$. Then $\mathscr{N}$ is a $\sigma$-algebra in $X$.*

*Proof.* Let $Y \in \mathscr{N}$. Then $\mathscr{C}Y \cap A_i = A_i - Y$. Hence $\mathscr{C}Y \in \mathscr{N}$. Let $Y$, $Z \in \mathscr{N}$. Then

$$(Y \cap Z) \cap A_i = (Y \cap A_i) \cap (Z \cap A_i)$$

whence $Y \cap Z$ is in $\mathcal{N}$. Let $\{Y_k\}$ be a sequence of subsets of $X$ in $\mathcal{N}$. Then

$$\left(\bigcup_{k=1}^{\infty} Y_k\right) \cap A_i = \bigcup_{k=1}^{\infty} (Y_k \cap A_i)$$

whence $\bigcup Y_k$ is in $\mathcal{N}$. This proves our lemma.

**Theorem 6.3.** *Let $\mathscr{A}$ be a subalgebra of $\mathcal{M}$, consisting of sets of finite measure, generating $\mathcal{M}$. Assume that $X$ is $\sigma$-finite with respect to $\mathscr{A}$. Then the space* $\mathrm{St}(\mathscr{A})$ *of step mappings with respect to $\mathscr{A}$ is dense in $\mathscr{L}^1(\mu, E)$. Furthermore, if $\{A_n\}$ is a sequence in $\mathscr{A}$ whose union is $X$, then for all $Y \in \mathcal{M}$, $\chi_{Y \cap A_n}$ lies in the $L^1$-closure of* $\mathrm{St}(\mathscr{A}_{A_n}, \mathbf{R})$, *for all $n$.*

*Proof.* We prove the second assertion first. By Lemma 6.1, we have a $\sigma$-algebra $\mathcal{N}_{A_n}$, and we apply Lemma 6.2. Every element of $\mathscr{A}$ is such that $A \cap A_n \in \mathcal{N}_{A_n}$, and since $\mathscr{A}$ generates $\mathcal{M}$, we conclude that $\mathcal{N} = \mathcal{M}$. Next, we prove a special case of our first statement:

*If $Y$ is a measurable set of finite measure, given $\varepsilon$ there exists a step function $\varphi$ with respect to $\mathscr{A}$ such that*

$$\|\chi_Y - \varphi\|_1 < \varepsilon.$$

Taking relative complements, we may assume the $A_n$ are disjoint. By Lemmas 6.1 and 6.2, for each $n$ there exists a step function $\varphi_n$ with respect to $\mathscr{A}$ such that

$$\|\chi_{Y \cap A_n} - \varphi_n\|_1 < \frac{\varepsilon}{2^n}.$$

Since $Y$ is the union of all sets $Y \cap A_n$, we can find some $n$ such that

$$\mu\left(Y - \bigcup_{k=1}^{n} (Y \cap A_k)\right) < \varepsilon,$$

or in other words such that

$$\left\|\chi_Y - \sum_{k=1}^{n} \chi_{Y \cap A_k}\right\|_1 < \varepsilon.$$

It follows that

$$\left\|\chi_Y - \sum_{k=1}^{n} \varphi_k\right\|_1 \leqq \left\|\chi_Y - \sum_{k=1}^{n} \chi_{Y \cap A_k}\right\|_1 + \left\|\sum_{k=1}^{n} \chi_{Y \cap A_k} - \sum_{k=1}^{n} \varphi_k\right\|_1 .$$
$$< 2\varepsilon.$$

This proves our special case.

The general case is now obvious: a step map $f \neq 0$ with respect to all sets of finite measure is a finite linear combination

$$f = \sum_{j=1}^{m} v_j \chi_{Y_j}$$

with $v_j \in E$, $v_j \neq 0$ for all $j$, and such that the sets $Y_j$ have finite measure. By definition, the space of these maps is $L^1$-dense in $\mathscr{L}^1(\mu, E)$. For each $\chi_{Y_j}$ we can find a step function $\varphi_j$ with respect to $\mathscr{A}$ such that

$$\|\chi_{Y_j} - \varphi_j\|_1 < \frac{\varepsilon}{m|v_j|}.$$

It follows immediately that

$$\left\| f - \sum_{j=1}^{m} v_j \varphi_j \right\|_1 < \varepsilon.$$

This proves our theorem.

We can now strengthen the corollaries of Theorem 5.15.

**Corollary 6.4.** *Let $\mathscr{A}$ be a subalgebra of $\mathscr{M}$, consisting of sets of finite measure, generating $\mathscr{M}$. Assume that $X$ is $\sigma$-finite with respect to $\mathscr{A}$. Let $f \in \mathscr{L}^1(\mu)$. If*

$$\int_A f \, d\mu = 0$$

*for all $A \in \mathscr{A}$, then $f$ is equal to $0$ almost everywhere.*

*Proof.* Our assumption implies by linearity that

$$\int_X f\varphi \, d\mu = 0$$

for all real step functions $\varphi$ with respect to $\mathscr{A}$. Let $Y$ be a set of finite measure. By Theorem 6.3 and Lemma 3.1, we can find a sequence of step functions $\{\varphi_n\}$ with respect to $\mathscr{A}$ which converges almost everywhere to $\chi_Y$ and is also $L^1$-convergent to $\chi_Y$. Taking $\inf(\varphi_n, 1)$ and $\sup(\varphi_n, 0)$ if necessary, we may assume without loss of generality that $0 \leqq \varphi_n \leqq 1$. Then

$$|f\varphi_n| \leqq |f|,$$

and $\{f\varphi_n\}$ converges almost everywhere to $f\chi_Y$. By the dominated con-

vergence theorem it follows that

$$0 = \int_X f\varphi_n \quad \text{converges to} \quad \int_X f\chi_Y = \int_Y f.$$

This proves that

$$\int_Y f = 0$$

for all sets of finite measure $Y$. By the $\sigma$-finiteness, every measurable set is a countable union of sets of finite measure. Since $f\chi_Y = 0$ almost everywhere by Theorem 5.15, we conclude that $f = 0$ almost everywhere, thus proving our corollary.

**Example.** Take $E = \mathbf{R}$ and let $X = \mathbf{R}$ also. Let $\mathscr{A}$ be the algebra consisting of sets which are finite unions of bounded intervals (obviously an algebra). We shall show in §9 that there is a unique measure on the $\sigma$-algebra generated by $\mathscr{A}$ such that the measure of an interval is its length. Thus we can develop integration theory on the reals, and we can apply the corollary to Theorem 6.3. Furthermore, the infinitely differentiable functions which vanish outside a compact set are dense in $\mathscr{L}^1$. In fact, given a characteristic function $\chi_Y$ of a finite interval, we can find a $C^\infty$ function $\varphi$ which is equal to $\chi_Y$ except in a given $\varepsilon$-neighborhood of the two end points of the interval, and $0 \leqq \varphi \leqq 1$. Thus as an application of our corollary, we see that if

$$\int_{\mathbf{R}} f\varphi = 0$$

for all $C^\infty$ functions $\varphi$ vanishing outside some compact set, then $f$ is equal to 0 almost everywhere. We shall state this result formally later in $\mathbf{R}^n$.

**Remark.** We observe that the domain of validity of Theorem 6.3 is actually greater than it seems, i.e. the hypothesis of $\sigma$-finiteness is to some extent superfluous. Indeed, every map in $\mathscr{L}^1(\mu)$ being a limit almost everywhere of step maps, must vanish outside some set which is a denumerable union of sets of finite measure. In determining a dense subset of $\mathscr{L}^1$, we are merely attempting to approximate each individual map $f$. Thus the hypothesis under which Theorem 6.3 holds can actually be weakened to the following:

*Every set of finite measure is contained in a countable union of sets of $\mathscr{A}$.*

All the applications I know of actually occur in the $\sigma$-finite case as we defined $\sigma$-finite, but one should keep in mind that in case of need, one

could take the preceding property as the definition of $\sigma$-finiteness with respect to $\mathscr{A}$, and still end up with the corresponding result. This remark is the analogue with respect to the domain set of the remark preceding Corollary 5.14, with respect to the image space. We see that the $\mathscr{L}^1$ theory has a built-in countability property for each one of its elements.

## VI, §7. EXTENSION OF POSITIVE MEASURES FROM ALGEBRAS TO σ-ALGEBRAS

In the previous sections, we started with a positive measure on a $\sigma$-algebra, and then defined the integral for certain limits of step maps. We now want to show how we can obtain such measures starting with fewer data.

We recall that an algebra $\mathscr{A}$ of subsets of $X$ is a collection of subsets containing the empty set, such that $\mathscr{A}$ is closed under finite unions and intersections, and such that if $A, B \in \mathscr{A}$, then $A - B \in \mathscr{A}$.

By a **positive measure** on an algebra $\mathscr{A}$, we mean a map

$$\mu: \mathscr{A} \to [0, \infty]$$

such that $\mu(\emptyset) = 0$, and such that $\mu$ is countably additive on $\mathscr{A}$. This means that if $\{A_n\}$ is a sequence of disjoint elements of $\mathscr{A}$, and if their union $\bigcup A_n$ is also in $\mathscr{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Under a suitable countability assumption, we shall prove that a measure on an algebra can be extended uniquely to a measure on the $\sigma$-algebra generated by $\mathscr{A}$. Observe that the countability condition is necessary for this to be possible, i.e. we could not merely assume that $\mu$ is finitely additive on $\mathscr{A}$. For instance, consider a denumerable set $X = \{x_n\}$, and let $\mathscr{A}$ be the algebra of all subsets. Let $x_n$ have measure $1/2^n$, and let a finite set have measure equal to the sum of the measures of its elements. Let an infinite set have infinite measure. Then we have defined a finitely additive function which is not a measure.

**Theorem 7.1 (Hahn).** *Let $\mu$ be a positive measure on an algebra $\mathscr{A}$ in $X$, and assume that $X$ can be expressed as a denumerable union of sets of $\mathscr{A}$. Then $\mu$ can be extended to a positive measure on the $\sigma$-algebra $\mathscr{M}$ generated by $\mathscr{A}$, so that for $Y \in \mathscr{M}$,*

$$\mu(Y) = \inf \sum_{n=1}^{\infty} \mu(A_n),$$

*the* inf *being taken over all sequences* $\{A_n\}$ *in* $\mathscr{A}$ *whose union contains* $Y$. *If* $X$ *can be expressed as a countable union of sets of finite measure in* $\mathscr{A}$, *then there exists a unique extension of* $\mu$ *to a positive measure on* $\mathscr{M}$.

*Proof.* The proof will proceed in two steps and needs the notion of an outer measure.

Let $\mathscr{N}$ be a $\sigma$-algebra in a set $X$. An **outer measure** $\mu$ **on** $\mathscr{N}$ is a function $\mu: \mathscr{N} \to [0, \infty]$ satisfying the conditions:

**OM 1.** *We have* $\mu(\varnothing) = 0$.

**OM 2.** *If* $A, B \in \mathscr{N}$ *and* $A \subset B$, *then* $\mu(A) \leqq \mu(B)$.

**OM 3.** *If* $\{A_n\}$ *is a sequence of elements of* $\mathscr{N}$, *then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} \mu(A_n).$$

**Lemma 7.2.** *Let* $\mu$ *be a positive measure on an algebra* $\mathscr{A}$ *in* $X$, *and assume that* $X$ *can be expressed as a denumerable union of sets of* $\mathscr{A}$. *On the* $\sigma$-*algebra of all subsets of* $X$, *define*

$$\mu^*(Y) = \inf \sum_{n=1}^{\infty} \mu(A_n),$$

*the* inf *being taken over all sequences* $\{A_n\}$ *of elements of* $\mathscr{A}$ *whose union contains* $Y$. *Then* $\mu^*$ *is an outer measure which extends* $\mu$.

*Proof.* We first show that if $A \in \mathscr{A}$, then $\mu^*(A) = \mu(A)$, in other words $\mu^*$ extends $\mu$. Since

$$A = A \cup \varnothing \cup \varnothing \cup \cdots$$

we see that $\mu^*(A) \leqq \mu(A)$. Conversely, given $\varepsilon$, let $\{A_n\}$ be a sequence of elements of $\mathscr{A}$ whose union covers $A$, and such that

$$\sum \mu(A_n) \leqq \mu^*(A) + \varepsilon.$$

Since $A = \bigcup (A_n \cap A)$ it follows that

$$\mu(A) \leqq \sum \mu(A_n \cap A) \leqq \sum \mu(A_n) \leqq \mu^*(A) + \varepsilon.$$

This proves that $\mu(A) \leqq \mu^*(A)$, whence $\mu(A) = \mu^*(A)$, as desired.

From now on we omit the * on $\mu$ since $\mu^*$ and $\mu$ take the same values on $\mathscr{A}$. We show that our extended $\mu$ is an outer measure. The first two

properties **OM 1** and **OM 2** are obvious. As for **OM 3**, it is clearly an $\varepsilon/2^n$ proof: let $\{Y_j\}$ be a sequence of subsets of $X$ which we may assume have finite measure. Given $\varepsilon$, for each $j$, let $\{A_n^{(j)}\}$ $(n = 1, 2, \ldots)$ be a sequence in $\mathscr{A}$ whose union covers $Y_j$ and such that

$$\sum_{n=1}^{\infty} \mu(A_n^{(j)}) \leqq \mu(Y_j) + \frac{\varepsilon}{2^j}.$$

Then the denumerable family $\{A_n^{(j)}\}$ (for $j$, $n$ positive integers) covers $\bigcup Y_j$, and we have

$$\mu(Y) \leqq \sum_{n,j} \mu(A_n^{(j)}) \leqq \sum_{j=1}^{\infty} \mu(Y_j) + \varepsilon.$$

This proves our proposition.

Let $\mu$ be an outer measure on the set of all subsets of $X$. We say that a subset $A$ of $X$ is **$\mu$-measurable** if for all subsets $Z$ of $X$ we have

$$\mu(Z) = \mu(Z \cap A) + \mu(Z \cap \mathscr{C}A).$$

**Lemma 7.3.** *Let $\mu$ be an outer measure on the subsets of $X$. Let $\mathscr{S}$ be the collection of all subsets of $X$ which are $\mu$-measurable. Then $\mathscr{S}$ is a $\sigma$-algebra, and $\mu$ is a positive measure on $\mathscr{S}$.*

*Proof.* Since we deal only with $\mu$, we omit the prefix $\mu$-. We first prove that $\mathscr{S}$ is an algebra. It obviously contains the empty set, and if $A$ is measurable, it is clear that $\mathscr{C}A$ is measurable (the definition of measurable is symmetric in $A$ and $\mathscr{C}A$). Let $A$, $B$ be measurable. We show that $A \cap B$ is measurable. Let $Z$ be any subset of $X$. Since $B$ is measurable, we get

$$\mu(Z \cap A \cap B) + \mu(Z \cap A \cap \mathscr{C}B) = \mu(Z \cap A).$$

Add $\mu(Z \cap \mathscr{C}A)$ to both sides. On the right we obtain $\mu(Z)$ because $A$ is measurable. To prove that $A \cap B$ is measurable, it will suffice to prove that

$$\mu(Z \cap \mathscr{C}(A \cap B)) = \mu(Z \cap A \cap \mathscr{C}B) + \mu(Z \cap \mathscr{C}A).$$

But this is seen by using the fact that $A$ is measurable, and writing

$$\mu(Z \cap \mathscr{C}(A \cap B)) = \mu(Z \cap \mathscr{C}(A \cap B) \cap A) + \mu(Z \cap \mathscr{C}(A \cap B) \cap \mathscr{C}A).$$

Thus $A \cap B$ is measurable.

Next we observe that if $A_1, \ldots, A_n$ are disjoint measurable sets, and $Z$ is arbitrary, then

$$\mu(Z \cap (A_1 \cup \cdots \cup A_n)) = \sum_{k=1}^{n} \mu(Z \cap A_k).$$

This follows for $n = 2$, replacing $Z$ by $Z \cap (A_1 \cup A_2)$ in the definition of measurability, and then by induction. Let now $\{A_n\}$ be a sequence of disjoint measurable sets, and let $A$ be their union. Using the fact that $\mu$ is an outer measure, we get for any subset $Z$:

$$\begin{aligned}\mu(Z) &= \mu(Z \cap (A_1 \cup \cdots \cup A_n)) + \mu(Z \cap \mathscr{C}(A_1 \cup \cdots \cup A_n)) \\ &\geqq \sum_{k=1}^{n} \mu(Z \cap A_k) + \mu(Z \cap \mathscr{C}A)\end{aligned}$$

for all $n$, whence

$$\begin{aligned}\mu(Z) &\geqq \sum_{k=1}^{\infty} \mu(Z \cap A_k) + \mu(Z \cap \mathscr{C}A) \\ &\geqq \mu(Z \cap A) + \mu(Z \cap \mathscr{C}A)\end{aligned}$$

because $\mu$ is an outer measure. The converse inequality

$$\mu(Z) \leqq \mu(Z \cap A) + \mu(Z \cap \mathscr{C}A)$$

is true again because $\mu$ is an outer measure. Thus we have equality. This proves both that $A$ is measurable, so the measurable sets form a $\sigma$-algebra, and that $\mu$ is countably additive on $\mathscr{S}$, thus concluding the proof of the lemma.

To prove the existence part of the theorem, all we need to show now is that the sets of our original algebra $\mathscr{A}$ are measurable. Let $A \in \mathscr{A}$ and let $Z$ be any subset of $X$. The inequality

$$\mu(Z) \leqq \mu(Z \cap A) + \mu(Z \cap \mathscr{C}A)$$

is true because $\mu$ is an outer measure. Conversely, given $\varepsilon$ let $\{A_n\}$ be a sequence in $\mathscr{A}$ whose union covers $Z$ and such that

$$\sum \mu(A_n) \leqq \mu(Z) + \varepsilon.$$

Then $Z \cap A$ is contained in the union of the sets $A_n \cap A$, and $Z \cap \mathscr{C}A$ is contained in the union of the sets $A_n \cap \mathscr{C}A = A_n - A$. Consequently

$$\begin{aligned}\mu(Z \cap A) + \mu(Z \cap \mathscr{C}A) &\leqq \sum \mu(A_n \cap A) + \sum \mu(A_n \cap \mathscr{C}A) \\ &= \sum \mu(A_n) \\ &\leqq \mu(Z) + \varepsilon.\end{aligned}$$

This proves the reverse inequality, and proves the existence of an extension of $\mu$ to a measure on $\mathscr{S}$, whence on the $\sigma$-algebra generated by $\mathscr{A}$.

Now for the uniqueness, we let $\mu$ be as we have just constructed it, and let $\nu$ be any positive measure on the $\sigma$-algebra $\mathscr{M}$ generated by $\mathscr{A}$, extending $\mu$ on $\mathscr{A}$. Let $\{A_n\}$ be a sequence in $\mathscr{A}$ of sets of finite $\mu$-measure, whose union is $X$. For any given $Y$ it suffices to prove that

$$\nu(Y \cap A_n) = \mu(Y \cap A_n).$$

Thus it suffices to prove: if $A \in \mathscr{A}$ has finite measure and $Y$ is in $\mathscr{M}$ and contained in $A$, then $\nu(Y) = \mu(Y)$. We have

$$\mu(Y) = \inf \sum \mu(B_n) = \inf \sum \nu(B_n)$$

the inf taken over all sequences $\{B_n\}$ in $\mathscr{A}$ whose union contains $Y$. This shows that $\nu(Y) \leqq \mu(Y)$. But then also,

$$\nu(A - Y) \leqq \mu(A - Y).$$

However

$$\mu(A) = \nu(A) = \nu(A - Y) + \nu(Y) \leqq \mu(A - Y) + \mu(Y) = \mu(A).$$

This proves that we must have $\nu(Y) = \mu(Y)$ and concludes the proof of the theorem. (For another proof of uniqueness, cf. Exercise 10(b).)

**Corollary 7.4.** *Let $(X, \mathscr{M}, \mu)$ be a measured space, and let $\mathscr{A}$ be a subalgebra of $\mathscr{M}$ consisting of sets of finite measure, generating $\mathscr{M}$, and such that $X$ is $\sigma$-finite with respect to $\mathscr{A}$. A subset $Z$ of $X$ has $\mu^*$-measure 0 if and only if given $\varepsilon$, there exists a sequence $\{A_n\}$ in $\mathscr{A}$ whose union covers $Z$ and such that*

$$\sum_{n=1}^{\infty} \mu(A_n) < \varepsilon.$$

*Similarly for a set $Z \in \mathscr{M}$ of measure 0.*

*Proof.* It is clear that a set satisfying the stated condition has measure 0. Conversely, we know from the theorem that the measure on $\mathscr{A}$ has a unique extension to $\mathscr{M}$, given as the outer measure. From this our assertion is obvious.

**Remark.** In euclidean space, with respect to Lebesgue measure (discussed later), the algebra $\mathscr{A}$ is that formed of finite disjoint unions of cubes. Thus a set has measure 0 in $\mathbf{R}^n$ if and only if given $\varepsilon$ it can be covered by a sequence of cubes, the sum of whose volumes is $< \varepsilon$. In many applications, one deals exclusively with sets of measure 0, and one does not need any fancy measure theory or integration theory. Thus the reader should keep this in mind so as to be more comfortable when meeting such applications.

## VI, §8. PRODUCT MEASURES AND INTEGRATION ON A PRODUCT SPACE

Let $X$, $Y$ be sets and $\mathscr{A}$, $\mathscr{B}$ algebras of subsets in $X$, $Y$ respectively. By a **rectangle** with respect to $\mathscr{A}$, $\mathscr{B}$ we mean a product $A \times B$ with $A \in \mathscr{A}$ and $B \in \mathscr{B}$. We let $\mathscr{A} \times \mathscr{B}$ denote the collection of all finite disjoint unions of rectangles with respect to $\mathscr{A}$, $\mathscr{B}$. (Unless needed for clarity, we omit the reference to $\mathscr{A}$, $\mathscr{B}$ in what follows.) *We contend that $\mathscr{A} \times \mathscr{B}$ is an algebra, in $X \times Y$.* This is easily proved as follows.

The empty set is in $\mathscr{A} \times \mathscr{B}$. We have the identities:

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

and

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 - A_2) \times B_1] \cup [(A_1 \cap A_2) \times (B_1 - B_2)].$$

If $P, Q \in \mathscr{A} \times \mathscr{B}$ these show that both $P \cap Q$ and $P - Q \in \mathscr{A} \times \mathscr{B}$. Since

$$P \cup Q = (P - Q) \cup Q$$

and $(P - Q) \cap Q$ is empty, it follows that $P \cup Q \in \mathscr{A} \times \mathscr{B}$. This proves that $\mathscr{A} \times \mathscr{B}$ is an algebra.

*We denote by $\mathscr{A} \otimes \mathscr{B}$ the $\sigma$-algebra generated by $\mathscr{A} \times \mathscr{B}$. Also, we denote by $\mathscr{A}^\sigma$ the $\sigma$-algebra generated by $\mathscr{A}$ in $X$. We have*

$$\mathscr{A}^\sigma \otimes \mathscr{B}^\sigma = (\mathscr{A} \times \mathscr{B})^\sigma.$$

*Proof.* Since $(\mathscr{A} \times \mathscr{B}) \subset (\mathscr{A}^\sigma \times \mathscr{B}^\sigma) \subset (\mathscr{A}^\sigma \otimes \mathscr{B}^\sigma)$ it follows that

$$(\mathscr{A} \times \mathscr{B})^\sigma \subset \mathscr{A}^\sigma \otimes \mathscr{B}^\sigma,$$

and we must prove the reverse inclusion. For each $B \in \mathscr{B}$ consider the $\sigma$-algebra in $X \times B$ generated by all sets $A \times B$ with $A \in \mathscr{A}$. It is contained in $(\mathscr{A} \times \mathscr{B})^\sigma$, which therefore contains $\mathscr{A}^\sigma \times \{B\}$ for all $B \in \mathscr{B}$. Now for any $A \in \mathscr{A}^\sigma$, it follows that $\{A\} \times \mathscr{B}^\sigma$ is contained in $(\mathscr{A} \times \mathscr{B})^\sigma$. Thus finally,

$$\mathscr{A}^\sigma \times \mathscr{B}^\sigma \subset (\mathscr{A} \times \mathscr{B})^\sigma,$$

whence the reverse inclusion

$$\mathscr{A}^\sigma \otimes \mathscr{B}^\sigma \subset (\mathscr{A} \times \mathscr{B})^\sigma,$$

which proves what we wanted.

**Lemma 8.1.** *Let $\mathcal{M}$ be a $\sigma$-algebra in $X$ and $\mathcal{N}$ a $\sigma$-algebra in $Y$.*

(i) *Let $Q \in \mathcal{M} \otimes \mathcal{N}$ and for each $x \in X$ let $Q_x$ be the set of $y$ such that $(x, y) \in Q$. Then $Q_x \in \mathcal{N}$.*
(ii) *Let $f: X \times Y \to Z$ be a $\mathcal{M} \otimes \mathcal{N}$ measurable map into a topological space $Z$. For each $x \in X$, the map*

$$f_x: Y \to Z$$

*given by $f_x(y) = f(x, y)$ is measurable.*

*Proof.* Let $\mathcal{S}$ be the collection of subsets $Q \in \mathcal{M} \otimes \mathcal{N}$ such that $Q_x \in \mathcal{N}$ for all $x$. Then $\mathcal{S}$ contains all rectangles $A \times B$ with $A \in \mathcal{M}$ and $B \in \mathcal{N}$. It will suffice to prove that $\mathcal{S}$ is a $\sigma$-algebra. The point is that the operation $Q \mapsto Q_x$ commutes with all the operations of set theory. Indeed, $X \times Y \in \mathcal{S}$. If $Q \in \mathcal{S}$, then $\mathcal{C}Q \in \mathcal{S}$ because $(\mathcal{C}Q)_x = \mathcal{C}(Q_x)$. If $Q$, $P$ are in $\mathcal{S}$, then

$$(P \cap Q)_x = P_x \cap Q_x.$$

If $\{Q_n\}$ is a sequence in $\mathcal{S}$, then $(\bigcup Q_n)_x = \bigcup (Q_n)_x$. Thus we see that $\mathcal{S}$ is a $\sigma$-algebra. This proves (i). As for (ii), if $V$ is open in $Z$, then

$$(f^{-1}(V))_x = f_x^{-1}(V),$$

so $f_x$ is measurable. This proves the lemma.

*For the rest of this section we let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measured spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be the algebras of sets of finite measure in $\mathcal{M}$ and $\mathcal{N}$ respectively.*

If $f$ is a step map with respect to $\mathcal{A} \times \mathcal{B}$, then we can define a repeated integral of $f$. Indeed, for each $x \in X$ the map $f_x$ is a step map on $Y$ with respect to $\mathcal{B}$. In fact, if

$$f = v\chi_{A \times B}$$

for some $v \in E$ and $A \in \mathcal{A}$, $B \in \mathcal{B}$, then

$$f_x = v(\chi_{A \times B})_x = v\chi_A(x)\chi_B \qquad \text{and} \qquad f_x(y) = v\chi_A(x)\chi_B(y).$$

Our assertion follows by linearity. Thus for each $x \in X$, we can form a first integral,

$$\int_Y f_x \, d\nu.$$

If $f = v\chi_{A \times B}$ as above, we see that

$$\int_Y f_x \, d\nu = v\chi_A(x)\nu(B).$$

If $f$ is a step map with respect to $\mathscr{A} \times \mathscr{B}$, we conclude that the map

$$x \mapsto \int_Y f_x \, d\nu$$

is a step map with respect to $\mathscr{A}$.

We may therefore integrate this map over $X$, with respect to $\mu$, and the repeated integral will be denoted by any one of the following notations:

$$\int_X \left[\int_Y f_x \, d\nu\right] d\mu(x), \qquad \int_X d\mu(x) \int_Y f_x \, d\nu,$$

$$\int_X \int_Y f(x, y) \, d\nu(y) \, d\mu(x), \qquad \int_X \int_Y f \, d\nu \, d\mu.$$

We use similar notation if we reverse the order of integration, and it is clear that on step maps, the repeated integrals are equal to each other, no matter what order of integration is chosen. In fact, we see at once that for $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have

$$\int_X \int_Y \chi_{A \times B} \, d\nu \, d\mu = \mu(A)\nu(B) = \int_Y \int_X \chi_{A \times B} \, d\mu \, d\nu.$$

*The repeated integral is linear on the space of step maps.*

*Proof.* Obvious, because each one of the single integrals is linear.

In particular, there is a unique finitely additive positive function $\mu \times \nu$ on $\mathscr{A} \times \mathscr{B}$ such that for $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B).$$

**Theorem 8.2.** *Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be $\sigma$-finite measured spaces. There exists a unique positive measure $\mu \otimes \nu$ on $\mathscr{M} \otimes \mathscr{N}$ such that for all sets $A$, $B$ of finite measure in $\mathscr{M}$ and $\mathscr{N}$ respectively we have*

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B).$$

*Proof.* By Hahn's theorem, it suffices to prove that $\mu \times \nu$ is countably additive on $\mathscr{A} \times \mathscr{B}$, i.e. is a measure on $\mathscr{A} \times \mathscr{B}$, where $\mathscr{A}$, $\mathscr{B}$ are the

algebras of sets of finite measure in $\mathcal{M}$ and $\mathcal{N}$ respectively. Let $\{Q_n\}$ be an increasing sequence in $\mathcal{A} \times \mathcal{B}$ whose union is an element $Q$ of $\mathcal{A} \times \mathcal{B}$. Let $f_n$ be the characteristic function of $Q_n$. Then $\{f_n\}$ is increasing to the characteristic function $f$ of $Q$. Furthermore, for each $x \in X$, the function $(f_n)_x$ is increasing to $f_x$. By the monotone convergence theorem with respect to $\nu$, we see that for each $x$,

$$\int_Y (f_n)_x \, d\nu \quad \text{is increasing to} \quad \int_Y f_x \, d\nu.$$

Now we apply the monotone convergence theorem with respect to $\mu$, to conclude that

$$\int_X \int_Y f_n \, d\nu \, d\mu \quad \text{converges to} \quad \int_X \int_Y f \, d\nu \, d\mu.$$

This proves our theorem.

**Lemma 8.3.** *Let $Z$ be a set of $(\mu \otimes \nu)$-measure 0 in $X \times Y$. Then for almost all $x \in X$ we have $\nu(Z_x) = 0$.*

*Proof.* For each positive integer $n$, let $S_n$ be the set of all $x$ such that $\nu(Z_x) \geqq 1/n$. Let $S = \bigcup S_n$. It will suffice to prove that $S$ is contained in a set of measure 0. Given $\varepsilon$, let $\{R_k\}$ be a sequence of rectangles whose union contains $Z$ and such that

$$\sum_{k=1}^{\infty} (\mu \times \nu)(R_k) < \frac{\varepsilon}{n2^n}.$$

Such $\{R_k\}$ exists by the corollary of Hahn's theorem. Then

$$Z_x \subset \bigcup_{k=1}^{\infty} R_{k,x}.$$

Let $T_n$ be the set of all $x$ such that

$$\frac{1}{n} \leqq \sum_{k=1}^{\infty} \nu(R_{k,x}).$$

Then $T_n$ is measurable, and $S_n \subset T_n$. Furthermore, the expression on the right is integrable with respect to $x$, and we find that

$$\frac{1}{n}\mu(T_n) \leqq \sum_{k=1}^{\infty} \int_X \nu(R_{k,x}) \, d\mu = \sum_{k=1}^{\infty} (\mu \times \nu)(R_k) < \frac{\varepsilon}{n2^n}.$$

This shows that $\mu(T_n) < \varepsilon/2^n$, whence $S$ is contained in a set of measure 0, thereby proving our lemma. The converse will follow from Corollary 8.5.

Suppose that $f$ is in $\mathscr{L}^1(\mu \otimes \nu, E)$ and let $g$ differ from $f$ only on a set of $(\mu \otimes \nu)$-measure 0, say $Z$. Then for each $x \in X$, the maps $f_x$ and $g_x$ differ only at those points $y \in Y$ such that $(x, y) \in Z$, i.e. those $y$ such that $y \in Z_x$. By the lemma, there exists a set $S$ of measure 0 in $X$ such that for all $x \notin S$ we have $\nu(Z_x) = 0$. From this we conclude that for such $x \notin S$, the maps $f_x$ and $g_x$ differ only on a set of measure 0. Thus $f_x$ is in $\mathscr{L}^1(\nu, E)$ if and only if $g_x$ is in $\mathscr{L}^1(\nu, E)$, and if this is the case, the integrals with respect to $\nu$ will be equal. This is the situation which we meet in the next theorem.

**Theorem 8.4 (Fubini's Theorem, Part 1).** *Let $f \in \mathscr{L}^1(\mu \otimes \nu)$. Then for almost all $x$, the map $f_x$ is in $\mathscr{L}^1(\nu)$, the map given by*

$$x \mapsto \int_Y f_x \, d\nu$$

*for almost all $x$ (and defined arbitrarily for other $x$) is in $\mathscr{L}^1(\mu)$; and we have*

$$\int_{X \times Y} f \, d(\mu \otimes \nu) = \int_X \int_Y f_x \, d\nu \, d\mu(x).$$

*There is a natural Banach space norm preserving isomorphism*

$$L^1(\mu \otimes \nu, E) \to L^1(\mu, L^1(\nu, E)).$$

*Proof.* By Theorem 6.3, we can find a sequence $\{\varphi_n\}$ of step mappings with respect to $\mathscr{A} \times \mathscr{B}$ which converges to $f$ both in $L^1$-seminorm and almost everywhere on $X \times Y$. As before, $\mathscr{A}$, $\mathscr{B}$ are the algebras of sets of finite measure in $\mathscr{M}$ and $\mathscr{N}$ respectively. We let $Z$ be a set of $(\mu \otimes \nu)$-measure 0 in $X \times Y$ such that $\{\varphi_n\}$ converges pointwise to $f$ outside $Z$. We let $S$ be a set of $\mu$-measure 0 in $X$ such that for $x \notin S$ we have $\nu(Z_x) = 0$.

If $x \notin S$, it follows that $\{\varphi_{n,x}\}$ converges pointwise to $f_x$ on the complement of $Z_x$.

Now we observe that for each $n$, the map

$$\Phi_n : x \mapsto \varphi_{n,x}$$

is a map of $X$ into $\mathrm{St}(\mathscr{B})$. Indeed, $\varphi_{n,x}$ is a step map with respect to $\mathscr{B}$,

and for $v \in E$, the formula

$$(v\chi_{A \times B})_x = v\chi_A(x)\chi_B$$

shows that $\Phi_n$ is step with respect to $\mathscr{A}$. We view the space $\mathrm{St}(\mathscr{B})$ as having the $L^1$-seminorm. We contend that $\{\Phi_n\}$ is a Cauchy sequence. This is easily seen, because

$$\begin{aligned}\|\Phi_n - \Phi_m\|_1 &= \int_X |\Phi_n - \Phi_m|_1 \, d\mu \\ &= \int_X \int_Y |\varphi_n(x, y) - \varphi_m(x, y)| \, d\nu(y) \, d\mu(x) \\ &= \|\varphi_n - \varphi_m\|_1 .\end{aligned}$$

(Of course, the $L^1$-seminorms taken on the right and left of the preceding equation refer to different spaces.)

By the fundamental Lemma 3.1, we may assume without loss of generality (using a subsequence if necessary) that there exists a set $T$ of measure 0 in $X$ such that for $x \notin T$ the sequence $\{\Phi_n(x)\}$ is Cauchy. [Lemma 3.1 and its proof are valid for values in a Banach space. For our purposes, we note that the proof of this lemma applies as well in the seminormed case to yield a pointwise Cauchy sequence for almost all $x$. Alternatively, we may also take the natural map of $\mathrm{St}(\mathscr{B})$ into $L^1(\nu)$ and apply the lemma with respect to the Banach space $L^1(\nu)$.] This means that for each $x \notin T$, the sequence

$$\{\Phi_n(x)\} = \{\varphi_{n,x}\}$$

is Cauchy (that is, $L^1$-Cauchy with respect to $\nu$). If $x \notin S \cup T$, we know that $\{\varphi_{n,x}(y)\}$ converges to $\{f_x(y)\}$ for almost all $y \in Y$. Hence by Corollary 5.10, we conclude that $f_x \in \mathscr{L}^1(\nu)$ and that $\{\varphi_{n,x}\}$ is $L^1$-convergent to $f_x$, so that

$$\int_Y \varphi_{n,x} \, d\nu \quad \text{converges to} \quad \int_Y f_x \, d\nu$$

for all $x \notin S \cup T$.

Finally, we note that the map

$$\Psi_n \colon x \mapsto \int_Y \varphi_{n,x} \, d\nu$$

is a step map with respect to $\mathscr{A}$. [It is in fact the composite map of $\Phi_n$

and the integral $\int_Y d\nu$.] Furthermore, the sequence $\{\Psi_n\}$ is Cauchy ($L^1$ with respect to $\mu$), as one sees by repeating the argument given above to show that $\{\Phi_n\}$ is Cauchy. Also for all $x \notin S \cup T$ we know that $\Psi_n(x)$ converges to the map $\Psi$ given by

$$\Psi(x) = \int_Y f_x \, d\nu.$$

Consequently $\{\Psi_n\}$ is $L^1$-convergent to $\Psi$, and as $n \to \infty$,

$$\int_X \int_Y \varphi_{n,x} \, d\nu \, d\mu(x) \quad \text{converges to} \quad \int_X \int_Y f_x \, d\nu \, d\mu(x).$$

Since $\varphi_n$ is a step map and

$$\int_X \int_Y \varphi_{n,x} \, d\nu \, d\mu(x) = \int_{X \times Y} \varphi_n \, d(\mu \otimes \nu),$$

we see that Fubini's theorem is proved.

**Corollary 8.5.** *Let $Q$ be a measurable subset of finite measure in $X \times Y$. Then*

$$\int_{X \times Y} \chi_Q \, d(\mu \otimes \nu) = \int_X \int_Y (\chi_Q)_x \, d\nu \, d\mu(x).$$

*Proof.* If $Q$ has finite measure, then $\chi_Q$ is in $\mathscr{L}^1(\mu \otimes \nu)$ and the theorem applies.

**Remark.** Our version of Fubini's theorem as it applies to the situation in the corollary does not yield the fact that the map

$$x \mapsto \int_Y (\chi_Q)_x \, d\nu = \nu(Q_x)$$

is measurable (only that it is $\mu$-measurable). It happens to be true that the map is in fact measurable. Cf. Exercise 11.

In Fubini's theorem, we start with a map $f \in \mathscr{L}^1(\mu \otimes \nu)$ and conclude that the various partial mappings arising from this $f$ are in the corresponding $\mathscr{L}^1$ spaces. One can ask for the converse, which is true, properly formulated.

**Lemma 8.6.** *Let $f\colon X \times Y \to E$ be a $(\mu \otimes \nu)$-measurable map. Then for almost all $x$, the map $f_x$ is $\nu$-measurable.*

*Proof.* Let $Z$ be a set of measure 0 in $X \times Y$ such that the restriction of $f$ to the complement of $Z$ is measurable, and the image of the complement of $Z$ in $X \times Y$ is separable. By Lemma 8.3, for almost all $x$ the set $Z_x$ has measure 0, and by Lemma 8.1 the restriction of $f_x$ to the complement of $Z_x$ in $Y$ is measurable, whence $\nu$-measurable by **M11**. This proves our lemma.

**Theorem 8.7 (Fubini's Theorem, Part 2).** *Let $f: X \times Y \to E$ be a $(\mu \otimes \nu)$-measurable map. Assume that for almost all $x \in X$ the map $f_x$ is in $\mathscr{L}^1(\nu)$, and that the map given by*

$$x \mapsto \int_Y |f_x| \, d\nu$$

*(for almost all $x$, and arbitrary otherwise) is in $\mathscr{L}^1(\mu, \mathbf{R})$. Then*

$$f \in \mathscr{L}^1(\mu \otimes \nu, E)$$

*and Part 1 of Fubini's theorem applies.*

*Proof.* By Corollary 5.9 of the dominated convergence theorem, it suffices to prove that $|f|$ is in $\mathscr{L}^1(\mu \otimes \nu, \mathbf{R})$, and thus we may assume without loss of generality that $f$ is a semipositive real function which is $(\mu \otimes \nu)$-measurable, satisfying the other hypotheses of the theorem. By condition **M9** of §1, we can find a sequence of positive simple functions $\{\varphi_n\}$ which is increasing to $f$ pointwise everywhere (changing $f$ if necessary on a set of measure 0). Using the $\sigma$-finiteness of $X \times Y$, we may assume further without loss of generality that each $\varphi_n$ vanishes outside a set of finite measure, i.e. is step. For each $x$ the sequence $\{\varphi_{n,x}\}$ is increasing to $f_x$. Whenever $x$ is such that $f_x$ is in $\mathscr{L}^1$, and $\varphi_{n,x}$ is $\nu$-measurable, it follows that as $n \to \infty$,

$$\int_Y \varphi_{n,x} \, d\nu \quad \text{is increasing and convergent to} \quad \int_Y f_x \, d\nu.$$

We can apply the corollary of Fubini's theorem (by linearity), and the monotone convergence theorem once more to conclude that the sequence given by

$$\left\{\int_{X \times Y} \varphi_n \, d(\mu \otimes \nu)\right\} = \left\{\int_X \int_Y \varphi_{n,x} \, d\nu \, d\mu(x)\right\}$$

is increasing and convergent to

$$\int_X \int_Y f_x \, d\nu \, d\mu(x).$$

A final application of the monotone convergence theorem shows that $f$ is in $\mathscr{L}^1$, thus proving our theorem.

## VI, §9. THE LEBESGUE INTEGRAL IN $\mathbf{R}^p$

We start with $\mathbf{R}$, and the algebra of subsets consisting of finite disjoint unions of intervals. The length function is easily seen to extend to a finitely additive function on this algebra. To get our theory going, we must show that it is a measure, i.e. countably additive. It is in fact just as convenient to prove a slightly more general statement.

**Theorem 9.1.** *Let $\{f_n\}$ be a sequence of functions $\geqq 0$ on a closed bounded interval $I$, decreasing monotonically to 0. Assume that each $f_n$ is a step function with respect to intervals. Then the sequence of (plain and ordinary) integrals*

$$\int_I f_n(x)\,dx$$

*decreases to 0.*

*Proof.* For each $n$, the intervals on which $f_n$ is constant have a finite number of end points. The union of such end points for all such intervals and all $n = 1, 2, \ldots$ is countable, and can therefore be covered by a sequence of open intervals $J_k$ such that

$$\sum_{k=1}^{\infty} l(J_k) < \varepsilon,$$

where $l$ is the length. Let $U = \bigcup J_k$. If $x \in I$ and $x \notin U$, there exists some $n_x$ and an open interval $V_x$ containing $x$ such that $f_{n_x}(t) < \varepsilon$ for all $t \in V_x$. Since the sequence $\{f_n\}$ is decreasing, it follows that $f_m(t) < \varepsilon$ for all $m \geqq n_x$ and all $t \in V_x$. The family of open sets

$$\{J_k, V_x\}, \qquad k = 1, 2, \ldots; \quad x \in I, \quad x \notin U,$$

covers $I$, and hence there exists a finite subcovering

$$\{J_{k_1}, \ldots, J_{k_r}, V_{x_1}, \ldots, V_{x_s}\}.$$

Let $N = \max(n_{x_1}, \ldots, n_{x_s})$. If $n \geqq N$, then

$$f_n(t) < \varepsilon \qquad \text{if} \quad t \in V_{x_1} \cup \cdots \cup V_{x_s}.$$

The integral $\int_I f_n(x)\,dx$ is bounded by the sum of the integrals of $f_n$ over the intervals $J_{k_1}, \ldots, J_{k_r}$, and over the union $V_{x_1} \cup \cdots \cup V_{x_s}$. If $C$ is a

bound for $f_1$ (and hence all $f_n$) we conclude that

$$\int_I f_n(x)\,dx \leqq C\varepsilon + l(I)\varepsilon,$$

which proves our theorem.

**Corollary 9.2.** *The length function of intervals extends uniquely to a measure on the algebra consisting of finite disjoint unions of bounded intervals.*

*Proof.* If $\{A_n\}$ is a disjoint sequence in the algebra, whose union is an element $A$ of the algebra, then the characteristic functions

$$\{\chi_A - \chi_{B_n}\} = \{\chi_{A-B_n}\} \qquad \text{with } B_n = A_1 \cup \cdots \cup A_n$$

forms a decreasing sequence of step functions, converging pointwise to 0, to which we can apply the theorem.

Having our measure on the algebra of finite disjoint unions of bounded intervals, we can first obtain a $\sigma$-algebra and a measure on it by Hahn's theorem. Then §3 gives us the integral on the reals. We can apply the theory of integration on product spaces to get the integral on $\mathbf{R}^p$, because $\mathbf{R}$ is obviously $\sigma$-finite with respect to bounded intervals. Thus we now have integration on $\mathbf{R}^p$. The $\sigma$-algebra of measurable sets in $\mathbf{R}^p$ obtained by the preceding procedure is that generated by the **rectangles** (i.e. $p$-fold Cartesian products of intervals), and is thus the $\sigma$-algebra of Borel sets in $\mathbf{R}^p$. The measure on this algebra obtained as above is called **Lebesgue measure**. One can also extend it to the completion of the $\sigma$-algebra generated by rectangles (cf. Exercise 7). This makes no difference concerning integration, and is in fact frequently very convenient.

We observe that for Lebesgue measure, rectangles have the expected measure, namely the product of the lengths of the sides.

*For the rest of this section, we let $\mu$ denote Lebesgue measure. One customarily writes $\mathscr{L}^1(\mathbf{R}^p)$ instead of $\mathscr{L}^1(\mu)$ in this case.*

It is clear that $\mathbf{R}^p$ is $\sigma$-finite, being a union of bounded rectangles, i.e. $p$-dimensional rectangles. Thus we can apply the density statement concerning step mappings with respect to finite unions of rectangles. We shall give an application of Corollary 6.4.

If $\varphi$ is a function on $\mathbf{R}^p$, we say that $\varphi$ has **compact support** if $\varphi(x) = 0$ for $x$ outside some compact set. We let $C_c^\infty(\mathbf{R}^p, \mathbf{C})$ be the space of $C^\infty$ (infinitely differentiable) functions (complex) with compact support. It is clearly a vector space.

**Theorem 9.3.** *Let*

$$f \in \mathscr{L}^1(\mu).$$

*If*

$$\int_{\mathbf{R}^p} f\varphi \, d\mu = 0$$

*for all* $\varphi \in C_c^\infty(\mathbf{R}^p, \mathbf{C})$, *then* $f$ *is equal to* 0 *almost everywhere.*

*Proof.* According to Corollary 6.4 it suffices to prove that

$$\int_A f \, d\mu = 0$$

for all bounded rectangles $A$. We shall recall below how to approximate a characteristic function $\chi_A$ of a rectangle by a $C^\infty$ function with compact support, both almost everywhere and for the $L^1$-seminorm. In other words, we can find a sequence $\{\varphi_n\}$ of $C^\infty$ functions with compact support which tends almost everywhere to $\chi_A$ and is bounded, say by a constant $C$. Then $\{\varphi_n f\}$ tends almost everywhere to $f_A = \chi_A f$, and each $\varphi_n f$ is in $\mathscr{L}^1$ by Corollary 5.11 of the dominated convergence theorem. Applying the dominated convergence theorem, we conclude that $\{\varphi_n f\}$ is $L^1$-convergent to $f_A$, whence

$$\int_{\mathbf{R}^p} \varphi_n f \, d\mu \quad \text{converges to} \quad \int_{\mathbf{R}^p} f_A \, d\mu.$$

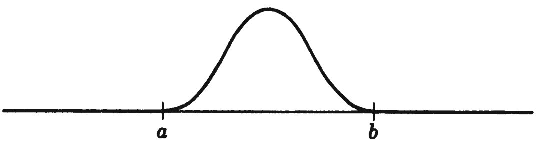
This proves what we wanted.

We now recall the construction mentioned in our proof. It is basically a one-dimensional construction. Let $a$, $b$ be real and $a < b$. The function

$$h(t) = e^{-1/(t-a)(b-t)} \qquad \text{if} \quad a < t < b,$$

$$h(t) = 0 \qquad \text{if} \quad t \leqq a \text{ or } t \geqq b,$$

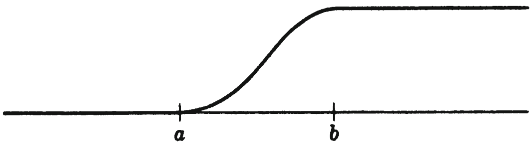
is a bell-shaped $C^\infty$ function which looks as follows:




The function

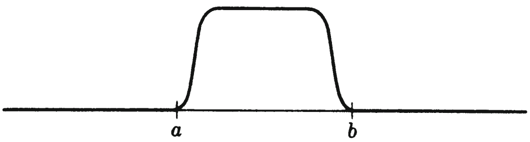
$$g \colon x \mapsto \int_{-\infty}^{x} h(t) \, dt = g(x)$$

then starts from 0 and climbs between $a$ and $b$ to a constant value, looking like this:




Multiplying by a positive constant, we can assume that the top value is equal to any given number $> 0$.

If we make a translation on $g$ we can assume for instance that $a = 0$. Considering the function $g(cx)$ instead of $g(x)$ where $c$ is a large constant, we can make the climb arbitrarily steep. Combining translations and such steep climbs, we can then find a function which is $C^\infty$ and looks like this:




In other words, this function approximates the characteristic function of $[a, b]$ from below. We can do the same thing from above. Taking suitable products to do the same thing in $p$-space, we end up with the following result.

**Lemma 9.4.** *Let $A$ be a bounded rectangle in $\mathbf{R}^p$. Given $\varepsilon$, there exist $C^\infty$ functions $\varphi$, $\psi$ having the following properties*:

(i) *We have*

$$0 \leqq \varphi \leqq \chi_A \leqq \psi \leqq 1.$$

(ii) *We have*

$$\int_{\mathbf{R}^p} (\psi - \varphi)\, d\mu < \varepsilon.$$

*In fact, if*

$$A = [a_1, b_1] \times \cdots \times [a_p, b_p],$$

*the function $\psi$ is 0 outside the rectangle*

$$[a_1 - \varepsilon, b_1 + \varepsilon] \times \cdots \times [a_p - \varepsilon, b_p + \varepsilon]$$

*and the function $\varphi$ is 1 on the rectangle*

$$[a_1 + \varepsilon, b_1 - \varepsilon] \times \cdots \times [a_p + \varepsilon, b_p - \varepsilon].$$

Observe that in deriving this lemma, we are dealing with the simplest case of Riemann integration. The lemma is at the level of elementary calculus.

The result of Theorem 9.3 really concerns the values of the map $f$ on bounded sets of $\mathbf{R}^p$. In many applications, it is not convenient to restrict oneself to elements of $\mathscr{L}^1$, and one needs a formulation which allows us to deal with maps locally. Thus we say that a map $f\colon \mathbf{R}^p \to E$ is **locally integrable** if for each compact set $K$ in $\mathbf{R}^p$ the map $f_K$ (equal to $f$ on $K$ and 0 outside $K$) is in $\mathscr{L}^1(\mu)$.

**Corollary 9.5.** *Let $f$ be a locally integrable map on $\mathbf{R}^p$ such that for all $\varphi \in C_c^\infty(\mathbf{R}^p, \mathbf{C})$ we have*

$$\int_{\mathbf{R}^p} f\varphi \, d\mu = 0.$$

*Then $f$ is equal to 0 almost everywhere.*

*Proof.* This is really what Theorem 9.3 proved, since all we have to consider is $f_A$ for every bounded rectangle $A$.

**Theorem 9.6.** *The space $C_c^\infty(\mathbf{R}^p)$ is dense in $\mathscr{L}^1(\mu, \mathbf{C})$.*

*Proof.* We may restrict ourselves to the real functions. We know that the step functions with respect to rectangles are dense in $\mathscr{L}^1$. On the other hand, the characteristic function of a rectangle can be approximated by $C^\infty$ functions with compact support, as we saw above for the proof of Theorem 9.3. The assertion of our corollary follows at once.

Let $f\colon \mathbf{R}^p \to E$ be a map, and let $a \in \mathbf{R}^p$. We define the **translation** $\tau_a f$, also written $f_a$, to be the map given by

$$(\tau_a f)(x) = f(x - a).$$

If $Y$ is a subset of $\mathbf{R}^p$, we define

$$Y_a = \tau_a(Y) = Y + a$$

to be the set of all points $x + a$ with $x \in Y$. Our definitions are adjusted in such a way that

$$\tau_a(\chi_Y) = \chi_{Y_a}.$$

**Theorem 9.7.** *The Lebesgue integral is translation invariant. This means: If $f \in \mathscr{L}^1(\mu)$, then for each $a \in \mathbf{R}^p$ the map $\tau_a f$* is in *$\mathscr{L}^1(\mu)$, and we have*

$$\int_{\mathbf{R}^p} \tau_a f \, d\mu = \int_{\mathbf{R}^p} f \, d\mu.$$

*Proof.* By Theorem 6.3 we can find a sequence $\{\varphi_n\}$ of step maps with respect to finite unions of rectangles which converges both $L^1$ and almost everywhere to $f$. If $\varphi$ is a step map as above, it is clear that its integral is the same as the integral of a translation $\tau_a\varphi$, because if $R$ is a rectangle, then

$$\mu(R) = \mu(\tau_a R).$$

But $\{\tau_a\varphi_n\}$ converges almost everywhere to $\tau_a f$, and by the preceding remark, $\{\tau_a\varphi_n\}$ is $L^1$-Cauchy, whence is also $L^1$-convergent to $\tau_a f$. Our theorem follows at once.

**Theorem 9.8.** *If $Y$ is a measurable set in $\mathbf{R}^p$, then we have:*

$$\mu(Y) = \inf \mu(U) \quad \textit{for } U \textit{ open}, \qquad U \supset Y,$$
$$\mu(Y) = \sup \mu(K) \quad \textit{for } K \textit{ compact}, \quad K \subset Y.$$

*Thus if $Y$ has finite measure, given $\varepsilon$ there exists an open set $U$ and a compact set $K$ such that*

$$K \subset Y \subset U \qquad \textit{and} \qquad \mu(U - K) < \varepsilon.$$

*Proof.* The statement concerning open sets is clear by applying the definition of our measure as an application of the Hahn theorem, giving $\mu$ as the outer measure with respect to bounded rectangles. We can always take the rectangles to be open to cover $Y$, since a closed rectangle is contained in an open one whose measure is at most $\varepsilon/2^n$ bigger. Concerning the statement about compact sets, suppose first that $Y$ is bounded, say contained in a closed bounded rectangle $R$. We find an open set $U$ containing $R - Y$ such that

$$\mu(U) < \mu(R - Y) + \varepsilon.$$

Let $K = R \cap \mathscr{C}U = R - U$. Then $K$ is compact and contained in $Y$. We have trivially:

$$\begin{aligned} \mu(K) \leqq \mu(Y) &= \mu(R) - \mu(R - Y) \\ &\leqq \mu(R) - \mu(U) + \varepsilon \\ &\leqq \mu(K) + \varepsilon. \end{aligned}$$

This proves our assertion when $Y$ is bounded. The general case follows at once by considering the intersections of $Y$ with a sequence $\{R_n\}$ of rectangles such that $R_n \subset R_{n+1}$ for all $n$, and such that the union of the $R_n$ is the entire euclidean space.

## VI, §10. EXERCISES

*Unless otherwise specified, $(X, \mathscr{M}, \mu)$ is a measured space.*

1. (a) Let $\mathscr{M}$ be a $\sigma$-algebra in a set $X$, and let

$$f\colon X \to Y \quad \text{and} \quad g\colon Y \to Z$$

be mappings. Show that

$$(g \circ f)_*(\mathscr{M}) = g_*(f_*(\mathscr{M})).$$

In other words, $(g \circ f)_* = g_* \circ f_*$.

(b) Let $\mathscr{M}$ be a $\sigma$-algebra in a set $X$, and let $\mu$ be a positive measure. Let $f\colon X \to Y$ be a mapping. Define the **direct image** $f_*\mu$ on $f_*\mathscr{M}$ by the condition

$$(f_*\mu)(B) = \mu(f^{-1}(B))$$

for all $B$ in $f_*\mathscr{M}$. Show that $f_*\mu$ is a positive measure.

2. **Egoroff's theorem.** Assume that $\mu$ is $\sigma$-finite. Let $f\colon X \to E$ be a map and assume that $f$ is the pointwise limit of a sequence of simple maps $\{\varphi_n\}$. Given $\varepsilon$, show that there exists a set $Z$ with $\mu(Z) < \varepsilon$ such that the convergence of $\{\varphi_n\}$ is uniform on the complement of $Z$. [*Hint*: Assume first that $\mu(X)$ is finite. Let $A_k$ be the set where $|f| \geqq k$. The intersection of all $A_k$ is empty so their measures tend to 0. Excluding a set of small measure, you can assume that $f$ is bounded, in which case $f$ is in $\mathscr{L}^1(\mu)$ and you can use the fundamental lemma of integration, or Theorem 5.2.]

3. Let $\{f_n\}$ be a sequence of measurable functions. Show that the set of those $x$ such that $\{f_n(x)\}$ converges is a measurable set.

4. Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$. View $[-\infty, \infty]$ as a toplogical space, neighborhoods of $-\infty$ being given by sets $[-\infty, a)$ for $a$ real, and similarly for neighborhoods of $\infty$. Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$. By

$$\limsup a_n$$

we mean the least upper bound of all points of accumulations of the sequence $\{a_n\}$. We allow $-\infty$ and $+\infty$ as points of accumulation, taking the obvious ordering in $[-\infty, \infty]$ where

$$-\infty < a < \infty$$

for all real $a$.

(a) Let $b = \limsup a_n$. Suppose that $b$ is a number. Show that given $\varepsilon$, there exists only a finite number of $n$ such that $a_n > b + \varepsilon$, and there exist infinitely many $n$ such that $a_n > b - \varepsilon$. Prove that this property characterizes the lim sup. Give a similar characterization when $b = \infty$.

(b) Charaterize lim inf similarly, and show that

$$\liminf a_n = -\limsup(-a_n).$$

(c) A sequence $\{a_n\}$ in $[-\infty, \infty]$ converges if and only if

$$\limsup a_n = \liminf a_n.$$

(d) If $\{f_n\}$ is a sequence of measurable maps of $X$ into $[-\infty, \infty]$, then its upper limit and lower limit are measurable. (By the way, the lim sup and lim inf of the sequence $\{f_n\}$ are defined pointwise.)

5. **Positive measurable maps.** A map $f: X \to [0, \infty]$ will be called positive.
(a) If $f, g: X \to [0, \infty]$ are measurable, show that $f + g$, $fg$ are measurable. If $\{f_n\}$ is a sequence of positive measurable maps, show that $\sup f_n$ and $\inf f_n$ are also measurable.
(b) If $\mu$ is $\sigma$-finite, show that $f$ is measurable if and only if $f$ is the limit of an increasing sequence of real valued step functions (0 outside a set of finite measure).
(c) For a positive measurable map $f: X \to [0, \infty]$ let $\{f_n\}$ be a sequence of positive simple functions (real valued) which is increasing to $f$. If the integrals

$$\int_X f_n \, d\mu$$

exist and are bounded (so in particular each $f_n$ is 0 outside a set of finite measure), define the integral of $f$ to be their least upper bound, and if unbounded, define the integral of $f$ to be $\infty$. Show that this is well defined, i.e. independent of the sequence $\{f_n\}$ increasing to $f$. Formulate and prove the monotone convergence theorem in this context. *Note:* Instead of redoing integration theory, you can quote results from the text to shorten the procedure.
(d) For each measurable $A$ and positive measurable map $f: X \to [0, \infty]$ define

$$\mu_f(A) = \int_A f \, d\mu.$$

Show that $\mu_f$ is a positive measure on $X$. If $g: X \to [0, \infty]$ is measurable, show that

$$\int_X g \, d\mu_f = \int_X fg \, d\mu.$$

6. Let $\{f_n\}$ be a sequence of continuous functions on $[0, 1]$ such that $0 \leqq f_n \leqq 1$ and such that $\{f_n(x)\}$ converges to 0 for every $x$ in $[0, 1]$. Show that

$$\lim \int_0^1 f_n \, d\mu = 0,$$

where $\mu$ is Lebesgue measure.

7. **Completion of a measure.**
(a) Let $\overline{\mathscr{M}}$ consist of all subsets $Y$ of $X$ which differ from an element of $\mathscr{M}$ by a set contained in a set of measure 0. In other words, there exists a set $A$ in $\mathscr{M}$ such that $(Y - A) \cup (A - Y)$ is contained in a set of measure

0. Show that $\overline{\mathcal{M}}$ is a $\sigma$-algebra. If we define $\bar{\mu}(Y) = \mu(A)$ for $Y$, $A$ as above, show that this is well defined on $\overline{\mathcal{M}}$, and that $\bar{\mu}$ is a measure on $\overline{\mathcal{M}}$. We call $(X, \overline{\mathcal{M}}, \bar{\mu})$ the **complete** measure space determined by $(X, \mathcal{M}, \mu)$, and we call $\bar{\mu}$ the completion of $\mu$.

(b) Let $(X_i, \mathcal{M}_i, \mu_i)$ $(i = 1, 2, 3)$ be measured spaces. Show that

$$(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3 = \mathcal{M}_1 \otimes (\mathcal{M}_2 \otimes \mathcal{M}_3),$$

$$(\mu_1 \otimes \mu_2) \otimes \mu_3 = \mu_1 \otimes (\mu_2 \otimes \mu_3).$$

If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measured spaces, show that

$$\overline{\overline{\mathcal{M}} \otimes \overline{\mathcal{N}}} = \overline{\mathcal{M} \otimes \mathcal{N}} \qquad \text{and} \qquad \overline{\bar{\mu} \otimes \bar{\nu}} = \overline{\mu \otimes \nu}.$$

8. (a) **Direct image of a measure.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Let $f: X \to Y$ be a map such that for each $B \in \mathcal{N}$ we have $f^{-1}(B) \in \mathcal{M}$. Let $\mu$ be a positive measure on $\mathcal{M}$, and let $\mu_* = f_*\mu$ be defined on $\mathcal{N}$ by $\mu_*(B) = \mu(f^{-1}(B))$. Show that $\mu_*$ is a measure. Show that if $g$ is in $\mathcal{L}^1(\mu_*)$, then $g \circ f$ is in $\mathcal{L}^1(\mu)$, and that

$$\int_X g \circ f \, d\mu = \int_Y g \, d\mu_*.$$

(b) Let $X$, $Y$ be topological spaces and $f: X \to Y$ a homeomorphism. Show that $f$ induces a bijective map

$$f^*: \mathcal{B}(Y) \to \mathcal{B}(X)$$

where $\mathcal{B}$ denotes the Borel algebra.

9. Let $E$ be a Hilbert space with countable base. A map $f: X \to E$ is called **weakly measurable** if for every functional $\lambda$ on $E$ the composite $\lambda \circ f$ is measurable. Let $f, g: X \to E$ be weakly measurable. Show that the map

$$x \mapsto \langle f(x), g(x) \rangle$$

is measurable. [*Hint*: Write the maps in terms of their component functions with respect to a Hilbert basis, so the scalar product becomes a limit of measurable functions.]

10. **Monotone families.**

(a) A collection $\mathcal{S}$ of subsets of $X$ is said to be **monotone** if, whenever $\{A_n\}$ is an increasing (resp. decreasing) sequence of subsets in $\mathcal{S}$, then

$$\bigcup A_n \quad (\text{resp. } \bigcap A_n)$$

also lies in $\mathcal{S}$. Let $\mathcal{A}$ be an algebra of subsets of $X$. Show that there exists a smallest monotone collection of subsets of $X$ containing $\mathcal{A}$. Denote it by $\mathcal{N}$. If $x \in X$, show that $\mathcal{N}$ is a $\sigma$-algebra, and is thus the smallest $\sigma$-algebra containing $\mathcal{A}$. [*Hint*: For each $A \in \mathcal{N}$, let $\mathcal{N}(A)$ be the collection of all sets $B \in \mathcal{N}$ such that $B \cup A$, $B - A$ and $A - B$ lie in $\mathcal{N}$. Then $\mathcal{N}(A)$ is monotone.]

(b) Assume that $X \in \mathcal{N}$. Let $\mu$ be a positive measure on $\mathcal{A}$. Show that an extension of $\mu$ to a positive measure on $\mathcal{N}$ is uniquely determined, by proving: If $\mu_1$, $\mu_2$ are extensions of $\mu$ to $\mathcal{N}$, then the collection of subsets $Y$ such that $\mu_1(Y) = \mu_2(Y)$ is monotone.

11. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measures spaces. If $Q \in \mathcal{M} \otimes \mathcal{N}$, show that the map

$$x \mapsto \nu(Q_x)$$

is measurable (with respect to $\mathcal{M}$). [*Hint*: Show that the set of $Q$ in $\mathcal{M} \otimes \mathcal{N}$ having the above property is a monotone family containing the rectangles.]

12. Show that if $c_n$ is the (Lebesgue) measure of the closed $n$-ball in $\mathbf{R}^n$ of radius 1, centered at the origin, then

$$c_n = c_{n-1} \int_{-\pi/2}^{\pi/2} \cos^n t \, dt,$$

and therefore

$$c_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

13. Let $T$ be a metric space and let $f$ be a map on $X \times T$ such that for each $t \in T$ the partial map

$$f_t \colon x \mapsto f(x, t)$$

is in $\mathcal{L}^1$. Assume that for each $x$ the map $t \mapsto f(x, t)$ is continuous. Finally assume that there is some $g \in \mathcal{L}^1(\mu, \mathbf{R})$ such that $|f(x, t)| \leqq |g(x)|$ for all $x$. Show that the function $\Phi$ given by

$$\Phi(t) = \int_X f(x, t) \, d\mu(x)$$

is continuous.

14. **Differentiating under the integral sign.** Let $T$ be open in some euclidean space. Let $f$ be a map on $X \times T$ satisfying:
(a) For each $t$ the map $x \mapsto f(x, t)$ is in $\mathcal{L}^1$.
(b) For each $x$, the map $f_x \colon t \mapsto f(x, t)$ is differentiable, and its derivative is continuous in $t$.
(c) The second partial $D_2 f(x, t)$ is in $\mathcal{L}^1$ for each $t$, and there exists an element $g \in \mathcal{L}^1(\mu, \mathbf{R})$ and $g \geqq 0$ such that

$$|D_2 f(x, t)| \leqq g(x)$$

for all $x$, $t$.

Then the map $\Phi$ as in the preceding exercise is differentiable, and its derivative is given by

$$D\Phi(t) = \int_X D_2 f(x, t) \, d\mu(x).$$

(If you prefer, take $T$ to be an open interval.)

15. Let $\mu$ be Lebesgue measure on $\mathbf{R}^p$. If $f, g \in \mathscr{L}^1(\mu)$, define $f * g$ by

$$f * g(x) = \int_{\mathbf{R}^p} f(t)g(x - t)\, d\mu(t).$$

(a) Show that $f * g \in \mathscr{L}^1(\mu)$ and that $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$. We call $f * g$ the **convolution** of $f$ and $g$.
(b) Show that convolution is commutative, associative, bilinear, and that $\mathscr{L}^1(\mu)$ is therefore a Banach algebra. Does there exist a unit element in this algebra?

16. Let $M$ be the set of all finite positive Borel measures on $\mathbf{R}^p$. For each $\mu \in M$ define $|\mu| = \mu(\mathbf{R}^p)$. For $\mu$, $\nu \in M$, and any Borel subset $A$ of $\mathbf{R}^p$ define

$$(\mu * \nu)(A) = (\mu \otimes \nu)(\sigma^{-1}(A))$$

where $\sigma: \mathbf{R}^p \times \mathbf{R}^p \to \mathbf{R}^p$ is the sum, that is $\sigma(x, y) = x + y$.
(a) Show that $\sigma^{-1}(A)$ is a Borel set in $\mathbf{R}^p \times \mathbf{R}^p$.
(b) Show that $\mu * \nu \in M$ and that $|\mu * \nu| \leqq |\mu|\,|\nu|$.
(c) Show that $\mu * \nu$ is the unique positive Borel measure $\tau$ such that

$$\int f\, d\tau = \iint f(x + y)\, d\nu(y)\, d\mu(x)$$

for every step function $f$ with respect to rectangles.
(d) The operation $(\mu, \nu) \mapsto \mu * \nu$ is called **convolution**. Show that it is commutative, associative, and bilinear.
(e) Show that there exists a unit element in $M$, i.e. an element $\delta$ such that $\delta * \mu = \mu * \delta = \mu$ for all $\mu \in M$.
(f) Let $\mu$ be Lebesgue measure, and let $f, g \in \mathscr{L}^1(\mu)$. Show that

$$\mu_f * \mu_g = \mu_{f*g}.$$

(g) After you have read about complex measures in the next chapter, show that all the previous properties apply as well to such measures, and that these measures therefore form a Banach algebra under convolution.

17. Let $X = [-\pi, \pi]$, and let $\mu$ be Lebesgue measure. Let $f \in \mathscr{L}^1(\mu, \mathbf{C})$. Show that one can define the Fourier coefficients of $f$ in the usual way, by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\, dx.$$

If $c_n = 0$ for all integers $n$, show that $f$ is equal to 0 almost everywhere.

18. **Riemann–Lebesgue lemma.** Let $f \in \mathscr{L}^1(\mathbf{R})$. Prove that

$$\lim_{t \to \infty} \int_{\mathbf{R}} f(x)e^{-itx}\, dx = 0.$$

[*Hint:* Approximate $f$ by a $C^\infty$ function with compact support, in which case integrate by parts.]

19. (a) Let $\mathbf{R}^*$ be the multiplicative group of non-zero real numbers. Show that the map

$$\psi \mapsto \int_{\mathbf{R}} \psi(t) \frac{dt}{|t|},$$

for $\psi$ a step function with respect to intervals not containing 0, defines a positive Borel measure on $\mathbf{R}^*$. We denote this measure by $\mu^*$. Show that a function $f$ is in $\mathscr{L}^1(\mu^*)$ if and only if $f(x)/|x|$ is in $\mathscr{L}^1(\mu)$, where $\mu$ is Lebesgue measure, and that in this case,

$$\int_{\mathbf{R}^*} f\, d\mu^* = \int_{\mathbf{R}-\{0\}} f(x)|x|^{-1}\, dx.$$

(b) Show $\mu^*$ is invariant under multiplicative translations, and so is the integral on $\mathbf{R}^*$ with respect to $\mu^*$. (Multiplicative translations are of type $x \mapsto ax$ for $a \neq 0$.)

20. Not all sets are measurable. Consider the reals modulo the rational numbers, and in each coset $x + \mathbf{Q}$, $x$ real, select an element $y$ such that $0 \leqq y < 1$. Show that the set consisting of all such elements cannot be Lebesgue measurable. [*Hint:* Use the countable additivity to show that this set cannot be measurable.]

21. Let $X$ be a measured space with finite measure $\mu(X)$. Let $f \in \mathscr{L}^1(\mu)$. Compute the limit

$$\lim_{n\to\infty} \int_X |f(x)|^{1/n}\, d\mu(x).$$

22. **Arbitrary products.** Let $(X_n, \mathscr{M}_n, \mu_n)$ be a family of measured spaces such that $\mu_n(X_n) = 1$ for almost all $n$ (meaning all but a finite number of $n$). Let

$$X = \prod X_n$$

be the product space. Let $\mathscr{M}$ be the $\sigma$-algebra generated by all sets of the form

$$A = \prod A_n,$$

where $A_n$ is measurable in $X_n$, and $A_n = X_n$ for almost all $n$. Then $(X, \mathscr{M})$ is a measurable space. A set $A$ as above is called **decomposable**.

(a) Show that there exists a unique measure $\mu$ on $(X, \mathscr{M})$ such that for every decomposable set as above, we have

$$\mu(A) = \prod \mu_n(A_n).$$

(b) Let $f_n \in \mathscr{L}^1(\mu_n)$ and assume that $f_n$ is the characteristic function of $X_n$ for almost all $n$. Show that the product function $f = \otimes f_n$ is in $\mathscr{L}^1(\mu)$, and

that

$$\int f\,d\mu = \prod \int f_n\,d\mu_n.$$

*Note:* Do this exercise first for finite products.

23. **Dini's theorem.** The dominated convergence therem is useful in almost all instances, but sometimes, one wants a more delicate criterion for the interchange of a limit and an integral. This may be provided by Dini's theorem, as follows.

**Dini's Theorem.** *Let $\{F_n\}$ be a sequence of functions on $[a, \infty)$ such that for all $n$ we have $F_n \in L^1([a, B])$ for all $B \geqq a$. Let*

$$I_n(x) = \int_a^x F_n.$$

*Assume:*

(a) *The sequence $\{F_n\}$ converges pointwise to a function $F$, uniformly on each finite interval $[a, B]$.*

(b) *The sequence $\{I_n\}$ converges uniformly on $[a, \infty)$.*

(c) *The improper integrals*

$$\int_a^\infty F_n = \lim_{B\to\infty} \int_a^B F_n \quad \textit{and} \quad \int_a^\infty F = \lim_{B\to\infty} \int_a^B F$$

*exist.*

*Then*

$$\lim_{n\to\infty} \int_a^\infty F_n = \int_a^\infty F.$$

*Proof.* Given $\varepsilon$, we consider

$$(1) \qquad \int_a^\infty (F_n - F_m) = \int_a^B (F_n - F_m) + \int_a^B (F_n - F) + \int_B^\infty (F_n - F).$$

By assumption (b) there exists $n_0$ such that for all $m, n \geqq n_0$ we have

$$\left|\int_a^B (F_n - F_m)\right| < \varepsilon \qquad \text{for all } B.$$

Let $n \geqq n_0$. By assumption (c), there is $B(n)$ and $B(\infty)$ such that

$$\left|\int_B^\infty F_n\right| < \varepsilon \quad \text{for } B \geqq B(n) \qquad \text{and} \qquad \left|\int_B^\infty F\right| < \varepsilon \quad \text{for } B \geqq B(\infty).$$

We now pick $B = \max(B(n), B(\infty))$. Then the third integral on the right of (1) is bounded by $2\varepsilon$. Using (a) we select $m$ sufficiently large so that the second integral in (1) is bounded by $\varepsilon$. This concludes the proof.

24. **Carathéodory's criterion.** Assume that $M$ is the $\sigma$-algebra of all $\mu$-measurable subsets of $X$ in the sense of §7. Let $f: X \to Y$ be a map of $X$ into a metric space $Y$. Prove that $f$ is measurable if and only if for every subset $Z$ of $X$ and every two subsets $B$, $C$ of $Y$ satisfying

$$\operatorname{dist}(B, C) > 0,$$

we have

$$(*) \qquad \mu(Z) \geqq \mu(Z \cap f^{-1}(B)) + \mu(Z \cap f^{-1}(C)).$$

[*Hint*: One direction is obvious. Conversely, assume $(*)$. Let $A$ be closed in $Y$. Let:

$$B_m = \left\{ y \in Y \,\middle|\, \frac{1}{m+1} \leqq \operatorname{dist}(y, A) \leqq \frac{1}{m} \right\},$$

$$C_m = \left\{ y \in Y \,|\, \operatorname{dist}(y, A) > \frac{1}{m} \right\},$$

$$B'_m = \left\{ y \in Y \,|\, 0 < \operatorname{dist}(y, A) \leqq \frac{1}{m} \right\}.$$

Then $C_m \cup B'_m = \mathscr{C}A$. Prove that for any subset $Z$ of $X$ we have

$$\mu(Z \cap f^{-1}(A)) + \mu(Z - f^{-1}(A)) \leqq \mu(z) + \mu(Z \cap f^{-1}(B'_m)).$$

Then show that

$$\lim_{m \to \infty} \mu(Z \cap f^{-1} B'_m) = 0$$

by considering the sums $\sum \mu(Z \cap f^{-1} B_k)$ for $k$ even and $k$ odd, and applying the hypothesis $(*)$.]

25. Let $X$ be a metric space and $\mathscr{F}$ a family of subsets of $X$ whose union covers $X$. Let

$$\varphi: \mathscr{F} \to \mathbf{R} \cup \{\infty\}$$

be a non-negative function. For every $c > 0$ and $A \subset X$, let

$$\mu_c(A) = \inf_{\mathscr{G}} \sum_{F \in \mathscr{G}} \varphi(F),$$

where $\mathscr{G}$ is a family of $\mathscr{F}$ such that:
(i) The union of the elements of $\mathscr{G}$ covers $A$.
(ii) If $F \in \mathscr{G}$, then $\operatorname{diam} F \leqq c$.

(a) Prove that $\mu_c$ is an outer measure on $X$. Define $\mu(A) = \lim_{c \to 0} \mu_c(A)$.
(b) Prove that $\mu$ is an outer measure, called the **Carathéodory measure associated with** $(\varphi, \mathscr{F})$.

26. Let $\mu$ be the Carathéodory measure associated with $(\varphi, \mathscr{F})$.
(a) Prove that the open sets of $X$ are $\mu$-measurable. (Use Carathéodory's criterion applied to the identity map.)

(b) If all elements of $\mathscr{F}$ are Borel sets, prove that for any subset $A$ of $X$ we have

$$\mu(A) = \inf \mu(B),$$

the inf being taken for all Borel sets $B$ containing $A$.

**Examples.** For $F \subset \mathbf{R}^m$ let

$$\varphi(F) = v_m 2^{-m}(\text{diam } F)^m,$$

where $v_m$ is the volume of the $m$-dimensional ball in $\mathbf{R}^m$, and let $\mathscr{F}$ be the family of all open sets of $\mathbf{R}^m$. The associated Carathéodory measure is called the $m$-dimensional **Hausdorff measure** $\mathscr{H}^m$.

CHAPTER VII

# Duality and Representation Theorems

*Throughout this chapter $(X, \mathcal{M}, \mu)$ is a measured space.*

## VII, §1. THE HILBERT SPACE $L^2(\mu)$

Consider first complex valued functions. We let $\mathcal{L}^2(\mu)$ be the set of all functions $f$ on $X$ that are limits almost everywhere of a sequence of step functions (i.e. $\mu$-measurable), and such that $|f|^2$ lies in $\mathcal{L}^1$. Thus

$$|f|^2 = f\bar{f}.$$

If we wish to consider a Hilbert space $E$ instead of $\mathbf{C}$, we let $\mathcal{L}^2(\mu, E)$ be the set of all maps $f\colon X \to E$ that are limits almost everywhere of a sequence of step maps, and such that $|f|^2$ lies in $\mathcal{L}^1$. There is no change from the preceding definition. In this case,

$$|f|^2 = \langle f, f \rangle,$$

the value of $\langle f, g \rangle$ at $x$ being given by the scalar product $\langle f(x), g(x) \rangle$ in $E$.

The reader interested only in the complex numbers can take $E = \mathbf{C}$ and the product to be $\langle f, g \rangle = f\bar{g}$, where the bar denotes complex conjugation. Not a *single* proof, however, will be made shorter or simpler.

**Theorem 1.1.** *The set $\mathcal{L}^2(\mu)$ is a vector space. If $f$, $g \in \mathcal{L}^2(\mu)$, then $\langle f, g \rangle$ is in $\mathcal{L}^1(\mu)$, and the map*

$$(f, g) \mapsto \int_X \langle f, g \rangle \, d\mu$$

*is a positive hermitian product on* $\mathscr{L}^2(\mu)$ *(not necessarily positive definite).*

*Proof.* The map $\langle f, g \rangle$ is obviously a limit almost everywhere of step maps, and we have

$$2|\langle f, g \rangle| \leqq |f|^2 + |g|^2.$$

Thus the absolute value is bounded by a function in $\mathscr{L}^1$, whence by Corollary 5.9 of the dominated convergence theorem, Chapter 11, it follows that $\langle f, g \rangle$ is in $\mathscr{L}^1$. As for the fact that $\mathscr{L}^2$ is a vector space, let $f, g \in \mathscr{L}^2$. We have

$$|f + g|^2 \leqq |f|^2 + 2|\langle f, g \rangle| + |g|^2,$$

whence the same reference shows that $f + g \in \mathscr{L}^2$. It is clear that if $\alpha$ is a number, then $\alpha f$ is in $\mathscr{L}^2$, so $\mathscr{L}^2$ is a vector space. The last assertion is now obvious.

We denote our hermitian product by

$$\langle f, g \rangle_\mu = \int_X \langle f, g \rangle \, d\mu.$$

We have the usual properties, like the Schwarz inequality. The **$L^2$-seminorm** is defined by

$$\|f\|_2 = \langle f, f \rangle_\mu^{1/2}.$$

**Corollary 1.2.** *We have* $\|f\|_2 = 0$ *if and only if* $f$ *is equal to* 0 *almost everywhere.*

*Proof.* This is really a statement about $|f|^2$, which is in $\mathscr{L}^1$, and we know this result already.

**Corollary 1.3.** *If* $X$ *has finite measure and* $f \in \mathscr{L}^2(\mu)$, *then actually* $f$ *is in* $\mathscr{L}^1(\mu)$ *and* $\|f\|_1 \leqq \|f\|_2 \|1_X\|_2$.

*Proof.* We apply the theorem and the Schwarz inequality to the pair $|f|$ and $1_X$ (the constant 1 on $X$).

We can form the space $L^2(\mu)$ of equivalence classes of maps in $\mathscr{L}^2$, differing only on a set of measure 0. We see that *the hermitian product is positive definite on* $L^2(\mu)$.

**Theorem 1.4.** *Let* $\{f_n\}$ *be an* $L^2$*-Cauchy sequence in* $\mathscr{L}^2$. *Then there exists some* $f$ *in* $\mathscr{L}^2$ *having the following properties:*

(i) *The sequence* $\{f_n\}$ *is* $L^2$-*convergent to* $f$, *so that* $\mathscr{L}^2$ *is complete, and* $L^2(\mu)$ *is a Hilbert space.*

*There exists a subsequence of* $\{f_n\}$ *having the following properties*:

(ii) *This subsequence converges almost everywhere to* $f$.
(iii) *Given* $\varepsilon$, *there exists a set* $Z$ *with* $\mu(Z) < \varepsilon$ *such that the convergence of this subsequence is uniform on the complement of* $Z$.

*Proof.* As before, we really prove these statements in reverse order. We may assume all $f_n$ measurable. Taking a subsequence if necessary we may assume that for $m \geqq n$ we have

$$\|f_n - f_m\|_2^2 < \frac{1}{2^{2n}}.$$

We let $Y_n$ be the set of $x \in X$ such that

$$|f_{n+1}(x) - f_n(x)|^2 \geqq \frac{1}{2^n}.$$

Then $Y_n$ has finite measure, and the proof of Lemma 3.1 in the preceding chapter goes through as before. We have $\mu(Y_n) \leqq 1/2^n$, and we let

$$Z_n = Y_n \cup Y_{n+1} \cup \cdots.$$

If $x \notin Z_n$, then for $k \geqq n$ we have

$$|f_{k+1}(x) - f_k(x)|^2 < \frac{1}{2^k}$$

so that the series

$$f_1 + \sum_{k=1}^{\infty} (f_{k+1} - f_k)$$

converges uniformly and absolutely on the complement of $Z_n$ for each $n$, whence pointwise and absolutely on the complement of $Z$ (intersection of all $Z_n$). This already proves (ii) and (iii).

Let $f(x)$ be the limit of $f_n(x)$ as $n \to \infty$ if $x \notin Z$, and let $f(x) = 0$ if $x \in Z$. There remains to prove that $f$ is in $\mathscr{L}^2$ and that $\{f_n\}$ is $L^2$-convergent to $f$. The expression

$$\int_X |f_n - f_m|^2 \, d\mu$$

is the $L^1$-seminorm of $|f_n - f_m|^2$. We fix $m$ and take the limit as $n \to \infty$. We can apply Fatou's lemma, and conclude that $|f - f_m|^2$ is in $\mathscr{L}^1$,

whence $f - f_m$ is in $\mathscr{L}^2$. Since $\mathscr{L}^2$ is a vector space, and $f_m \in \mathscr{L}^2$, we conclude that $f \in \mathscr{L}^2$. Fatou's lemma also shows that

$$\int_X |f - f_m|^2 \, d\mu \leqq \lim_{n\to\infty} \inf_{k \geqq n} \int_X |f_k - f_m|^2 \, d\mu,$$

so that for large $m$ we see that $\|f - f_m\|_2$ is small, i.e. the sequence $\{f_m\}$ is $L^2$-convergent to $f$. This proves our theorem.

**Corollary 1.5.** *If $\{f_n\}$ is an $L^2$-Cauchy sequence in $\mathscr{L}^2$ and if $\{f_n\}$ converges almost everywhere to a map $f$, then $f$ is in $\mathscr{L}^2$ and $\{f_n\}$ is also $L^2$-convergent to $f$.*

*Proof.* Obvious.

**Theorem 1.6 (Dominated Convergence Theorem for $L^2$).** *Let $\{f_n\}$ be a sequence in $\mathscr{L}^2$ which converges pointwise almost everywhere to $f$. Assume that there exists $g \in \mathscr{L}^2(\mu, \mathbf{R})$ such that $g \geqq 0$ and such that $|f_n| \leqq g$. Then $f$ is in $\mathscr{L}^2$ and $\{f_n\}$ is $L^2$-convergent to $f$.*

*Proof.* The proof is essentially the same as in the $\mathscr{L}^1$ case. For each positive integer $k$ let

$$g_k = \sup_{m,n \geqq k} |f_n - f_m|.$$

Then $\{g_k\}$ is a decreasing sequence of real valued functions, and for $m$, $n \geqq k$ we have $|f_n - f_m| \leqq 2g$. Therefore by Corollary 5.9 of the monotone convergence theorem (Chapter VI) it follows that

$$g_k^2 = \sup_{m,n \geqq k} |f_n - f_m|^2$$

is in $\mathscr{L}^1$. By the monotone convergence theorem and the hypothesis, the sequence $\{g_k\}$ converges almost everywhere to 0. Hence $\{f_n\}$ is actually an $L^2$-Cauchy sequence, and we can apply Corollary 1.5 to conclude the proof.

**Corollary 1.7.** *The step maps are dense in $\mathscr{L}^2$.*

*Proof.* Let $\{\varphi_n\}$ be a sequence of step maps converging pointwise to an element $f$ of $\mathscr{L}^2$. Then $f$ is measurable. Define

$$\psi_n(x) = \varphi_n(x) \quad \text{if } |\varphi_n(x)| \leqq 2|f(x)|,$$
$$\psi_n(x) = 0 \quad \text{if } |\varphi_n(x)| > 2|f(x)|.$$

Then $\psi_n$ is a step map for each $n$, and the sequence $\{\psi_n\}$ converges pointwise to $f$. Furthermore, $|\psi_n| \leqq 2|f|$ for all $n$. The theorem shows that $\{\psi_n\}$ is $L^2$-convergent to $f$. Any element of $\mathscr{L}^2$ is equivalent to one for which you can find a sequence $\{\varphi_n\}$ as above. Hence our corollary is proved.

## VII, §2. DUALITY BETWEEN $L^1(\mu)$ AND $L^\infty(\mu)$

As Corollary 5.11 of the dominated convergence theorem, in Chapter VI we found that if $f \in \mathscr{L}^1$ and $g$ is a bounded $\mu$-measurable function, then $fg$ is in $\mathscr{L}^1$. We now investigate this property more closely. Half of what we do in this section will be valid in Hilbert space without changing the proofs at all, but again the reader who wishes to understand everything in terms of complex or real valued functions is welcome to do so throughout.

We could put the sup norm on the space of step maps, but it is convenient to adjust this norm in terms of the given measure $\mu$, and thus define what is called the essential sup, as well as the completion of the space of step maps under this seminorm. We define $\mathscr{L}^\infty(\mu)$ to be the vector space of maps $f$ such that there exists a bounded $\mu$-measurable $g$ equal to $f$ almost everywhere. Properties relating to the integral with respect to $\mu$ hold for equivalence classes of such maps. Therefore, if $f \in \mathscr{L}^\infty(\mu)$ it is natural of define its **essential sup** to be

$$\text{ess sup}(f) = \|f\|_\infty = \inf_g \|g\|,$$

where $\| \ \|$ is the sup norm, and the inf is taken over all bounded $\mu$-measurable maps $g$ equal to $f$ almost everywhere. Alternatively, for each $c \geqq 0$ let $S_c$ be the set of all $x$ such that $|f(x)| \geqq c$. We could have defined $\|f\|_\infty$ by the condition:

$$\|f\|_\infty = \inf \textit{ of the set of all numbers } c \textit{ such that } \mu(S_c) = 0.$$

The equivalence between the two conditions is immediately verified. (For instance if $c > b$ and $\mu(S_c) > 0$, then $|f(x)| \geqq c$ for all $x$ in a set of measure $> 0$, so that for all $g$ equivalent to $f$ we must have $\|g\| \geqq c$ also. This proves that $c \leqq b$. The reverse inequality is equally clear.) We also see at once that $\| \ \|_\infty$ is a seminorm on $\mathscr{L}^\infty(\mu)$. By definition, the set of $x$ such that $|f(x)| > \|f\|_\infty$ has measure 0.

If $f \in \mathscr{L}^\infty(\mu)$, it is clear that we have $\|f\|_\infty = 0$ if and only if $f$ is equal to 0 almost everywhere. Consequently, we can form the space $L^\infty(\mu)$ of equivalence classes of elements of $\mathscr{L}^\infty(\mu)$, and we shall see in a moment that $L^\infty(\mu)$ is a Banach space.

**Theorem 2.1.**

(i) *The space* $\mathscr{L}^\infty(\mu)$ *is complete. If* $\{f_n\}$ *is an* $L^\infty$*-Cauchy sequence in* $\mathscr{L}^\infty(\mu)$, *then there exists a set* $Z$ *of measure* 0 *such that the convergence of* $\{f_n\}$ *is uniform on the complement of* $Z$.

(ii) *If* $E$ *is finite dimensional, then the simple maps are dense in* $L^\infty(\mu, E)$.

(iii) *If* $\mu(X)$ *is finite, then given* $\varepsilon$ *and* $f \in \mathscr{L}^\infty(\mu)$, *there exists a step map* $\varphi$ *and a set* $Z$ *with* $\mu(Z) < \varepsilon$ *such that*

$$|f - \varphi| < \varepsilon \qquad \textit{on the complement of } Z.$$

*Proof.* To prove the first statement, let $\{f_n\}$ be an $L^\infty$-Cauchy sequence in $\mathscr{L}^\infty(\mu)$. Let $Z$ be the set of all $x$ such that we have

$$|f_n(x)| > \|f_n\|_\infty$$

or

$$|f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty$$

for some $n$, or some pair $m$, $n$. Then $Z$ has measure 0, and the convergence of the sequence is uniform on the complement of $Z$. We let $f$ have value 0 in $Z$ and be the uniform limit of the sequence $\{f_n\}$ on the complement of $Z$. Then $f \in \mathscr{L}^\infty(\mu)$, and clearly is the $L^\infty$ limit of $\{f_n\}$.

Now assume that $E$ is finite dimensional, or say equal to the complex numbers for concreteness. Let $f \in \mathscr{L}^\infty(\mu, \mathbf{C})$. After replacing $f$ by an equivalent function, we may assume that $f$ is measurable and bounded. Say the values of $f$ are contained in a square. We cut up the square into small $\varepsilon$-squares which are disjoint, and take their inverse images in $X$. These give a partition of $X$ and we can define a simple function with respect to this partition by giving the function any one of its values in a given square. To get our small squares, let, say, $e_1$, $e_2$ be the standard unit vectors in $\mathbf{C} = \mathbf{R}^2$, and let $S$ be the square

$$te_1 + ue_2$$

with $0 \leqq t < \varepsilon$ and $0 \leqq u < \varepsilon$. The translates

$$S + n\varepsilon e_1 + m\varepsilon e_2 = S_{m,n}$$

with integers $m$, $n$, are disjoint. If $N$ is large and we take

$$-N \leqq n \leqq N \qquad \text{and} \qquad -N \leqq m \leqq N,$$

then our small squares $S_{m,n}$ cover the image of $f$ as desired. The argument also works in any finite dimensional space, taking unit vectors $e_1, \ldots, e_p$ in $\mathbf{R}^p$.

Finally, for the third part of the theorem, if $\mu(X)$ is finite, then any element of $\mathscr{L}^\infty(\mu)$ is in $\mathscr{L}^1(\mu)$, and our assertion follows from the fact that elements of $\mathscr{L}^1$ are $L^1$-limits of step maps, together with the fundamental lemma of integration, or Theorem 5.2 of Chapter VI.

**Remark.** We phrased the density of (ii) in terms of simple maps. Recall that step maps are assumed to be equal to 0 outside a set of finite measure. Thus the step maps cannot possibly be dense in $L^\infty(\mu)$ if $\mu(X)$ is infinite, since the constant function 1 cannot be uniformly approximated by step functions in that case. If we restrict our attention to the case when $\mu(X)$ is finite, then step maps and simple maps coincide. In applications this suffices, since one deals mostly with $\sigma$-finite measures, and certain problems can be reduced to the case of finite measures. The density statement of (iii) is also useful in the infinite dimensional case.

Consider now the case of functions (complex valued, say). We have a bilinear map

$$\mathscr{L}^1(\mu, \mathbf{C}) \times \mathscr{L}^\infty(\mu, \mathbf{C}) \to \mathbf{C}$$

given by

$$(f, g) \mapsto \int_X fg \, d\mu = [f, g]_\mu$$

This arises from Corollary 5.11 of the dominated convergence theorem (Chapter VI). It is clear that the value of this map on $(f, g)$ depends only on the equivalence class of $f$ and $g$, respectively, and thus defines a bilinear map

$$L^1(\mu) \times L^\infty(\mu) \to \mathbf{C}.$$

Without changing anything above except the notation slightly, if we write $\langle f, g \rangle$ instead of $fg$, and take the values of $f$, $g$ in a Hilbert space $E$, then what we said holds, except that as usual, the map is not bilinear but *sesquilinear* (i.e. linear in its first variable, but anti-linear in its second variable, that is a complex conjugation occurs when we multiply $g$ by a constant). For the convenience of the reader, we shall state our results first for functions, and then for the Hilbert case. There will be absolutely no difference in the proofs except for this change between $fg$ and $\langle f, g \rangle$.

Quite generally, let

$$\tau \colon F \times G \to H$$

be a bilinear map of vector spaces into another vector space. If $v \in F$ and $w \in G$ we write $v \perp w$ and say that $v$ is **orthogonal** to $w$ if $\tau(v, w) = 0$. We define the **kernel** on the left to consist of all $v \in F$ such that $v \perp G$, i.e. $v$ is orthogonal to all elements of $G$, and similarly we define the

kernel on the right. These kernels are clearly subspaces of $F$ and $G$, respectively. We say that the bilinear map is **non-degenerate** if the kernels on the left and right are equal to 0. Suppose that $F$, $G$ are normed vector spaces (or semi-normed) and that the bilinear map is continuous. In applications, the condition

$$|\tau(v, w)| \leqq |v|\,|w|$$

is even satisfied. Then we obtain corresponding mappings of $F$ and $G$ into each other's dual spaces, namely each $v \in F$ gives rise to the functional $\lambda_v \in G'$ given by

$$\lambda_v(w) = \tau(v, w).$$

Similarly, each $w \in G$ gives rise to the function $\lambda_w$ in $F'$ given by

$$\lambda_w(v) = \tau(v, w).$$

We investigate this situation when we deal with the spaces $L^1(\mu)$ and $L^\infty(\mu)$.

**Theorem 2.2.** *Let $\mu$ be $\sigma$-finite. The kernels on the right and left of the bilinear map*

$$L^1(\mu) \times L^\infty(\mu) \to \mathbf{C}$$

*are 0. This map satisfies the product inequality*

$$|[f, g]_\mu| \leqq \|fg\|_1 \leqq \|f\|_1 \|g\|_\infty .$$

*The maps $g \mapsto \lambda_g$ and $f \mapsto \lambda_f$ for $g \in \mathscr{L}^\infty(\mu)$ and $f \in \mathscr{L}^1(\mu)$ induce norm-preserving linear maps of $L^\infty(\mu)$ and $L^1(\mu)$, respectively, into the other's dual space. In the case of $L^\infty(\mu)$, the map $g \mapsto \lambda_g$ is a norm-preserving isomorphism between $L^\infty(\mu)$ and the dual space of $L^1(\mu)$, i.e. the map is surjective.*

*Proof.* Let $f \in \mathscr{L}^1(\mu)$ be orthogonal to $\mathscr{L}^\infty(\mu)$. Then $f$ is 0 almost everywhere by Corollary 5.19 of Chapter VI (the averaging theorem). The other side works similarly as follows. If $g$ is bounded $\mu$-measurable, then for every measurable subset $A$ of finite measure, the map $g_A = g\chi_A$ is in $\mathscr{L}^1$. We can therefore apply the same argument, and see that $g_A$ is 0 almost everywhere, whence $g$ is 0 almost everywhere since $\mu$ is assumed $\sigma$-finite.

Let $C$ be a bound for $g$. Then

$$\|fg\|_1 = \int_X |fg|\, d\mu \leqq C \int_X |f|\, d\mu = C\|f\|_1 .$$

This implies our inequality

$$|[f, g]_\mu| \leqq \|fg\|_1 \leqq \|f\|_1 \|g\|_\infty,$$

and shows that $|\lambda_g| \leqq \|g\|_\infty$. For the reverse, let $b = |\lambda_g|$. For each subset $A$ of finite measure, we have

$$\left| \int_X \chi_A g \, d\mu \right| \leqq b\mu(A).$$

By Corollary 5.18 of the averaging theorem of Chapter VI, we conclude that $|g(x)| \leqq b$ for almost all $x$, whence $\|g\|_\infty \leqq b$. Therefore $|\lambda_g| = \|g\|_\infty$.

Now on the other side, let $f \in \mathscr{L}^1(\mu)$, and define $1/|f|$ to be the map having value $1/|f(x)|$ if $f(x) \neq 0$ and $0$ if $f(x) = 0$. Then $1/|f|$ is $\mu$-measurable, and $\bar{f}/|f|$ is $\mu$-measurable and bounded. Let $g = \bar{f}/|f|$. Then from $\|g\|_\infty = 1$ or $0$, we get

$$\|f\|_1 = \int_X fg \, d\mu = \lambda_f(g) \leqq |\lambda_f|.$$

This proves the reverse inequality, whence $\|f\|_1 = |\lambda_f|$.

This proves all our statements except the last, that $L^\infty(\mu)$ provides us with all functionals on $L^1(\mu)$. To see this, we give the argument of von Neumann (originally applied to the Radon–Nikodym theorem, see below). Assume first that $X$ has finite measure. Let $\lambda: L^1(\mu) \to \mathbf{C}$ be a functional, and let $b$ be its norm so that we have

$$|\lambda f| \leqq b\|f\|_1$$

for all $f \in \mathscr{L}^1(\mu)$. The functional $\lambda$ can actually be viewed as defined on $\mathscr{L}^2(\mu)$ because any map $g$ in $\mathscr{L}^2$ on a set of finite measure $X$ is in $\mathscr{L}^1$ (use the Schwarz inequality on the pair $|g|$ and the function $1_X$). Thus we obtain

$$|\lambda f| \leqq b\|f\|_1 = b \int_X |f| \cdot 1_X \, d\mu \leqq \|f\|_2 \|1_X\|_2.$$

This shows that $\lambda$ is continuous with respect to the $L^2$-seminorm. Since $L^2(\mu)$ is a Hilbert space, there exists $g \in \mathscr{L}^2(\mu)$ such that we have

$$\lambda f = \int_X fg \, d\mu$$

for all step maps $f$. For any measurable set $A$ of finite measure, we then

obtain

$$\left|\int_X \chi_A g \, d\mu\right| = |\lambda(\chi_A)| \leqq b\|\chi_A\|_1 \leqq b\mu(A).$$

By Corollary 5.18 of the averaging theorem (Theorem 5.15, Chapter VI), it follows that $|g(x)| \leqq b$ for almost all $x$, whence $g$ is in fact in $\mathscr{L}^\infty(\mu)$ and $\|g\|_\infty \leqq b$. Since

$$\lambda f = [f, g]_\mu$$

for all step maps $f$, this same relation must hold true for all $f \in \mathscr{L}^1(\mu)$ because the step maps are dense in $\mathscr{L}^1$. This proves our last assertion when $X$ has finite measure.

The general case when $\mu$ is $\sigma$-finite follows easily. We write $X$ as a disjoint union of sets of finite measure $X_k$ $(k = 1, 2, \ldots)$. Let $f \in \mathscr{L}^1(\mu)$, and let $f_k = f_{X_k}$ be the same as $f$ on $X_k$ and 0 outside $X_k$. Then the series

$$\sum_{k=1}^{\infty} f_k$$

is $L^1$-convergent to $f$, say by the dominated convergence theorem, and therefore by the continuity of $\lambda$ we have

$$\lambda f = \sum_{k=1}^{\infty} \lambda(f_k).$$

For each $k$ there exists a $\mu$-measurable map $g_k$ on $X_k$, bounded by $b$, and 0 outside $X_k$ such that

$$\lambda(f_k) = \int_X f_k g_k \, d\mu.$$

We let

$$g = \sum_{k=1}^{\infty} g_k$$

(pointwise). Then $g$ is bounded by $b$, $\mu$-measurable, and it is clear that $\lambda = \lambda_g$, thus concluding the proof of our theorem.

We now repeat the statement of Theorem 2.2 for the Hilbert case.

**Theorem 2.3 (Hilbert Case).** *Assume that $\mu$ is $\sigma$-finite. Let $E$ be a Hilbert space. We have a sesquilinear map*

$$L^1(\mu, E) \times L^\infty(\mu, E) \to \mathbf{C}$$

*defined for $f \in \mathscr{L}^1(\mu, E)$ and $g \in \mathscr{L}^\infty(\mu, E)$ by*

$$(f, g) \mapsto \langle f, g \rangle_\mu = \int_A \langle f, g \rangle \, d\mu.$$

*The kernels on both sides are 0. The map $g \mapsto \lambda_g$ induces a norm-preserving linear map of $L^\infty(\mu, E)$ onto the dual of $L^1(\mu, E)$ (so the map is surjective), and the map $f \mapsto \lambda_f$ induces a norm-preserving antilinear map of $L^1(\mu, E)$ into the dual of $L^\infty(\mu, E)$ (not necessarily surjective).*

*Proof.* Exactly the same, except that when for instance we considered

$$\left| \int_X \chi_A g \, d\mu \right| \leqq b\mu(A)$$

in the proof of Theorem 2.2, we now have to write $\langle e\chi_A, g \rangle$ for some unit vector $e \in E$, and apply Corollary 5.20 instead of Corollary 5.18 of the averaging theorem.

We wish to characterize those elements of the dual of $L^\infty(\mu)$ which can be represented by some element in $\mathscr{L}^1(\mu)$. Over the complex numbers, the classical Radon–Nikodym theorem achieves this purpose, and can be viewed as stating that if a functional on $L^\infty(\mu)$ can be represented by a finite measure, then it already can be represented by a function. We first make some comments in this case.

Let $\nu$ be a positive measure on $\mathscr{M}$. We say that $\nu$ is **absolutely continuous with respect to** $\mu$ or $\mu$**-absolutely continuous**, if we have $\nu(A) = 0$ whenever $\mu(A) = 0$. (Cf. Exercise 1.) We say that a functional $\lambda$ on $L^\infty(\mu)$ can be **represented by a positive measure** $\nu$ if the functional has the form

$$\lambda \colon g \mapsto \int_X g \, d\nu.$$

We then write $\lambda = d\nu$. If the functional can be so represented, then $\nu$ is necessarily absolutely continuous with respect to $\mu$, because the functional vanishes on characteristic functions of measurable sets $A$ such that $\mu(A) = 0$. The Radon–Nikodym theorem in its classical form states:

*If $\nu$ is a finite positive measure on $\mathscr{M}$ which is $\mu$-absolutely continuous, then there exists some $f \in \mathscr{L}^1(\mu)$ such that for all $A \in \mathscr{M}$ we have*

$$\nu(A) = \int_A f \, d\mu.$$

This measure is conveniently denoted by $\mu_f$.

A functional $d\nu$ can be viewed as functional on various spaces (e.g. spaces of step maps, $L^\infty(\mu)$, etc.). We shall always make it explicit on which space we intend this functional to be. We observe that a functional on $L^\infty(\mu)$ represented by a map in $\mathscr{L}^1$ is determined by its values on step maps. Thus actually we can limit our attention to step maps.

But it is reasonable also to ask for functionals on simple maps, without any reference to the measure $\mu$, and with continuity with respect to the sup norm. For this purpose, we need another definition.

A positive measure $\nu$ on $\mathcal{M}$ is said to be **concentrated** or **carried** in a measurable set $A$ if $\nu(Y) = 0$ for all $Y$ in the complement of $A$. If $\nu_1$, $\nu_2$ are two positive measures on $\mathcal{M}$ we say that they are **orthogonal** or **singular** to each other and write $\nu_1 \perp \nu_2$ if there exists a decomposition

$$X = A \cup B$$

of $X$ into a disjoint union of measurable sets such that $\nu_1$ is concentrated in $A$ and $\nu_2$ is concentrated in $B$.

Let $\mathcal{L}^\infty(\mathcal{M}, \mathbf{C})$ denote the space of bounded measurable functions on $X$. We make no reference to any measure here at all, and we take the sup norm on this space. Let $\nu$ be a finite positive measure on $\mathcal{M}$. This means that $\nu(X) < \infty$. Then $\nu$ gives rise to a functional on $\mathcal{L}^\infty(\mathcal{M}, \mathbf{C})$ by the map

$$g \mapsto \int_X g \, d\nu$$

satisfying the bound

$$\left| \int_X g \, d\nu \right| \leqq \|g\| \, \nu(X).$$

**Theorem 2.4 (Radon–Nikodym and Lebesgue).** *Assume that $\mu$ is $\sigma$-finite, and let $\nu$ be a finite positive measure on $\mathcal{M}$. Then there exists a unique decomposition*

$$\nu = \nu_a + \nu_s$$

*as a sum of positive measures, such that $\nu_a$ is absolutely continuous with respect to $\mu$, and $\nu_s$ is singular with respect to $\mu$. We have $\nu_a \perp \nu_s$. Finally, if $\nu$ is absolutely continuous with respect to $\mu$, then there exists an element $f \in \mathcal{L}^1(\mu)$ such that $\nu = \mu_f$, and $f$ is uniquely determined up to equivalence. Furthermore, the functional on $L^\infty(\mu)$ represented by $\nu$ is then also represented by $f$, i.e. we have $d\nu = f \, d\mu$ on $L^\infty(\mu)$.*

*Proof* (von Neumann). The uniqueness is essentially obvious. If we can write

$$d\nu = f_1 \, d\mu + d\nu_s = f_2 \, d\mu + d\nu_s'$$

with $f_1, f_2 \in \mathcal{L}^1(\mu)$, and $\nu_s$, $\nu_s'$ singular with respect to $\mu$, then

$$(f_1 - f_2) \, d\mu = d\nu_s' - d\nu_s,$$

whence $f_1 - f_2$ is 0 almost everywhere, and $\nu_s' = \nu_s$.

Now for existence, we assume first that $\mu(X)$ is finite. Then $\mu + \nu$ is a finite positive measure, and we consider the integral with respect to $\mu + \nu$ on $\mathscr{L}^\infty(\mathscr{M})$, i.e. on the bounded measurable functions. Since all sets have finite measure, we don't need to specify that we deal with step functions vanishing outside a set of finite measure. Using the Schwarz inequality with respect to $L^2(\mu + \nu)$, we have for any step function $\varphi$:

$$\left|\int_X \varphi \, d\nu\right| \leqq \int_X |\varphi| \, d\nu \leqq \int_X |\varphi| \, d(\mu + \nu)$$
$$\leqq \|\varphi\|_2 \|1_X\|_2,$$

where $1_X$ is the function equal to 1 on $X$. Hence the map

$$\varphi \mapsto \int_X \varphi \, d\nu$$

is $L^2(\mu + \nu)$-continuous on step functions, whence it extends uniquely to a functional on $L^2(\mu + \nu)$. By the $L^2$ duality, there exists a function

$$h \in \mathscr{L}^2(\mu + \nu)$$

(uniquely determined up to equivalence) such that for all step functions $\varphi$ we have

$$\int_X \varphi \, d\nu = \int_X \varphi h \, d(\mu + \nu).$$

Letting $\varphi$ be the characteristic function of a measurable set $A$, we find

$$\int_A h \, d(\mu + \nu) = \nu(A) \leqq (\mu + \nu)(A).$$

By the averaging theorem (Theorem 5.15 of Chapter VI) we may assume without loss of generality that $0 \leqq h \leqq 1$, and setting $h$ equal to 0 on a set of $(\mu + \nu)$-measure 0, we may also assume that $h$ is measurable.

For step functions $\varphi$ we have

$$\int_X \varphi \, d(\mu + \nu) = \int_X \varphi \, d\mu + \int_X \varphi \, d\nu,$$

whence the same holds if $\varphi$ is any bounded measurable function, by the dominated convergence theorem. Consequently for $g \in \mathscr{L}^\infty(\mathscr{M})$ we have

$$(1) \qquad \int_X g \, d\nu = \int_X gh \, d\mu + \int_X gh \, d\nu.$$

Let $Y$ be the set of all $x \in X$ such that $0 \leqq h(x) < 1$, and let $Z$ be the set of all $x \in X$ such that $h(x) = 1$. First let $g$ be the characteristic function of $Z$. From (1) we see that $\mu(Z) = 0$. Let $g$ be arbitrary (bounded measurable) and iterate (1). By induction we obtain

$$\int_X g\,d\nu = \int_X g(h + h^2 + \cdots + h^n)\,d\mu + \int_X gh^n\,d\nu. \tag{2}$$

Take the limit as $n \to \infty$. The dominated convergence theorem shows that

$$\int_X gh^n\,d\nu \to \int_Z g\,d\nu \qquad \text{as} \quad n \to \infty.$$

Let

$$f = \frac{h}{1 - h}$$

on $Y$ and 0 outside $Y$. Since $\mu(Z) = 0$, the first integral on the right is really carried by $Y$, and taking the limit yields

$$\int_X g\,d\nu = \int_Y gf\,d\mu + \int_Z g\,d\nu.$$

We define $\nu_s$ to be the measure obtained from $\nu$ by

$$\nu_s(A) = \nu(A \cap Z).$$

We could also write $\nu_s = \nu_Z$. We let $\nu_a$ be the measure represented by $\mu_f$ on $Y$ and 0 outside $Y$. We see that our theorem is proved in the finite case.

The extension to the $\sigma$-finite case follows easily as in Theorem 2.2. We express $X$ as a disjoint union of measurable sets $\{X_k\}$ of finite measure, apply the finite result to each piece, and see that we get the expected convergence.

The Lebesgue part is the decomposition into absolutely continuous and singular measures. The representation of $\nu$ by $f$ is the Radon–Nikodym part of the theorem. We look further into this. It is reasonable to expect it to hold in Hilbert space, in the sense that if a functional on $L^\infty(\mu, E)$ can be represented by a "measure", then it can be represented by some $f \in \mathscr{L}^1(\mu)$. When I mentioned this to Palais, he pointed out to me that if one takes the right definition of measure, then the result follows at once from the positive case, and I am indebted to him for the following corollary.

If $\nu$ is a positive measure on $\mathscr{M}$, absolutely continuous with respect to

$\mu$, and $h \in \mathscr{L}^1(\nu, E)$ where $E$ is a Hilbert space, then we get a functional $\lambda$ on $L^\infty(\mu, E)$ by

$$g \mapsto \int_X \langle g, h \rangle \, d\nu = \lambda(g).$$

It will be convenient to denote this functional by $h\, d\nu$. In this case, we also say that $\lambda$ can be **represented by a finite ($E$-valued) measure**. This terminology will be justified in the next section.

**Corollary 2.5.** *Assume that $\mu$ is $\sigma$-finite. Let $E$ be a Hilbert space and let $\nu$ be a positive measure on $\mathscr{M}$, absolutely continuous with respect to $\mu$. Let $h \in \mathscr{L}^1(\nu, E)$. Then there exists $f \in \mathscr{L}^1(\mu, E)$, uniquely determined up to equivalence, such that $h\, d\nu = f\, d\mu$. In other words, if a functional on $\mathscr{L}^\infty(\mu, E)$ can be represented by a finite E-valued measure, then it can be represented by a map $f$ in $\mathscr{L}^1(\mu, E)$.*

*Proof.* Let $1/|h|$ denote the function equal to 0 at a point $x$ such that $h(x) = 0$, and equal to $1/|h(x)|$ if $h(x) \neq 0$. Then $h/|h|$ is $\mu$-measurable and bounded, and $|h|$ is in $\mathscr{L}^1(\nu, \mathbf{R})$. Then $|h|\, d\nu$ is a positive measure on $\mathscr{M}$, which is absolutely continuous with respect to $\mu$. By the positive Radon–Nikodym theorem, we conclude that $|h|\, d\nu = k\, d\mu$, where $k$ is positive and in $\mathscr{L}^1(\mu)$. Then

$$f = \frac{h}{|h|} k$$

is in $\mathscr{L}^1(\mu, E)$, being the product of a bounded $\mu$-measurable map and an element in $\mathscr{L}^1(\mu)$. It is then clear that this $f$ satisfies our requirements.

We shall see in the next section that, in fact, we can start from the "measure" point of view to arrive at our functionals.

## VII, §3. COMPLEX AND VECTORIAL MEASURES

*Let $\mathscr{M}$ be a $\sigma$-algebra in $X$ and $E$ a Banach space.*

Instead of considering real positive valued measures, we wish to investigate complex valued measures satisfying the same countable additivity property. It is then clearer to start with Banach valued measures, so that we see clearly where the property of finite dimensionality is used for certain results peculiar to the complex numbers. Again, no proof would be made shorter if we were to assume from the start that $E = \mathbf{C}$. In any case, finite or infinite dimensional spaces are useful in a number of applications.

By a **decomposition** of a measurable set $A$, we mean a sequence $\{A_n\}$

of disjoint measurable sets whose union is $A$. (We don't use the word partition, which was used for a finite decomposition of a set of finite measure with respect to a positive measure.) A map

$$\nu: \mathcal{M} \to E$$

is called **countably additive** if $\nu(\emptyset) = 0$, and if for every $A \in \mathcal{M}$ and every decomposition $\{A_n\}$ of $A$ we have

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A_n).$$

This infinite sum is to be interpreted as convergent to the same value, independent of the ordering of the terms. Its value is in $E$. We now consider properties of such a countably additive map.

The limiting properties of a positive measure are again satisfied in the present case, namely:

*Let $\{Y_n\}$ be an increasing sequence of measurable sets such that $\bigcup Y_n = Y$. Then*

$$\lim_{n\to\infty} \nu(Y_n) = \nu(Y).$$

*Similarly, if $\{Y_n\}$ is a decreasing sequence of measurable sets, and $Y = \bigcap Y_n$, then*

$$\lim_{n\to\infty} \nu(Y_n) = \nu(Y).$$

The proof is obvious, as for the positive measures. We define a function

$$|\nu|: \mathcal{M} \to [0, \infty]$$

by letting

$$|\nu|(A) = \sup \sum_{n=1}^{\infty} |\nu(A_n)|,$$

the sup being taken over all decompositions $\{A_n\}$ of $A$. We shall prove that $|\nu|$ is a positive measure, and that if $E = \mathbf{C}$ (or is finite dimensional), then $|\nu|$ is in fact real valued, i.e. finite.

*We observe that if $A \subset B$ are measurable, then*

$$|\nu|(A) \leqq |\nu|(B).$$

This is obvious, because if $\{A_n\}$ is a decomposition of $A$, then $\{A_n, B - A\}$ is a decomposition of $B$. In particular, if $|\nu|(B)$ is finite, so is $|\nu|(A)$.

**Theorem 3.1.** *Let $v: \mathcal{M} \to E$ be countably additive. Then $|v|$ is a positive measure.*

*Proof.* Let $\{A_n\}$ be a decomposition of $A \in \mathcal{M}$. Let $b_n$ be a real number $\geqq 0$ such that $b_n \leqq |v|(A_n)$. Let $\{A_{nj}\}$ be a decomposition of $A_n$ such that

$$b_n - \frac{\varepsilon}{2^n} \leqq \sum_{j=1}^{\infty} |v(A_{nj})|.$$

Then we may view $\{A_{nj}\}$ $(n, j = 1, 2, \ldots)$ as a decomposition of $A$, and therefore summing over $n$ yields

$$\sum_n b_n - \varepsilon \leqq \sum_n \sum_j |v(A_{nj})| \leqq |v|(A).$$

Taking the sup over all $\{b_n\}$ and letting $\varepsilon \to 0$, we get

$$\sum_n |v|(A_n) \leqq |v|(A).$$

Conversely, let $\{B_j\}$ be any decomposition of $A$. By the countable additivity of $v$ applied to the decomposition $\{A_n \cap B_j\}$ $(n = 1, 2, \ldots)$ of $B_j$, we get

$$\sum_j |v(B_j)| = \sum_j \left| \sum_n v(A_n \cap B_i) \right|$$

$$\leqq \sum_{j,n} |v(A_n \cap B_j)| \leqq \sum_n |v|(A_n).$$

This is true for all decompositions $\{B_j\}$ of $A$, whence we get the reverse inequality

$$|v|(A) \leqq \sum_n |v|(A_n),$$

thus proving our theorem.

The measure $|v|$ is sometimes called the **total variation** of $v$.

**Theorem 3.2.** *If $E$ is finite dimensional, and $v: \mathcal{M} \to E$ is countably additive, then $|v|$ is real valued, i.e. finite.*

*Proof.* The general case reduces at once to the real case (componentwise). We deal with the real case as in Saks [Sa]. Suppose that $|v|(X) = \infty$. We first observe that there exist measurable subsets of $X$ whose measures have arbitrarily large absolute values. This is seen as follows. We take a decomposition $\{X_n\}$ of $X$ such that

$$\sum_{n=1}^{\infty} |v(X_n)|$$

is large. We combine all those terms with indices $n$ such that $\nu(X_n)$ have the same sign. For either + or −, the corresponding sum will be large. We take a finite number of such $n$, but sufficiently many so that the sum of the corresponding $X_n$ is a subset $B$ with $|\nu(B)|$ large. All we need here is the finite additivity of $\nu$.

Now we construct a decreasing sequence of subsets of $X$ having measures whose absolute values tend to infinity. Let $X = A_1$. By what we have just seen, there exists a subset $B \subset A_1$ such that

$$|\nu(B)| \geqq |\nu(A_1)| + 2.$$

If $|\nu|(B) = \infty$, we let $A_2 = B$. If $|\nu|(B)$ is finite, then

$$|\nu|(A_1 - B) = \infty$$

and we let $A_2 = A_1 - B$. Then

$$|\nu(A_2)| \geqq |\nu(B)| - |\nu(A_1)| \geqq 2.$$

It is clear that we could have replaced 2 by any number. Repeating the procedure inductively, we get a decreasing sequence

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

such that $|\nu(A_n)| \geqq n$. Let $A = \bigcap A_n$. The countable additivity of $\nu$ now yields a contradiction, because

$$\nu(A) = \lim \nu(A_n).$$

This proves our theorem.

**Example.** The following is an example in which the conclusion of Theorem 3.2 fails. It is already in a paper of Birkhoff (*Trans. Amer. Math. Soc.*, **38** (1935) pp. 357–378). Let $E = l^2$ be the space of sequences $\{a_n\}$ of (say) real numbers such that $\sum a_n^2$ converges, with the standard scalar product. Then $E$ is a Hilbert space. Let $\mu$ be Lebesgue measure on the line. For each positive integer $n$ and measurable set $A$ let

$$\nu_n(A) = \frac{1}{n}\mu(A \cap [n-1, n]).$$

Let $\nu(A)$ be the sequence whose $n$-th term is $\nu_n(A)$. It is clear that the total variation of $\nu$ is infinite and that $\nu$ is countably additive on the positive line $X$ consisting of all real numbers $\geqq 0$.

By a **vectorial measure** on $\mathscr{M}$ we shall mean a countably additive map

$$\nu: \mathscr{M} \to E$$

such that $|\nu|(X)$ is finite, i.e. such that $|\nu|$ is a real valued positive measure. [Recall that if $A \subset B$, then $|\nu|(A) \leqq |\nu|(B)$.] For simplicity, we also call a vectorial measure a measure, and when we have to make a distinction with the objects discussed in Chapter VI or the preceding sections, we emphasize this and say **positive measure** for the former object. Another way of making the distinction is to say (even more correctly) an ***E*-valued measure** for our map $\nu: \mathscr{M} \to E$.

It is clear that $E$-valued measures form a vector space denoted by $M^1(\mathscr{M}, E)$, or simply $M^1$. For such a measure, we define

$$\|\nu\| = |\nu|(X).$$

Then it is verified at once that $\| \ \|$ is a norm (not merely a seminorm) on $M^1$. In fact, $M^1$ is **complete**, *i.e. a Banach space*. The proof is a routine $\varepsilon/2^n$ proof which we leave to the reader. Theorem 3.2 shows that the complex measures on $\mathscr{M}$ are precisely the complex valued, countably additive functions on $\mathscr{M}$.

**Note.** Our terminology is adjusted to the applications we are going to make. It would be more proper to define an $E$-valued measure to be simply a countably additive map $\nu: \mathscr{M} \to E$ such that $\nu(\emptyset) = 0$, and define then a **bounded** measure to be such a map that $|\nu|(X) < \infty$. In the sequel, we are concerned only with bounded measures or with complex measures (which are automatically bounded), so that we have taken the convention as described above.

**Example.** Let $\mu$ be a positive measure on $\mathscr{M}$ and let $f \in \mathscr{L}^1(\mu)$. Define $\mu_f$ by

$$\mu_f(A) = \int_A f \, d\mu.$$

Then it is immediately verified that $\mu_f$ is a measure, and that

$$\|\mu_f\| \leqq \|f\|_1.$$

We shall prove the reverse inequality after a remark, which it is convenient to formulate in a slightly more general context than we need for the next theorem. Let $\mu$ be a positive measure, and let $\nu: \mathscr{M} \to E$ be an $E$-valued measure. We say that $\nu$ is **absolutely continuous** with respect to $\mu$, or to put it shortly is **$\mu$-continuous**, if either one of the following two conditions is satisfied.

**AC 1.** *If $A \in \mathcal{M}$ and $\mu(A) = 0$, then $\nu(A) = 0$.*

**AC 2.** *Given $\varepsilon$, there exists $\delta$ such that $\mu(A) < \delta$ implies that $|\nu(A)| < \varepsilon$.*

We shall prove that these two conditions are equivalent. It is clear that **AC 2** implies **AC 1**. Conversely, assume **AC 1**. If **AC 2** is false, for each positive integer $n$ there exists a set $Y_n$ such that $\mu(Y_n) < 1/2^n$, but $|\nu(Y_n)| > \varepsilon$. Then $|\nu|(Y_n) > \varepsilon$. Let

$$Z_n = Y_n \cup Y_{n+1} \cup \cdots$$

and let $Z = \bigcap Z_n$. Then $\mu(Z) = 0$, but

$$|\nu|(Z) = \lim |\nu|(Z_n) \geqq \varepsilon$$

because $Y_n \subset Z_n$. Hence there is some measurable subset $Z'$ of $Z$ such that $\nu(Z') \neq 0$, contradicting **AC 1** because $\mu(Z') = 0$. This proves the equivalence between our two conditions.

**Remark 1.** *If $f \in \mathcal{L}^1(\mu)$, then the measure $\mu_f$ is obviously $\mu$-continuous.*

**Remark 2.** *The measure $\nu$ is $\mu$-continuous if and only if $|\nu|$ is $\mu$-continuous.*

**Theorem 3.3.** *Let $\mu$ be a positive measure on $\mathcal{M}$ and let $f \in \mathcal{L}^1(\mu)$. Then*

$$\|\mu_f\| = \|f\|_1.$$

*The map $f \mapsto \mu_f$ is a norm-preserving embedding $L^1(\mu) \to M^1$.*

*Proof.* It suffices to prove the inequality $\|f\|_1 \leqq \|\mu_f\|$. We may assume $\|f\|_1 > 0$. Given $\|f\|_1$, there exists a set $A$ of finite measure such that

$$\int_X |f| \, d\mu - \varepsilon \leqq \int_A |f| \, d\mu.$$

By $\mu$-continuity, there exists $\delta$ such that if $Z$ is a set with $\mu(Z) < \delta$, then

$$\int_Z |f| \, d\mu < \varepsilon.$$

By the fundamental lemma of integration (Lemma 3.1 of Chapter VI), there exists a step map $\varphi$ and a set $Z$ of measure $< \delta$ such that $Z$ is contained in $A$ and such that we have for all $x \in A - Z$:

$$|f(x) - \varphi(x)| < \frac{\varepsilon}{\mu(A)}.$$

Write

$$\varphi = \sum_{i=1}^{n} v_i \chi_{A_i},$$

where $\{A_i\}$ is a partition of $A - Z$, and let $\varphi$ be 0 outside $A$. We have:

$$\begin{aligned}\int_A |f|\, d\mu - \varepsilon \leqq \sum_i \int_{A_i} |f|\, d\mu &\leqq \sum_i \int_{A_i} \left(|\varphi| + \frac{\varepsilon}{\mu(A)}\right) d\mu \\ &= \sum_i \left|\int_{A_i} \varphi\, d\mu\right| + \sum_i \frac{\varepsilon\mu(A_i)}{\mu(A)} \\ &\leqq \sum_i \left|\int_{A_i} f\, d\mu\right| + 2\varepsilon \\ &\leqq |\mu_f|(A) + 2\varepsilon \leqq |\mu_f|(X) + 2\varepsilon.\end{aligned}$$

This proves our theorem.

**Corollary 3.4.** *For any step map g we have*

$$\int_X g\, d|\mu_f| = \int_X g|f|\, d\mu.$$

*Or symbolically, on step maps,*

$$d|\mu_f| = |f|\, d\mu.$$

*Proof.* For each measurable set $A$, we can apply Theorem 3.3 with respect to $A$ and get

$$|\mu_f|(A) = \int_A |f|\, d\mu.$$

The result for step maps follows by linearity.

We shall interpret measures as functionals. Let $E$ be a *Hilbert space* or the complex numbers and let $\nu$ be an $E$-valued measure on $\mathcal{M}$. We first view $\nu$ as inducing a linear map on step mappings with respect to $|\nu|$. Let $\varphi \in \mathrm{St}(|\nu|)$ and let us write

$$\varphi = \sum_{i=1}^{n} v_i \chi_{A_i}$$

where $\{A_1, \ldots, A_n\}$ is a partition of a set $A$ having finite $|\nu|$-measure. We define $d\nu$ by

$$\int_X \varphi\, d\nu = \langle \varphi, d\nu \rangle = \sum_{i=1}^{n} v_i \nu(A_i).$$

This obviously satisfies properties similar to those of the integral of Chapter VI, §2. Note that we wrote $v_i\nu(A_i)$ instead of

$$\langle v_i, \nu(A_i)\rangle$$

to fit the notation of functions better. In particular, since

$$|\nu(A_i)| \leqq |\nu|(A_i),$$

we have the inequality

$$|\langle \varphi, d\nu\rangle| \leqq \int_X |\varphi|\, d|\nu| \leqq \|\varphi\|_2 \|1_X\|_2$$

where the $L^2$-seminorm is taken with respect to $|\nu|$. (There is no other positive measure floating around at the moment.) Consequently $d\nu$ is $L^2$-continuous on $\mathrm{St}(|\nu|)$ and can thus be extended to a unique functional on $L^2(|\nu|)$ since the step maps are dense. By the $L^2$-duality, we know that there exists a unique (up to a set of $|\nu|$-measure 0) map $h \in \mathscr{L}^2(|\nu|)$ such that on all step maps,

$$d\nu = h\, d|\nu|.$$

In other words, for all step maps $\varphi$ we have

$$\langle \varphi, d\nu\rangle = \int_X \varphi h\, d|\nu|.$$

We shall say that $d\nu$ **is represented** by $h$. Since $|\nu|$ is finite, we know that $h$ is in $\mathscr{L}^1(|\nu|)$ (Schwarz inequality on $|h|$ and $1_X$).

We state the next theorem first for the complex numbers, for the convenience of the reader interested only in the complex case.

**Theorem 3.5.** *Let $\nu$ be a complex measure on $\mathscr{M}$. There exists a measurable function $h$ on $X$ such that $|h| = 1$ and such that for all*

$$\varphi \in \mathrm{St}(|\nu|, \mathbf{C})$$

*we have*

$$\langle \varphi, d\nu\rangle = \int_X \varphi h\, d|\nu|.$$

*This function $h$ is uniquely determined up to $|\nu|$-equivalence.*

*Proof.* We have already found such an $h$ in $\mathscr{L}^1$ and we must show that $|h| = 1$. We may assume that $h$ is measurable. For $r > 0$ let $S_r$ be

the set of all $x \in X$ such that $|h(x)| < r$. Let $\{A_n\}$ be a decomposition of $S_r$. Then

$$\sum_n |\nu(A_n)| = \sum_n \left| \int_X \chi_{A_n} h \, d|\nu| \right| \leqq \sum_n r|\nu|(A_n) = r|\nu|(S_r).$$

This shows that $|\nu|(S_r) \leqq r|\nu|(S_r)$. If $r < 1$, we must have $|\nu|(S_r) = 0$. Hence $|h(x)| \geqq 1$ for almost all $x$. Changing $h$ on a set of measure 0, we may assume $|h(x)| \geqq 1$ for all $x$.

For the reverse inequality, let $A$ be a measurable set. Then from the definition of $h$ we have

$$\int_X \chi_A h \, d|\nu| = \langle \chi_A, d\nu \rangle = |\nu(A)| \leqq |\nu|(A).$$

The averaging theorem (Theorem 5.15 of Chapter VI and its corollaries) shows that $|h| \leqq 1$ almost everywhere. This proves our theorem.

**Corollary 3.6 (Hahn–Jordan Decomposition of a Measure).** *Let $\nu$ be a real valued measure on $\mathcal{M}$ and define*

$$\nu^+ = \tfrac{1}{2}(|\nu| + \nu) \qquad \textit{and} \qquad \nu^- = \tfrac{1}{2}(|\nu| - \nu).$$

*Then the expression $\nu = \nu^+ - \nu^-$ gives a decomposition of $\nu$ into a difference of two mutually singular positive measures, and any such decomposition is uniquely determined. If $X = A \cup B$ is a decomposition into two disjoint measurable sets such that $\nu^+$ is carried by $A$ and $\nu^-$ is carried by $B$, then*

$$\nu^+(Y) = \sup \nu(Z) \quad \textit{for} \quad Z \in \mathcal{M} \quad \textit{and} \quad Z \subset Y;$$
$$-\nu^-(Y) = \inf \nu(Z) \quad \textit{for} \quad Z \in \mathcal{M} \quad \textit{and} \quad Z \subset Y.$$

*Proof.* We sketch the proof. By Theorem 3.5, there exists a real valued function $h$ such that $|h| = 1$ and

$$d\nu = h \, d|\nu|.$$

Then $h$ takes on only the values 1 and $-1$. Let $A$ be the set of points where $h$ takes the value 1, and let $B$ be the set where $h$ takes the value $-1$. Let $\nu_h^+$ and $\nu_h^-$ now be defined by the formulas

$$\nu_h^+ = |\nu|_{h_A} \qquad \text{and} \qquad \nu_h^- = |\nu|_{h_B}.$$

It is then clear that $\nu_h^+$ and $\nu_h^-$ are mutually singular, and it is immedi-

ately verified that $\nu_h^+ = \nu^+$, $\nu_h^- = \nu^-$. We leave the uniqueness and the proof of the last properties as an exercise.

## VII, §4. COMPLEX OR VECTORIAL MEASURES AND DUALITY

In this section we discuss the duality arising from complex or Hilbert space valued measures. We let $E$ be a Hilbert space, which the reader may assume to be $\mathbf{C}$ in first reading, although as usual, no changes would be needed.

**Theorem 4.1 (Hilbert Case).** *Let $E$ be a Hilbert space and let $\nu$ be an $E$-valued measure on $\mathcal{M}$. There exists a measurable map $h: X \to E$ such that $|h| = 1$ and such that for all $\varphi \in \mathrm{St}(|\nu|, E)$ we have*

$$\langle \varphi, d\nu \rangle = \int_X \langle \varphi, h \rangle \, d|\nu|.$$

*This map $h$ is uniquely determined up to $|\nu|$-equivalence.*

*Proof.* Identical with that of Theorem 3.5, except that we must insert unit vectors $e$ and write $e\chi_{A_n}$ or $e\chi_A$ in the appropriate place.

**Corollary 4.2 (Radon–Nikodym, Hilbert Case).** *Let $E$ be a Hilbert space. Let $\mu$ be a $\sigma$-finite positive measure on $\mathcal{M}$, and let $\nu$ be an $E$-valued measure on $\mathcal{M}$ such that $\nu$ is $\mu$-continuous. Then there exists $f \in \mathcal{L}^1(\mu, E)$ such that $\nu = \mu_f$, uniquely determined up to $\mu$-equivalence.*

*Proof.* We can write (by the real form of Radon–Nikodym)

$$d|\nu| = k \, d\mu$$

with some positive $k$ in $\mathcal{L}^1(\mu, \mathbf{R})$, whence by the theorem, on step maps we get (cf. Exercise 15)

$$d\nu = h \, d|\nu| = hk \, d\mu,$$

as was to be shown.

If $\mu$ is a positive measure on $\mathcal{M}$, we can now associate with each $\mu$-continuous $E$-valued measure $\nu$ on $\mathcal{M}$ a functional, again denoted by $d\nu$, on $L^\infty(\mu, E)$. Indeed, if we write

$$\nu = \mu_f$$

with $f \in \mathscr{L}^1(\mu, E)$, then we define $dv$ by

$$\langle g, dv\rangle = \int_X \langle g, f\rangle \, d\mu.$$

Let us denote by $M^1(\mu, E)$ the vector space of $\mu$-continuous $E$-valued measures on $\mathscr{M}$.

**Corollary 4.3.** *Let $\mu$ be $\sigma$-finite, and $E$ a Hilbert space. We have arrows:*

$$L^1(\mu, E) \overset{\approx}{\leftrightarrow} M^1(\mu, E) \to L^\infty(\mu, E)'.$$

*The first arrow, given by $f \mapsto \mu_f$, is a norm-preserving isomorphism, between $L^1(\mu, E)$ and $M^1(\mu, E)$. The second, given by $v \mapsto dv$, is a norm-preserving anti-linear map of $M^1(\mu, E)$ into the dual of $L^\infty(\mu, E)$. If $v = \mu_f$ with $f \in \mathscr{L}^1(\mu, E)$, then*

$$|dv| = \|v\| = \|f\|_1.$$

*Proof.* The norm statements are obtained by combining Theorem 3.3 and Theorem 2.3. All other statements summarize what has already been proved.

We now determine a necessary and sufficient condition for a functional on $L^\infty(\mu, E)$ to be expressible in the form $f\,d\mu$, with some $f$ in $\mathscr{L}^1(\mu, E)$. In other words, we characterize the image of the map

$$M^1(\mu, E) \to L^\infty(\mu, E)'.$$

We shall say that a functional

$$\lambda\colon \mathrm{St}(\mu, E) \to \mathbf{C}$$

is ***μ*-continuous** if there exists a positive real valued function $\tau$ on $\mathscr{M}$ such that

$$\lim_{\mu(A)\to 0} \tau(A) = 0,$$

and such that for every $g \in \mathrm{St}(\mu, E)$ we have

$$|\lambda(g_A)| \leqq \|g\|_\infty \tau(A).$$

Similarly we define $\mu$-continuity on the bounded measurable maps, taking $g$ to be such a map. We recall that $g_A = \chi_A g$.

**Corollary 4.4.** *Assume that $\mu(X)$ is finite. Every $\mu$-continuous functional on the step maps* $\mathrm{St}(\mu, E)$ *has a unique extension to a $\mu$-continuous functional on $L^\infty(\mu, E)$. A functional $\lambda$ on $L^\infty(\mu, E)$ can be written in the form $f\, d\mu$ with some $f \in \mathscr{L}^1(\mu, E)$ if and only if it is $\mu$-continuous.*

*Proof.* If $f \in \mathscr{L}^1(\mu, E)$, then for any bounded measurable $g$ we have

$$\left| \int_A \langle g, f \rangle \, d\mu \right| \leqq \|g\|_\infty \int_A |f| \, d\mu$$

so that our condition of $\mu$-continuity is satisfied. Conversely, let $\lambda$ be a $\mu$-continuous functional on the step maps $\mathrm{St}(\mu, E)$. To see that an extension to $L^\infty(\mu, E)$ is unique we note that if $g \in \mathscr{L}^\infty(\mu, E)$, then given $\varepsilon$ there exists a set $Z$ with $\mu(Z) < \varepsilon$ and a sequence of step maps $\{\varphi_n\}$ which converges uniformly to $g$ outside $Z$. This is true because on a set of finite measure, every bounded measurable map is in $\mathscr{L}^1$, and the fundamental lemma of integration (Lemma 3.1 of Chapter VI) gives us such approximation. It follows that any $\mu$-continuous extension of $\lambda$ to all of $\mathscr{L}^\infty(\mu, E)$ is uniquely determined.

We prove existence by representing $\lambda$ on the step maps as $d\nu$ for some measure $\nu$. For each fixed measurable $A$ we consider the map

$$v \mapsto \lambda(v\chi_A), \qquad v \in E.$$

This map is obviously a functional on $E$, and hence by the self duality of Hilbert space there exists a unique vector $\nu(A)$ such that for all $v \in E$ we have

$$\lambda(v\chi_A) = \langle v, \nu(A) \rangle.$$

The finite additivity of $\nu$ follows from the additivity of $\lambda$. Furthermore, we have the estimate

$$|\langle v, \nu A)\rangle| = |\lambda(v\chi_A)| \leqq |v|\,\tau(A).$$

This yields

$$|\nu(A)| \leqq \tau(A).$$

Let $\{A_n\}$ be a decomposition of $A$, and let

$$B_n = A_1 \cup \cdots \cup A_n.$$

Then $\nu(B_n) = \nu(A_1) + \cdots + \nu(A_n)$, and

$$|\nu(A) - \nu(B_n)| = |\nu(A - B_n)| \leqq \tau(A - B_n).$$

The right-hand side tends to 0 as $n \to \infty$, so that $\nu$ is countable additive.

As for its total variation, let $e_n$ be the unit vector in the direction of $\nu(A_n)$, and consider the series

$$g = \sum_{k=1}^{\infty} e_k \chi_{A_k}.$$

It is a measurable, bounded map. If $n \geqq m$, we have

$$\left| \lambda\left( \sum_{k=m}^{n} e_k \chi_{A_k} \right) \right| \leqq \tau(A_{mn})$$

where $A_{mn} = A_m \cup \cdots \cup A_n$. Applying the Cauchy criterion, we have:

$$\lambda(g) = \sum_{k=1}^{\infty} \lambda(e_k \chi_{A_k}) = \sum_{k=1}^{\infty} |\nu(A_k)| = |\lambda(g)|,$$

and also by the hypothesis on $\lambda$,

$$|\lambda(g)| \leqq \tau(A).$$

Taking $A = X$ shows that the total variation is finite, whence $\nu$ is a measure.

Finally, it is clear from the definition of $\nu$ that $\lambda = d\nu$ on the step maps. By Corollaries 4.2 and 4.3, if we write $d\nu = f\,d\mu$ for some $f$ in $\mathscr{L}^1(\mu, E)$ then we can extend $d\nu$ to a $\mu$-continuous functional on $\mathscr{L}^\infty(\mu, E)$. This proves Corollary 4.4.

**Example.** Let $X = [0, 1]$ with Lebesgue measure $\mu$. Let $F$ be the space of continuous functions on $X$, so that $F$ is a subspace of $\mathscr{L}^\infty(\mu, \mathbf{C})$. It is easy to verify that if $f, g \in F$ are equivalent (i.e. equal almost everywhere), then they are equal, so that $F$ is a subspace of $L^\infty(\mu, \mathbf{C})$. Let $\nu$ be the measure which gives the set $\{0\}$ measure 1, and gives a subset of $[0, 1]$ measure 0 if this subset does not contain 0. For any $f \in F$ we have

$$\int_X f\,d\nu = f(0).$$

This measure $\nu$ is obviously not $\mu$-continuous, but $d\nu$ is continuous for the $L^\infty(\mu)$-seminorm (actually a norm on $F$). We can extend the functional $d\nu$ on $F$ to all of $L^\infty(\mu, \mathbf{C})$ by the Hahn–Banach theorem to give examples of functionals on $L^\infty(\mu, \mathbf{C})$ which cannot be represented by $\mu$-continuous measures.

**Remark.** The part of the proof showing that $\nu$ is a measure does not depend in an essential way on the assumption that $E$ is a Hilbert space,

and goes through with very minor modifications in the arbitrary Banach case. The definition of $\mu$-continuity of a functional $\lambda$ applies in this case, and one can characterize such functionals as measures in the following manner:

*Assume that $\mu(X)$ is finite. Let $E$ be a Banach space and $E'$ its dual space. There exists a unique norm-preserving linear map*

$$M^1(\mu, E') \to L^\infty(\mu, E)'$$

*from the space of $\mu$-continuous $E'$-valued measures into the dual space of $L^\infty(\mu, E)$, denoted by $\nu \mapsto d\nu$, whose image is the space of $\mu$-continuous functional on $L^\infty(\mu, E)$, and such that on step maps $v\chi_A$ ($v \in E$ and $A$ measurable) we have*

$$\langle v\chi_A \, d\nu \rangle = \langle v, \nu(A) \rangle.$$

The crucial part of the proof of the preceding statement, namely that a $\mu$-continuous functional can be written as $d\nu$, follows closely the Hilbert case proof of Corollary 4.4. See Exercises 16 and 17.

There remains to determine when a given measure $\nu$ can be written in the form $\mu_f$ for some $f \in \mathscr{L}^1(\mu, E)$, and $E$ is an arbitrary Banach space. A complete answer is given in Rieffel's paper [Ri], as follows:

**Rieffel's Theorem.** *Let $\mu$ be $\sigma$-finite and let $E$ be a Banach space. Let $m$ be an $E$-valued measure, which is $\mu$-continuous. Either one of the following conditions is necessary and sufficient that $m$ can be written in the form $\mu_f$ for some $f \in \mathscr{L}^1(\mu, E)$:*

**R.** *Given $A$ measurable and $0 < \mu(A) < \infty$, there exists $B \subset A$ with $\mu(B) > 0$ such that the average set*

$$\mathrm{Av}_B(m) = \text{set of all } m(Y)/\mu(Y), \qquad Y \subset B,\ \mu(Y) > 0$$

*is relatively compact.*

**R′.** *Given $A$ measurable with $0 < \mu(A) < \infty$, there is some $B \subset A$ and a compact subset $K$ of $E$ not containing $0$ such that $\mu(B) > 0$ and $m(Y)$ is contained in the cone generated by $K$ for all $Y \subset B$.*

**Note.** The cone generated by $K$ is the set of all positive finite linear combinations of elements of $K$. Condition $R'$ may be expressed by saying that $m$ has compact direction locally somewhere. Condition $R'$ is obviously satisfied in the finite dimensional case. A discussion of the literature and applications will also be found in Rieffel's paper. For an example when the measure $m$ cannot be written as $\mu_f$, even though it is $\mu$-continuous, cf. Exercise 21.

For more on the Radon–Nykodum derivative in euclidean space, and the relation between differentiation and integration, I recommend Smith's book [Smi].

## VII, §5. THE $L^p$ SPACES, $1 < p < \infty$

*We let $(X, \mathcal{M}, \mu)$ be a measured space.*

In this section we give results analogous to those concerning $L^2$, replacing 2 by a real number $p$ with $1 < p$. We need some inequalities to replace the Schwarz inequality. Throughout we let $q$ be the positive number (necessarily $> 1$) such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and call $q$ the **dual exponent of** $p$.

We have the basic inequalities for real $a, b > 0$:

$$(*) \qquad a^{1/p}b^{1/q} \leqq \frac{a}{p} + \frac{b}{q},$$

$$(**) \qquad \left(\frac{a+b}{2}\right)^p \leqq \frac{1}{2}(a^p + b^p).$$

There are several easy proofs for this. Either take the log of both sides and use the convexity of the log, or proceed as follows. If $t \geqq 1$, then

$$t^{1/p} \leqq \frac{t}{p} + \frac{1}{q}$$

as one sees by differentiating both sides, evaluating at $t = 1$, and seeing that the derivative on the right is bigger than the derivative on the left. Suppose now that $a/b \geqq 1$; the inequality $(*)$ drops out at once. The other is proved similarly.

We let $\mathcal{L}^p(\mu)$ be the set of maps $f$ on $X$ which are $\mu$-measurable, and such that $|f|^p$ lies in $\mathcal{L}^1$.

**Theorem 5.1.** *Let $1 < p < \infty$. Then $\mathcal{L}^p(\mu)$ is a vector space. If we define*

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p},$$

*then $\| \ \|_p$ is a seminorm on $\mathcal{L}^p$. If $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$, then $|f||g|$*

*is in $\mathscr{L}^1$ and Hölder's inequality holds, namely*

$$\int_X |f||g|\, d\mu \leqq \|f\|_p \|g\|_q.$$

*Proof.* We see that $\mathscr{L}^p$ is a vector space directly by applying the inequality (**). If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $f$ or $g$ is 0 almost everywhere and the Hölder inequality is obvious. Suppose that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. Let

$$a = \frac{|f|^p}{\|f\|_p^p} \quad \text{and} \quad b = \frac{|g|^q}{\|g\|_q^q}.$$

Using inequality (*), we find that

$$\frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leqq \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q}{\|g\|_q^q}.$$

First this shows that $|f||g|$ is in $\mathscr{L}^1$ (corollary of the dominated convergence theorem), and second it yields the last inequality stated in the theorem, after we integrate over $X$. To show that $\| \ \|_p$ is a seminorm, write

$$|f + g|^p \leqq |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

Integrating and using Hölder's inequality yields the fact that $\| \ \|_p$ is a seminorm, and concludes the proof of the theorem.

We are now in a position to prove many of the results of §1 hold if one replaces $L^2$ by $L^p$ with $1 < p < \infty$.

**Theorem 5.2.** *Let $\{f_n\}$ be an $L^p$-Cauchy sequence in $\mathscr{L}^p$. Then there exists some $f \in \mathscr{L}^p$ having the following properties:*

(i) *The sequence $\{f_n\}$ is $L^p$-convergent to $f$, so that $\mathscr{L}^p$ is complete.*

*There exists a subsequence having the following properties:*

(ii) *This subsequence of $\{f_n\}$ converges almost everywhere to $f$.*
(iii) *Given $\varepsilon$, there exists a set $Z$ with $\mu(Z) < \varepsilon$ such that the convergence of this subsequence is uniform on the complement of $Z$.*

*Proof.* Identical with that of Theorem 1.4.

**Theorem 5.3 (Dominated Convergence Theorem for $L^p$).** *Let $\{f_n\}$ be a sequence in $\mathscr{L}^p$ which converges pointwise almost everywhere to $f$. Assume that there exists $g \in \mathscr{L}^p(\mu, \mathbf{R})$ such that $g \geqq 0$ and such that $|f_n| \leqq g$. Then $f$ is in $\mathscr{L}^p$ and $\{f_n\}$ is $L^p$-convergent to $f$.*

*Proof.* As before.

**Corollary 5.4.** *The step maps are dense in $\mathscr{L}^p$, and $L^p(\mu)$ is the completion of the step maps in the $L^p$-seminorm.*

*Proof.* As before.

Finally, the duality statement holds. Over **C**, we let $\langle f, g\rangle = f\bar{g}$. Theorem 5.5 and its proof are true as usual for a Hilbert space $E$ and $E$-valued maps $f$, $g$, with $\langle f, g\rangle$ denoting the scalar product of the values of $f$ and $g$.

**Theorem 5.5.** *Assume that $\mu$ is $\sigma$-finite. For $f \in \mathscr{L}^p(\mu)$ and $g \in \mathscr{L}^q(\mu)$, we let*

$$\langle f, g\rangle_\mu = \int_X \langle f, g\rangle \, d\mu,$$

*and define $\lambda_g$ by $\lambda_g(f) = \langle f, g\rangle_\mu$. Then the map $g \mapsto \lambda_g$ is norm-preserving isomorphism of $L^q(\mu)$ onto the dual space of $L^p(\mu)$.*

*Proof.* We consider first as usual the case when $\mu(X)$ is finite. Our map $g \mapsto \lambda_g$ is certainly an injective linear map, and we have

$$|\lambda_g| \leqq \|g\|_q$$

by Hölder's inequality. Let us prove that it is surjective. Let $\lambda$ be a functional on $L^p(\mu)$. Then $\lambda$ can be viewed as a $\mu$-continuous, $\mu$-bounded functional on $L^\infty(\mu)$ because if $g$ is a bounded measurable map, then $g$ is in $\mathscr{L}^p(\mu)$ and if $C = |\lambda|$, then

$$|\lambda(g)| \leqq C\|g\|_p = C\left(\int_X |g|^p \, d\mu\right)^{1/p} \leqq C\|g\|_\infty \mu(X)^{1/p}.$$

If we replace $X$ by $A$ for any measurable $A$, and $g$ by $g_A$, we get the same estimate with $\mu(X)$ replaced by $\mu(A)$. We can therefore apply Corollary 4.4 of the Radon–Nikodym theorem (vectorial case). There exists a map $f \in \mathscr{L}^1(\mu)$ such that $d\nu = f\, d\mu$ as a functional on $L^\infty(\mu)$. We shall prove that in fact, $f$ lies in $\mathscr{L}^q(\mu)$. Let $Y_n$ be the set of $x$ such that $|f(x)| \leqq n$. We first get a bound for the integral of $|f|^p$ over $Y_n$. Let

$$g = \frac{f}{|f|}|f|^{q-1}$$

and let $g_n$ be equal to $g$ on $Y_n$ and 0 outside $Y_n$. (That is $g_n = \chi_{Y_n} g$.) As usual, dividing by $|f|$ is to be understood as $1/|f(x)|$ if $f(x) \neq 0$ and 0 if

$f(x) = 0$. Then $g_n$ is bounded, $|g|^p = |f|^q$, and

$$\text{(1)} \qquad \int_X \langle g_n, f\rangle \, d\mu = \int_{Y_n} |f|^q \, d\mu$$

$$= \lambda(g_n) \leqq C\|g_n\|_p \leqq C\left(\int_{Y_n} |f|^q \, d\mu\right)^{1/p}.$$

From this we conclude that

$$\left(\int_X \chi_{Y_n} |f|^q \, d\mu\right)^{1/q} \leqq C.$$

By the monotone convergence theorem, it follows that $|f|^q$ lies in $\mathscr{L}^1$, whence $|f|$ lies in $\mathscr{L}^q$ and $\|f\|_q \leqq |\lambda|$.

The functionals $\lambda$ and $f\, d\mu$ have the same effect on step maps, which are dense in $\mathscr{L}^p$. Therefore they are equal on $\mathscr{L}^p(\mu)$. This proves our theorem when $\mu(X)$ is finite.

As for the $\sigma$-finite case, we consider a decomposition $X = \bigcup X_k$ (disjoint union of sets of finite measure). For each $X_k$ we can find a function $f_k$ that lies in $\mathscr{L}^q(\mu)$ and is 0 outside $X_k$, and such that $f_k \, d\mu$ represents $\lambda$ over $X_k$. Let $h \in \mathscr{L}^p(\mu)$ be arbitrary and let $h_k$ be the same map as $h$ on $X_k$ and 0 outside $X_k$. Then the series

$$\sum_{k=1}^{\infty} h_k$$

is $L^p$-convergent to $h$, say by the dominated convergence theorem, and therefore by the continuity of $\lambda$ we have

$$\lambda h = \sum_{k=1}^{\infty} \lambda(h_k).$$

For each $k$ we have $\lambda h_k = \langle h_k, f_k\rangle_\mu$. If we let $f = \sum f_k$, it follows that $\lambda = f\, d\mu$ on $\mathscr{L}^p(\mu)$. This concludes the proof of the $L^p$-duality theorem.

**Remark.** The proof follows the classical pattern (see Rudin [Ru 1] or Loomis [Lo]), granted the $L^2$ and $(L^1, L^\infty)$-duality theorem. For the general case when $E$ is a Banach space, and one wants $L^q(\mu, E')$ to be dual to $L^p(\mu, E)$ for $1 \leqq p < \infty$, cf. Dinculeanu [Din], §13, Corollary 1 of Theorem 8, where this is proved under some countability assumption.

The next theorem gives an example of an integral operator in the fairly general setting of $L^p$-spaces.

**Theorem 5.6.** *Let $1 \leqq p \leqq \infty$ and $C > 0$. Let $K$ be a measurable function on $X \times X$ such that*

$$\int_X |K(x, y)|\, d\mu(y) \leqq C \qquad \text{for all} \quad x \in X$$

*and*

$$\int_X |K(x, y)|\, d\mu(x) \leqq C \qquad \text{for all} \quad y \in X.$$

*Let $f \in L^p(\mu)$. Then the function $S_K f$ defined by*

$$S_K f(x) = \int_X K(x, y) f(y)\, d\mu(y)$$

*is defined for almost all $x$, and is in $L^p(\mu)$. Furthermore,*

$$\|S_K f\|_p \leqq C \|f\|_p.$$

*Proof.* We leave the proof as an exercise. The $L^2$-case is especially interesting. Cf. Exercises 9–13 of Chapter XVII.

## VII, §6. THE LAW OF LARGE NUMBERS

I cannot resist giving an application of integration theory to a probabilistic setting which shows integration theory at work. This consists of the "law of large numbers" in a suitable formulation. I follow the exposition of [La–T]. This section can be read immediately after §1, as an application of the definitions and convergence theorems in §1 concerning $L^2$.

We assume that the reader has done the exercise of extending the notion of product measures to denumerable products. Specifically, we use the following theorem.

*Let $(X_n, \mathcal{M}_n, \mu_n)$ be a sequence of measured spaces such that $\mu_n(X_n) = 1$ for almost all $n$ (meaning for all but a finite number of $n$). Let $\mathcal{M}$ be the $\sigma$-algebra in the product space*

$$X = \prod X_n$$

*generated by all sets*

$$A = \prod A_n,$$

*where $A_n \in \mathcal{M}_n$, and $A_n = X_n$ for almost all $n$. Then there exists a*

*unique measure $\mu$ on $(X, \mathcal{M})$ such that for all such sets $A$ we have*

$$\mu(A) = \prod \mu_n(A_n).$$

We call $\mu = \bigotimes \mu_n$ the **product measure.**

In the sequel we assume that in fact $\mu_n(X_n) = 1$ for *all* $n$. We view $X$ as our **probability space.**

**Theorem 6.1.** *Suppose given a measurable subset $S_n$ of $X_n$ for each $n$. Assume that the limit exists,*

$$\lim_{n\to\infty} \mu_n(S_n) = L.$$

*Then for almost all elements (sequences) $x = \{x_n\}$ in $X$, the density of $n$ such that $x_n \in S_n$ exists and is equal to $L$. This means:*

$$\lim_{N\to\infty} \frac{\#\{n \leqq N, x_n \in S_n\}}{N} = L.$$

The above theorem has a simple intuitive content, but some applications require a stronger version, as follows.

**Theorem 6.2.** *Suppose given a measurable subset $S_n$ of $X_n$ for each $n$. Let $\{b_n\}$ be a sequence of positive real numbers tending monotonically to infinity. Assume that*

$$\sum \frac{1}{b_n^2} \mu_n(S_n) < \infty.$$

*Then for almost all sequences $x$ we have*

$$\#\{n \leqq N, x_n \in S_n\} = \sum_{n=1}^{N} \mu_n(S_n) + o(b_N),$$

*where $o(b_N)/b_N \to 0$ as $N \to \infty$.*

The first theorem is obtained from the second by putting $b_n = n$. We shall now prove the theorem.

The first lemma, due to Kolmogoroff and formulated by him in probabilistic terms, will be a refinement of the fundamental lemma of integration theory, which asserts that given an $L^1$ (or $L^2$) Cauchy sequence, there exists a subsequence that converges absolutely almost everywhere. Here we give up on absolute convergence, but have conditions which make the full sequence converge pointwise almost everywhere.

**Lemma 6.3.** *For each n let $h_n$ be a function on $X_n$, also viewed as a function on X by projection on the n-th factor. Assume that*

$$\int h_n \, d\mu_n = 0.$$

*Let*

$$H_n(x) = \sum_{k=1}^{n} h_k(x)$$

*be the partial sum. Assume that $\sum \|h_k\|_2^2$ converges. Then the limit*

$$\lim_{n\to\infty} H_n(x)$$

*exists for almost all $x \in X$.*

*Proof.* We first note that the functions $h_n$ are mutually orthogonal on $X$. The heart of the proof lies in the next statement.

**Kolmogoroff's Inequality.** *Given ε, let*

$$Z = \left\{ x \in X, \max_{1 \leqq k \leqq n} H_k^2(x) \geqq \varepsilon \right\}.$$

*Then*

$$\varepsilon\mu(Z) \leqq \sum_{k=1}^{n} \|h_n\|_2^2.$$

*Proof.* Let

$$Y_k = \{x \in X \text{ such that } H_k^2(x) \geqq \varepsilon \text{ and } H_i^2(x) < \varepsilon \text{ for all } i < k\}.$$

In other words, $Y_k$ is the set of points $x$ such that $H_k^2(x)$ is the first partial sum at least equal to ε. Then the sets $Y_k$ are disjoint, and we get the inequality

$$\varepsilon \sum_{k \leqq n} \mu(Y_k) \leqq \sum_{k \leqq n} \int_{Y_k} H_k^2.$$

Write

$$H_k^2 = H_n^2 - 2H_k(H_n - H_k) - (H_n - H_k)^2.$$

The last term is negative, and we shall leave it out when we integrate. On the other hand, the middle term gives

$$\int_{Y_k} H_k(H_n - H_k)\, d\mu = 0.$$

This holds because $H_k$ is effectively a function of only the first $k$ variables, whereas $H_n - H_k$ is effectively a function of only the last $n - k$ variables. The integral splits into a product of integrals over the distinct variables, and is immediately seen to yield 0, as desired. Therefore we can replace $H_k^2$ by $H_n^2$ and then integrate over all of $X$, thereby by giving as bound the square of the $L^2$-norm of $\sum h_k$, which proves the asserted inequality.

We have assumed that $\sum h_k$ is in $L^2$, that is

$$\sum_{k=1}^{\infty} \|h_k\|_2^2 < \infty.$$

This means that for $m_0$ sufficiently large, and $n \geqq m \geqq m_0$, we get

$$\mu\{x \in X, \max(H_n - H_m)^2(x) \geqq \varepsilon\} \leqq \frac{1}{\varepsilon} \sum_{n=m}^{\infty} \|h_n\|_2^2 < \varepsilon.$$

Define

$$Z_i = \{x \in X, (H_n - H_m)^2(x) \geqq 1/2^i \text{ if } m, n \geqq m_0(i)\}.$$

Then $Z_i$ has measure $\leqq 1/2^i$ if we pick $m_0(i)$ sufficiently large. Let

$$W_n = Z_n \cup Z_{n+1} \cup \cdots$$

for large $n$, so that $W_n$ has measure $\leqq 1/2^{n-1}$. Then the partial sums $\sum h_k(x)$ converge for $x$ not in $W_n$. Hence if we let $W$ be the intersection

$$W = \bigcap W_n,$$

then these partial sums converge for $x$ not in $W$, and $W$ has measure zero, thereby proving the lemma.

The next theorem is also due to Kolmogoroff, in that generality.

**Theorem 6.4.** *For each $n$ let $f_n$ be a function on $X_n$, and assume that*

$$\int f_n \, d\mu_n = 0.$$

*Let $\{b_n\}$ be a sequence of positive real numbers monotonically increasing to infinity. If*

$$\sum \frac{1}{b_n^2} \|f_n\|_2^2 < \infty,$$

*then for almost all $x$ the partial sums*

$$F_n(x) = \sum_{k=1}^{n} f_k(x)$$

*satisfy the estimate*

$$F_n(x) = o(b_n).$$

*Proof.* Let $h_n = f_n/b_n$ and apply the lemma to the partial sums

$$H_n(x) = \sum_{k=1}^{n} h_k(x) = \sum_{k=1}^{n} \frac{f_k(x)}{b_k}.$$

The lemma says that these partial sums converge for almost all $x$. It is a trivial fact (proved by summation by parts) that if $\sum a_k$ is a convergent sequence, then

$$\sum_{k=1}^{n} a_k b_k = o(b_n).$$

Applying this fact when $a_k = h_k(x)$ proves the theorem.

We have stated Theorem 6.4 under the normalization that the integral of the functions $f_n$ is 0. This is of course not satisfied in general, but a translation reduces the general case to this special case. Indeed, suppose that $\psi_n$ are functions such that

$$\int \psi_n \, d\mu_n = c_n$$

is a constant $c_n$. Define

$$f_n = \psi_n - c_n.$$

Then the integral of $f_n$ is 0. In particular, suppose that $\psi_n$ is the characteristic function of some subset $S_n$ of $X_n$. Then

$$\|f_n\|_2^2 = \int (\psi_n - c_n)^2 \, d\mu_n = c_n - c_n^2.$$

Applying Theorem 6.4 in this situation yields Theorem 6.2, as desired.

## VII, §7. EXERCISES

*Unless otherwise specified, $(X, \mathcal{M}, \mu)$ is a measured space.*

1. Let $\mathcal{A}$ be an algebra in $X$ and $\mu$ a positive measure on $\mathcal{A}$. Assume that all elements of $\mathcal{A}$ have finite measure. For $A, B \in \mathcal{A}$ define

$$d(A, B) = \mu(A - B) + \mu(B - A).$$

Show that $d$ is a semimetric [in the obvious sense, that is $d(A, B) \geqq 0$,

$$d(A, B) = d(B, A),$$

and the triangle inequality is satisfied]. The only difference from a metric is that we may have $d(A, B) = 0$ and yet $A \neq B$. In this way, $\mathscr{A}$ becomes a topological space, and $\mu$-continuity corresponds to the topological notion.

2. **Radon–Nikodym derivative.** Let $\mu$, $\nu$ be positive measures, and let $m$ be a complex measure. Suppose that $dm = f\,d\mu$, where $f \in \mathscr{L}^1(\mu)$ and $d\mu = g\,d\nu$ where $g \in \mathscr{L}^1(\nu, \mathbf{R})$. Prove that $fg \in \mathscr{L}^1(\nu)$, and that $dm = fg\,d\nu$. If we use the notation $dm/d\mu = f$ and $d\mu/d\nu = g$, then we have the old formalism

$$\frac{dm}{d\mu}\frac{d\mu}{d\nu} = \frac{dm}{d\nu}.$$

One sometimes calls $f$ the **Radon–Nikodym derivative** of $m$ with respect to $\mu$. [By the way, you may view $dm$ or $d\mu$, $d\nu$ as linear maps on step functions, which amounts to considering measures $m = \mu_f$ or $\mu = \nu_g$.]

3. Let $X$ consist of two points $x$ and $y$. Define $\mu(\{x\}) = 1$ and

$$\mu(\{y\}) = \mu(X) = \infty.$$

Determine whether $L^\infty(\mu, \mathbf{R})$ is the dual of $L^1(\mu, \mathbf{R})$.

4. Let $F$ be a subspace of $L^2(\mu, \mathbf{C})$ and assume that there is some number $c > 0$ such that for all $f$ in $F$ we have

$$|f| \leqq c\|f\|_2.$$

Assume that $\mu(X) < \infty$. Show that $F$ is finite dimensional and that

$$\dim F \leqq c\mu(X).$$

[*Hint* (Moser): Let $f_1, \ldots, f_n$ be orthonormal elements in $F$. Let

$$b^2 = \text{ess sup}(|f_1|^2 + \cdots + |f_n|^2).$$

Let $x_0$ be a point such that

$$\sum |f_k(x_0)|^2 \geqq b^2(1 - \varepsilon) \qquad \text{and also} \qquad \geqq b^2 - \varepsilon b.$$

Consider the function $f = \sum \alpha_k f_k$ with $\alpha_k = \overline{f_k(x_0)}/b$.]

5. Let $X$ be the set of positive integers, and let $\mu$ be the counting measure on $X$ which gives each point measure 1. Let $l^1$ and $l^\infty$ denote $L^1(\mu, \mathbf{C})$ and $L^\infty(\mu, \mathbf{C})$.
(a) Show that $l^1$ consists of all complex sequences $\alpha = \{a_n\}$ with norm

$$\|\alpha\|_1 = \sum |a_n| < \infty.$$

Show that $l^\infty$ consists of all complex sequences $\alpha = \{a_n\}$ with norm

$$\|\alpha\|_\infty = \sup|a_n| < \infty.$$

(b) Show that $l^1$ is the dual of the subspace $C_0$ of $l^\infty$ consisting of all sequences $\alpha = \{a_n\}$ such that $a_n \to 0$ as $n \to \infty$. Show that the dual of $l^1$ is $l^\infty$ quoting any theorem from the text.

(c) Show that $C_0$ and $l^1$ are separable, but that $l^\infty$ is not separable.

**The space $H_s$.** (For applications to PDE, cf. $SL_2(\mathbf{R})$, Appendix 4.)

6. Let $s$ be an integer. On the integers $\mathbf{Z}$ define

$$\mu_s(n) = (1 + n^2)^s.$$

*Then $\mu_s$ is a measure on* $\mathbf{Z}$.

(a) Define the space $H_s$ to be the space of functions on $\mathbf{Z}$, written in the form of sequences $\{a_n\}$, such that the sum

$$\sum (1 + n^2)^s |a_n|^2$$

converges. If $f = \{a_n\}$ and $g = \{b_n\}$, define the scalar product in $H_s$ to be

$$\langle f, g\rangle = \sum a_n \bar{b}_n (1 + n^2)^s.$$

Show that $H_s = L^2(\mathbf{Z}, \mu_s)$, and in particular is complete for the norm associated with this scalar product.

(b) Show that the finite sequences $f = \{a_n\}$ such that $a_n = 0$ for all but a finite number of $n$ form a dense subspace of $H_s$.

7. For each function $f \in C^\infty(\mathbf{T})$, where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the circle, or if you wish, for each $C^\infty$ function on $\mathbf{R}$, periodic of period 1, associate the Fourier series

$$f(x) = \sum a_n e^{2\pi i n x} \qquad \text{where} \quad a_n = \int_0^1 f(t) e^{-2\pi i n x}\, dx.$$

(a) Integrating by parts, show that the coefficients satisfy the inequality

$$|a_n| \ll \frac{1}{|n|^k}$$

for each positive integer $k$. The symbol $\ll$ means that the left-hand side is less than some constant times the right-hand side for $|n| \to \infty$.

(b) Prove that $C^\infty(\mathbf{T}) \subset L^2(\mathbf{Z}, \mu_s)$ for all $s \in \mathbf{Z}$, and that $C^\infty(\mathbf{T})$ is dense in this space $L^2$. [Look at the finite Fourier series

$$\sum_{|n| \leqq N} a_n e^{2\pi i n x}.]$$

8. Let $r < s$. Prove that the unit ball in $H_s$ is relatively compact in $H_r$, in other words that this unit ball is totally bounded in $H_r$.

9. Let $\{f_n\}$ be a sequence in $\mathscr{L}^2(X, \mu)$ such that $\|f_n\|_2 \to 0$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \int_X |f_n(x)| \log(1 + |f_n(x)|) \, d\mu(x) = 0.$$

10. Let $1 \leqq p < \infty$ and let $f \in \mathscr{L}^p(\mathbf{R})$ (for Lebesgue measure). For $a \in \mathbf{R}$ let $T_a$ be the translation by $a$, that is $T_a f(x) = f(x - a)$. Prove that $T_a f$ converges to $f$ in $L^p$ as $a \to 0$. Is the conclusion still true if $p = \infty$?

11. Let $\mu$ be a $\sigma$-finite positive measure on the Borel sets in $\mathbf{R}$, and suppose $L^1(\mathbf{R}, \mu) \subset L^\infty(\mathbf{R}, \mu)$. Show that there exists $c > 0$ such that if $A$ is a Borel set with $\mu(A) > 0$ then $\mu(A) \geqq c$.

12. Prove Theorem 5.6. [*Hint*: Use Hölder's inequality and Fubini's theorem.]

13. Let $T: L^2(X, \mu) \to L^2(X, \mu)$ be a continuous linear map, and assume that $X$ is $\sigma$-finite. Assume that $T$ commutes with all operators $M_g$ such that $M_g(f) = gf$, for $g \in \mathscr{L}^\infty$ and $f \in \mathscr{L}^2$. Prove that $T = M_g$ for some $g$. [*Hint*: Write $X$ as a disjoint union of sets of finite measure $X_n$ and let $\varphi$ be the function which is the constant $1/n^2\mu(X_n)^{1/2}$ on $X_n$. For $f \in \mathscr{L}^\infty \cap \mathscr{L}^2$, we have

$$T(\varphi f) = \varphi T(f) = f T(\varphi).$$

Let $g = T\varphi/\varphi$. Then $Tf = gf$. Prove that $g$ is bounded as follows. If it is not, given $N$ there is a subset of finite positive measure $Y$ such that $|g| \geqq N$ on $Y$. Consider $T\big((\bar{g}/g)\, \chi_Y\big)$ to contradict the boundedness of $T$.]

For an application, see $SL_2(\mathbf{R})$, Lemma 4 of Theorem 4, Chapter XI, §3.

14. Let $E$ be a Banach space and let $\nu: \mathscr{M} \to E$ be an $E$-valued measure. Show that one can define (in a manner similar to that in the text) a linear map

$$\mathrm{St}(|\nu|, \mathbf{C}) \to E,$$

and that this map is $L^1(|\nu|)$-continuous. This linear map can therefore be extended linearly by continuity to $\mathscr{L}^1(|\nu|, \mathbf{C})$, thus allowing you to define $\int f \, d\nu$, for $f \in \mathscr{L}^1(|\nu|, \mathbf{C})$.

15. Let $E$ be a Banach space and $E'$ its dual. In the bilinear map

$$L^1(\mu, E) \times L^\infty(\mu, E') \to \mathbf{C}$$

given by

$$(f, g) \mapsto \langle f, g \rangle_\mu = \int_X \langle f, g \rangle \, d\mu$$

show that $|\lambda_f| = \|f\|_1$ and $|\lambda_g| = \|g\|_\infty$, just as in the Hilbert case. [*Hint*: Use step maps, and for a constant map, use the Hahn–Banach theorem to see that given $v \in E$, there exists $v' \in E'$ such that $|v'| = |v|$ and $\langle v, v' \rangle = |v|$.]

16. Assume that $\mu(X)$ is finite. Let $E$ be a Banach space and $E'$ its dual space. The definition of a $\mu$-continuous functional $\lambda$ on $L^\infty(\mu, E)$ is as in the text. Show that such a functional can be written in the form $\lambda = d\nu$ for some

$E'$-valued measure $\nu$, in the sense that on a map $v\chi_A$ ($v \in E$ and $A$ measurable) we have

$$\lambda(v\chi_A) = \langle v, \nu(A)\rangle.$$

17. Prove the statement included in the remark following Corollary 4.4 of Theorem 4.1, concerning that part of the dual of $L^\infty(\mu, E)$ represented by a measure in $E'$.

18. Let $f$ be a $\mu$-measurable map of $X$ into a Banach space $E$. Given a measurable set $A$ with $\mu(A)$ finite, and $\varepsilon$, show that there exists $Z \subset A$ such that $\mu(Z) < \varepsilon$ and $f(A - Z)$ is relatively compact (or equivalently, totally bounded). We may say that $f$ is locally almost compact valued.

19. **The essential image.** Let $E$ be a Banach space. Let $f$ be a measurable map and let $A$ be a measurable set. The **essential image** of $f$ on $A$ is defined to be the set of all $v \in E$ such that for every $r > 0$ the measure of the set

$$A \cap f^{-1}(B_r(v))$$

is strictly positive. We denote it by $\mathrm{ei}_A(f)$.
 (i) The essential image is closed.
 (ii) If $\mu(A) > 0$, then $\mathrm{ei}_A(f)$ intersects the image $f(A)$.
 (iii) The set $Z$ of elements $x \in A$ such that $f(x)$ does not lie in $\mathrm{ei}_A(f)$ has measure 0.
 (iv) Let $A = \bigcup A_n$ be a denumerable union of measurable sets. Show that

$$\mathrm{ei}_A(f) = \text{closure of } \bigcup_{n=1}^{\infty} \mathrm{ei}_{A_n}(f).$$

20. Let $E$ be a real Banach space, and $f: X \to E$ any map. Let $g: X \to \mathbf{R}$ be a real positive function on $X$ which is in $\mathscr{L}^1(\mu, \mathbf{R})$ and such that

$$\int_X g\, d\mu > 0.$$

Assume that $gf$ is in $\mathscr{L}^1(\mu, E)$. If $\lambda$ is a functional on $E$, and $c \in \mathbf{R}$, we define a half space $H^+(\lambda, c)$ to consist of all $v \in E$ such that $\lambda v \geqq c$. Let $H$ be such a half space containing $f(X)$. Show that

$$\frac{\int_X gf\, d\mu}{\int_X f\, d\mu}$$

belongs to $H$.

In view of the result on convex sets in §2 of the Appendix to Chapter 4, it follows that the above "average" in fact lies in the closure of the convex set generated by the image $f(X)$, i.e. the smallest closed convex set containing $f(X)$.

21. Let $E = L^1(\mu, \mathbf{C})$ where $X = [0, 1]$ and $\mu$ is Lebesgue measure on the algebra of Borel sets. For each Borel set $A$ let

$$m(A) = \text{class of } \chi_A \text{ in } L^1(\mu, \mathbf{C}).$$

(a) Show that $|m|$ is Lebesgue measure itself. (b) Show that $m$ is an $E$-valued measure which cannot be written as $\mu_f$. [*Hint:* View $dm$ as a functional on step functions, say real valued, so that for any step function $\varphi$ and measurable set $A$ we have

$$\int_A \varphi \, dm = \int_A \varphi f \, d\mu.]$$

22. Let $E$ be a Banach space. Let $P$ denote the set of all partitions, i.e. collections $\pi$ consisting of a finite number of disjoint measurable sets of finite measure. We let $\pi_1 \geqq \pi$ if every element of $\pi$ is, up to a set of measure 0, the union of elements of $\pi_1$. For each $\pi \in P$ and $f \in \mathscr{L}^1(\mu, E)$ we define

$$f_\pi = T_\pi f = \sum_{A \in \pi} [\mu_f(A)/\mu(A)]\chi_A$$

where $\mu_f(A) = \int_A f \, d\mu$.

(a) Show that $T_\pi\colon L^1(\mu, E) \to L^1(\mu, E)$ is a continuous linear map of norm 1, and that $T_\pi f$ is $L^1$-convergent to $f$ in the following sense: Given $\varepsilon$ there exists $\pi_0$ such that for all $\pi \geqq \pi_0$ we have

$$\|T_\pi f - f\|_1 < \varepsilon.$$

(b) Prove the same thing replacing 1 by $p$ for $1 < p < \infty$.

# CHAPTER VIII

# Some Applications of Integration

After the abstract theory on arbitrary measured spaces, it is a relief to get into some classical situations on $\mathbf{R}^n$ where we see the integral at work. None of this chapter will be used later, except for the approximation by Dirac families in the uniqueness proof for the spectral measure of Chapter 20.

In this chapter we deal with the Fourier transform in a context of absolute convergence. In Chapter 10, §2 we shall deal with a more delicate context, involving oscillatory convergence.

## VIII, §1. CONVOLUTION

Suppose first we deal with functions $f$, $g$ on the real line. We shall study their **convolution**, defined by the integral

$$f * g(y) = \int_{-\infty}^{\infty} f(x)g(y-x)\,dx.$$

Of course, the integral must be convergent, or even absolutely convergent. Theorems 1.1 and 1.2 will give conditions for such convergence. Furthermore, we don't need to work only on $\mathbf{R}$, and we shall express the results on $\mathbf{R}^n$, abbreviating

$$\int_{\mathbf{R}^n} f(x)g(y-x)\,dx = \int f(x)g(y-x)\,dx.$$

In most applications, one of the two functions $f$ or $g$ is continuous or

even $C^\infty$, and the resulting convolution is also continuous or $C^\infty$. To see this one must be able to take a limit or differentiate under the integral sign, and the next section gives basic conditions under which this is legitimate. We shall see several examples after the main approximation theorem is proved in Theorem 3.1.

We now come to the basic tests for absolute convergence of the convolution integral.

**Theorem 1.1.** *Let $f, g \in \mathscr{L}^1(\mathbf{R}^n)$. Then for almost all $y \in \mathbf{R}^n$ the function*

$$x \mapsto f(x)g(y-x)$$

*is in $\mathscr{L}^1(\mathbf{R}^n)$. The convolution $f * g$ given for almost all $y$ by*

$$f * g(y) = \int f(x)g(y-x)\,dx$$

*is also in $\mathscr{L}^1$. The association $(f, g) \mapsto f * g$ is an associative, commutative bilinear map, satisfying*

$$\|f * g\|_1 \leqq \|f\|_1 \|g\|_1.$$

*Thus $\mathscr{L}^1(\mathbf{R}^n)$ is a Banach algebra under the convolution product.*

*Proof.* We integrate $|f(x)|\,|g(y-x)|$ first with respect to $y$, and then with respect to $x$. We apply part 2 of Fubini's theorem, Theorem 8.7 of Chapter VI. We then conclude that $f * g$ is in $\mathscr{L}^1$. The last inequality in the statement of the theorem follows at once. The bilinearity is obvious, and so is commutativity. The associativity is proved using Fubini's theorem, and is left to the reader.

**Theorem 1.2.** *Let $f \in \mathscr{L}^1(\mathbf{R}^n)$ and $g \in \mathscr{L}^p(\mathbf{R}^n)$ with $1 \leqq p \leqq \infty$. Then $f * g(y)$ is defined by the integral for almost all $y$ and is in $\mathscr{L}^p$. We have*

$$\|f * g\|_p \leqq \|f\|_1 \|g\|_p.$$

*Proof.* The case $p = 1$ is treated in Theorem 1.1. Suppose that $p = \infty$. If $f \in \mathscr{L}^1(\mathbf{R}^n)$ and $g \in \mathscr{L}^\infty(\mathbf{R}^n)$, so we may assume $g$ is a bounded measurable function, then we may also form the **convolution** $f * g$ given by the same formula

$$f * g(y) = \int f(x)g(y-x)\,dx = \int f(y-x)g(x)\,dx.$$

The integrals converge absolutely, and we have the trivial estimate from

the first integral, replacing $g$ by its bound $\|g\|_\infty$, namely

$$\|f * g\|_\infty \leqq \|f\|_1 \|g\|_\infty$$

as desired.

Finally suppose that $1 < p$. Let $q$ be as usual such that $1/p + 1/q = 1$. Then we have the inequality

$$\int |f(x)|^{1/p} |g(y - x)| \, |f(x)|^{1/q} \, dx$$
$$\leqq \left[\int |f(x)| \, |g(y - x)|^p \, dx\right]^{1/p} \left[\int |f(x)| \, dx\right]^{1/q},$$

from which we see that $f * g$ is defined for almost all elements of $\mathbf{R}^n$, and also that

$$|(f * g)(y)|^p \leqq \left[\int |f(x)| \, |g(y - x)|^p \, dx\right] \|f\|_1^{p/q}.$$

We integrate and use Fubini's theorem, obtaining

$$\|f * g\|_p^p \leqq \|g_x\|_p^p \|f\|_1 \|f\|_1^{p/q}.$$

But the $L^p$-seminorm is invariant under translations, i.e. $\|g_x\|_p = \|g\|_p$. Since $1 + p/q = p$, we take the $p$-th root to obtain

$$\|f * g\|_p \leqq \|g\|_p \|f\|_1,$$

thus proving our theorem.

## VIII, §2. CONTINUITY AND DIFFERENTIATION UNDER THE INTEGRAL SIGN

**Lemma 2.1.** *Let $X$ be a measured space with positive measure $\mu$. Let $U$ be an open subset of $\mathbf{R}^n$. Let $f$ be function on $X \times U$. Assume:*

(i) *For each $y \in U$ the function $x \mapsto f(x, y)$ is in $\mathscr{L}^1(\mu)$.*
(ii) *For each $x \in X$ and $y_0 \in U$, we have*

$$\lim_{y \to y_0} f(x, y) = f(x, y_0).$$

(iii) *There exists a function $f_1 \in \mathscr{L}^1(\mu)$ such that for all $y \in U$,*

$$|f(x, y)| \leqq |f_1(x)|.$$

*Then the function*

$$y \mapsto \int_X f(x, y)\, d\mu(x)$$

*is continuous.*

*Proof.* It suffices to prove that for any sequence $\{y_k\}$ converging to $y$,

$$\int_X f(x, y_k)\, d\mu(x) \quad \text{converges to} \quad \int_X f(x, y)\, d\mu(x).$$

Let $f_k(x) = f(x, y_k)$. Then $\{f_k\}$ converges pointwise to the function

$$x \mapsto f(x, y),$$

and by (iii), we can apply the dominated convergence theorem to conclude the proof.

**Lemma 2.2.** *Let $X$ be a measured space with positive measure $\mu$. Let $U$ be an open subset of $\mathbf{R}^n$. Let $f$ be a function on $X \times U$. Assume:*

(i) *For each $y \in U$ the function $x \mapsto f(x, y)$ is in $\mathscr{L}^1(\mu)$.*
(ii) *For each $y \in U$, each partial derivative $D_j f(x, y)$ (taken with respect to the j-th y-variable) is in $\mathscr{L}^1(\mu)$.*
(iii) *There exists a function $f_1 \in \mathscr{L}^1(\mu)$ such that for all $y \in U$,*

$$|D_j f(x, y)| \leqq |f_1(x)|.$$

*Let*

$$\Phi(y) = \int_X f(x, y)\, d\mu(x).$$

*Then $D_j\Phi$ exists and we have*

$$D_j\Phi(y) = \int_X D_j f(x, y)\, d\mu(x).$$

*Proof.* Let $e_j$ be the usual $j$-th unit vector in $\mathbf{R}^n$. We have

$$\frac{\Phi(y + he_j) - \Phi(y)}{h} = \int_X \frac{1}{h}[f(x, y + he_j) - f(x, y)]\, d\mu(x).$$

Using the mean value theorem and (iii), together with the dominated convergence theorem, we conclude that the right-hand side has a limit, equal to

$$\int_X D_j f(x, y)\, d\mu(x).$$

[As in the previous proof, we have to use the device of taking a sequence $\{h_k\}$ to apply the dominated convergence theorem in its standard form.]

**Theorem 2.3.** *Let $f \in \mathscr{L}^1(\mathbf{R}^n)$ and let $\varphi$ be a $C^\infty$ function with compact support. Then $f * \varphi$ is $C^\infty$ and in fact*

$$D^p(f * \varphi) = f * D^p\varphi.$$

*Proof.* We can form the convolutions by using Theorem 1.1 and we have

$$f * \varphi(y) = \int f(x)\varphi(y - x)\,dx.$$

Lemmas 2.1 and 2.2 show that $f * \varphi$ is $C^\infty$, and allow us to differentiate repeatedly under the integral sign.

## VIII, §3. DIRAC SEQUENCES

By a **Dirac sequence** on $\mathbf{R}^n$ we shall mean a sequence of functions $\{\varphi_k\}$ on $\mathbf{R}^n$ real valued, continuous, satisfying the following properties:

**DIR 1.** *We have $\varphi_k \geqq 0$ for all $k$.*

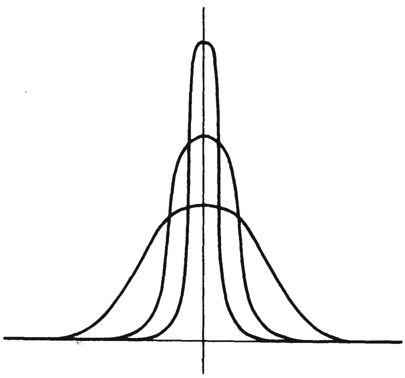
**DIR 2.** *For all $k$ we have $\int \varphi_k(x)\,dx = 1$.*

**DIR 3.** *Given $\varepsilon$, $\delta > 0$ there exists $k_0$ such that*

$$\int_{|x| \geqq \delta} \varphi_k(x)\,dx < \varepsilon$$

*for all $k \geqq k_0$.*

The third condition shows that for large $k$, the volume under $\varphi_k$ is concentrated near the origin. Thus in one variable, the sequence looks like this:



We have drawn the picture so that the sequence satisfies a property somewhat stronger than what is expressed in **DIR 3**, namely the support of $\varphi_k$ tends to 0 as $k \to \infty$. We state this condition formally. By a **Dirac sequence with shrinking support**, we mean a sequence satisfying **DIR 1**, **DIR 2**, and the third condition:

**DIR 3s.** *Each $\varphi_k$ has compact support, and given $\delta$, the support of $\varphi_k$ is contained in the ball of radius $\delta$, centered at the origin, for all $k$ sufficient large.*

To construct such a sequence we can start with a positive function $\varphi$, continuous or even infinitely differentiable, having support in the ball of radius 1, centered at the origin, and such that

$$\int \varphi(x)\, dx = 1.$$

We then let $\varphi_k(x) = k^n \varphi(kx)$. A sequence constructed in this manner will be called a **regularizing sequence**. It has additional properties besides those three of the Dirac sequence namely: the support of $\varphi_k$ is determined explicitly in terms of the support of $\varphi$, and is contained in the ball of radius $1/k$; in fact, it is contained in $(1/k)\operatorname{supp} \varphi$. In addition, the partial derivatives of $\varphi_k$ can be easily estimated in terms of those of $\varphi$ and $k$. This is frequently useful in applications when one has to make careful estimates on such derivatives.

We now show how a Dirac sequence can be used to approximate a function. We shall prove first the main approximation theorem for $\mathscr{L}^\infty$, with condition **DIR 3**, and give some applications. Then we prove an analogous theorem with condition **DIR 3s**, and see how it implies approximation results for functions in $\mathscr{L}^p$ with $p < \infty$.

**Theorem 3.1.** *Let $f$ be a bounded measurable function on $\mathbf{R}^n$. Let $A$ be a compact set on which $f$ is continuous. Let $\{\varphi_k\}$ be a sequence satisfying* **DIR 1, DIR 2, DIR 3**. *Then $\varphi_k * f$ converges to $f$ uniformly on $A$.*

*Proof.* For $x \in A$ we have:

$$\varphi_k * f(x) = \int \varphi_k(y) f(x - y)\, dy$$

and by **DIR 2**,

$$f(x) = f(x) \int \varphi_k(y)\, dy = \int f(x)\varphi_k(y)\, dy.$$

Hence

$$\varphi_k * f(x) - f(x) = \int \varphi_k(y)[f(x-y) - f(x)]\,dy.$$

By the relative uniform continuity of $f$ on $A$, given $\varepsilon$, there exists $\delta$ such that if $|y| < \delta$ then for all $x \in A$ we have

$$|f(x-y) - f(x)| < \varepsilon.$$

We then write

$$\varphi_k * f(x) - f(x) = \int_{|y|<\delta} + \int_{|y| \geqq \delta} \varphi_k(y)[f(x-y) - f(x)]\,dy.$$

The integral over $|y| < \delta$ is then bounded by $\varepsilon$. For the other integral with $|y| \geqq \delta$, we use **DIR 3** to conclude that this integral is bounded by $2\|f\|_\infty \varepsilon$ for $k$ sufficiently large. This concludes the proof.

We shall now give classical examples of Theorem 3.1.

**Example 1 (The Landau Sequence and Weierstrass' Approximation Theorem).** By means of a suitable Dirac sequence one can give an explicit proof for Weierstrass' theorem that a continuous function can be uniformly approximated by a polynomial on an closed interval. Suppose $f$ is continuous on $[a, b]$. Making a translation and dilation of the variable if necessary we may assume that $[a, b] = [0, 1]$. Let $y = L(x)$ be the equation of the straight line passing through the end points of the graph of the function. Then $L$ is a polynomial (of degree $\leqq 1$), and $f - L$ has the additional property that $f - L$ vanishes at 0 and 1. Thus without loss of generality, to prove Weierstrass' theorem, we may assume that $f(0) = f(1) = 0$. We then extend $f$ to $\mathbf{R}$ by defining $f(x) = 0$ for $x \notin [0, 1]$.

We now define the **Landau functions**

$$\varphi_k(x) = \frac{1}{c_k}(1 - x^2)^k \qquad \text{for} \quad |x| \leqq 1,$$

where the constant $c_k$ is taken to be

$$c_k = \int_{-1}^{1} (1 - x^2)^k\,dx,$$

so that

$$\int_{-1}^{1} \varphi_k(x)\,dx = 1.$$

We define $\varphi_k(x) = 0$ if $x$ is outside the interval $[-1, 1]$. It is an easy exercise to show that $\{\varphi_k\}$ is a Dirac sequence. We have

$$\varphi_k * f(x) = \int_{-\infty}^{\infty} \varphi_k(x-t)f(t)\,dt = \int_0^1 \varphi_k(x-t)f(t)\,dt.$$

Furthermore, for $x, t \in [0, 1]$ the function $\varphi_k(x-t)$ is a polynomial, namely

$$\varphi_k(x-t) = c_k(1-(x-t)^2)^k = \sum_{j=0}^{2k} u_j(x)t^j$$

where each $u_j$ is a polynomial. Hence

$$\varphi_k * f(x) = \sum \alpha_j u_j(x) \qquad \text{where} \qquad \alpha_j = \int_0^1 t^j f(t)\,dt,$$

and therefore $\varphi_k * f$ is a polynomial. By Theorem 3.1 the sequence $\{\varphi_k * f\}$ converges to $f$ uniformly on $[0, 1]$, thus proving Weierstrass' theorem.

In some applications, we deal with periodic functions of period $2\pi$, and in this case a Dirac sequence is defined in an analogous way, taking integrals over an interval of length $2\pi$. The next example is of this type.

**Example 2 (Cesaro Summation).** Let $f$ be a period continuous function of period $2\pi$. Let $S_{n,f}$ be the $n$-th partial sum of the Fourier series for $f$, that is

$$S_{n,f}(x) = \sum_{k=-n}^{n} c_k e^{ikx} \qquad \text{where} \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt.$$

Let $A_n$ be the average of these partial sums, that is

$$A_n = \frac{1}{n}(S_0 + \cdots + S_{n-1}).$$

Then a theorem of Fejer–Cesaro asserts that $\{A_n\}$ converges uniformly to $f$. This result is a special case of Theorem 3.1 as follows. Let

$$D_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx},$$

$$K_n(x) = \frac{1}{2\pi n}\sum_{m=0}^{n-1}\sum_{k=-m}^{m} e^{ikx} = \frac{1}{n}(D_0 + \cdots + D_{n-1}).$$

Then simple manipulations will prove the identity

$$K_n(x) = \frac{1}{2\pi n} \frac{\sin^2 nx/2}{\sin^2 x/2}.$$

It is then easy to verify that $\{K_n\}$ is a Dirac sequence, that $D_n * f$ is the $n$-th partial sum of the Fourier series of $f$, and that $K_n * f$ is the average of the partial sums. Therefore by Theorem 3.1, the averages of the partial sums of the Fourier series converge uniformly to $f$. Do Exercise 2 to carry out the details of this proof.

In some cases, instead of considering a sequence of functions, one considers a family of functions indexed by some real numbers, as in the next example.

**Example 3 (Harmonic Functions and the Poisson Family).** For $0 \leqq r < 1$, define the **Poisson family** to be

$$P_r(\theta) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Then $P_r(\theta)$ satisfies the three conditions **DIR 1**, **DIR 2**, **DIR 3** where $k$ is replaced by $r$ and $r \to 1$ instead of $k \to \infty$. In other words:

**DIR 1.** *We have* $P_r(\theta) \geqq 0$ *for all* $r$ *and all* $\theta$.

**DIR 2.** *For all* $r$ *we have*

$$\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1.$$

**DIR 3.** *Given* $\varepsilon$ *and* $\delta$, *there exists* $r_0$, $0 < r_0 < 1$, *such that if* $r_0 < r < 1$, *then*

$$\int_{-\pi}^{-\delta} P_r + \int_{\delta}^{\pi} P_r < \varepsilon.$$

For **DIR 3** you will prove and use the formula

$$P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Theorem 3.1 concerning Dirac sequences applies to the family $\{P_r\}$, again letting $r \to 1$ instead of $k \to \infty$. In other words, let $f$ be a bounded

measurable function on **R** which is periodic. Let $S$ be a compact set on which $f$ is continuous. Let

$$f_r = P_r * f.$$

Then $\{f_r\}$ converges to $f$ uniformly on $S$ as $r \to 1$.

The use of the Poisson family comes from the desire to solve a boundary value problem as follows. We are given a function $f$, viewed as a function on the circle, that is $f(\theta)$ is periodic, as usual. We want to find a function on the disc, that is a function $u(r, \theta)$ with $0 \leqq r \leqq 1$, satisfying the Laplace equation $\Delta u = 0$, where $\Delta$ is the Laplace operator, given in polar coordinates by

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

and such that $u$ has period $2\pi$ in its second variable, that is

$$u(r, \theta) = u(r, \theta + 2\pi).$$

We want $u$ to be continuous, and we want $u(1, \theta)$ to be as much like $f(\theta)$ as possible. If $f$ is continuous on the circle, then we want $u(1, \theta) = f(\theta)$. The convolution

$$u(r, \theta) = (P_r * f)(\theta)$$

solves the problem, because $\Delta P = 0$, so by differentiating under the integral sign we find

$$\Delta(P * f) = (\Delta P) * f = 0.$$

Carry out the details as Exercise 3.

**Example 4 (The Heat Equation for the Laplace Operator).** For $t > 0$, and a real variable $x$, define

$$K_t(x) = K(x, t) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}.$$

Then $\{K_t\}$ is a Dirac family, replacing $k$ by $t$ in the definition of Dirac sequences, and letting $t \to 0$ instead of $k \to \infty$. We define the **heat operator** to be

$$\mathbf{H} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$$

on functions of two variables $(x, t)$. You can easily verify that $\mathbf{H}K = 0$. Thus $K$ satisfies the heat equation. On $\mathbf{R}^n$, we can define $K_t$ and $\mathbf{H}$

similarly, by

$$K_t(x) = K(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-x^2/4t}.$$

Here $x \in \mathbf{R}^n$ is an $n$-tuple, and $x^2$ is the dot product of $x$ with itself. The heat operator is then written

$$\mathbf{H} = \Delta - \frac{\partial}{\partial t},$$

where $\Delta$ is the Laplace operator, $\Delta = \sum (\partial/\partial x_i)^2$. Again $\mathbf{H}K = 0$.

It is Exercise 4 to verify that $\{K_t\}$ is a Dirac family. By arguments similar to those of Example 3, one verifies that for any bounded continuous function $f$, the function $(x, t) \mapsto (K_t * f)(x)$ is a solution of the heat equation on $\mathbf{R}^n \times \mathbf{R}$. The Dirac family $\{K_t\}$ is called the **fundamental heat family**.

For an example of a solution of the heat equation for periodic functions, see Exercise 7.

**Example 5 (The Heat Equation for the Schroedinger Operator).** Recall first the elementary definitions of hyperbolic trigonometric functions:

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2},$$

$$\coth t = \frac{\cosh t}{\sinh t}, \qquad \operatorname{csch} t = \frac{1}{\sinh t}.$$

Let $A_t$ be the $2 \times 2$ matrix given by

$$A_t = \begin{pmatrix} \coth t & -\operatorname{csch} t \\ -\operatorname{csch} t & \coth t \end{pmatrix}.$$

Define the **Mehler family** $\{F_t\}$ on $\mathbf{R}^2$ by the formula

$$F_t(X) = (\operatorname{csch} t)^{1/2} e^{-\pi\langle A_t X, X\rangle} \qquad \text{for} \quad x \in \mathbf{R}^2.$$

Thus $F_t$ is positive, real valued, and we can write

$$F_t(X) = F(X, t) = F(x, y, t) = F_{t,x}(y)$$

if $X$ is the transpose of the vector $(x, y)$ in $\mathbf{R}^2$. Then:

(a) For each fixed $x$, the family $\{F_{t,x}\}$ is a Dirac family for $t \to 0$.

(b) Let

$$L = \frac{1}{4\pi}\left(\frac{\partial}{\partial x}\right)^2 - \pi x^2 - \frac{\partial}{\partial t}.$$

Thus $L$ is the **heat operator** for the **Schroedinger operator**

$$\frac{1}{4\pi}\left(\frac{\partial}{\partial x}\right)^2 - \pi x^2.$$

Then $LF = 0$, where the partial differentiation is with respect to the variables $(x, t)$.

(c) For any reasonable function $f$ of a variable $y$, we have the integral operation

$$F * f(x, t) = \int_{\mathbf{R}} F(x, y, t) f(y)\, dy.$$

Suppose $f$ is bounded and continuous. Then $L(F * f) = 0$, i.e. $F * f$ satisfies the heat equation for the Schroedinger operator.

The proofs are left as Exercise 5. Readers who want to see a general context for this example are referred to Howe's article [How], Section 5. See also [HowT], Chapter III, §2, and Exercise 5 of that chapter. For a somewhat different context, see [BGV], 4.2, p. 154.

For an example when most of the formalism of a Dirac family works but the family is not positive, see Exercise 7.

Next we consider the use of Dirac sequences for approximation in $\mathscr{L}^p$ with $p < \infty$. We shall use **DIR 3s**, i.e. functions with shrinking support.

**Theorem 3.2.** *Let $f$ be in $\mathscr{L}^1(\mathbf{R}^n)$, and let $A$ be a compact set on which $f$ is continuous. Let $\{\varphi_k\}$ be a Dirac sequence with shrinking support. Then $\varphi_k * f$ converges to $f$ uniformly on $A$.*

*Proof.* The proof is identical to the proof of Theorem 3.1 except for the following final modification. At the very last estimate, for $k$ large, the support of $\varphi_k$ is contained in the ball of radius $\delta$, whence the integral expressing $\varphi_k * f(y) - f(y)$ is concentrated on that ball, and is obviously estimated by $\varepsilon$, thus proving our theorem.

**Corollary 3.3.** *The support of $\varphi_k * f$ is contained in*

$$\operatorname{supp} f + \operatorname{supp} \varphi_k.$$

*If $f$ is continuous with compact support, then $\{\varphi_k * f\}$ converges uniformly to $f$ on $\mathbf{R}^n$.*

*Proof.* We have

$$\varphi_k * f(x) = \int \varphi_k(t) f(x - t)\, dt,$$

and this integral is concentrated on the support of $\varphi_k$. If

$$\varphi_k * f(x) \neq 0,$$

then we must have $x - t \in \operatorname{supp} f$, for some $t \in \operatorname{supp} \varphi_k$. Hence

$$x \in \operatorname{supp} f + \operatorname{supp} \varphi_k,$$

thus proving our first assertion. The second assertion follows at once from the first, and from the theorem (both $f$ and $\varphi_k$ being equal to 0 outside some fixed compact set).

**Corollary 3.4.** *Let $f \in \mathscr{L}^p(\mathbf{R}^n)$ for $1 \leqq p < \infty$. Then $\{\varphi_k * f\}$ is $L^p$ convergent to $f$.*

*Proof.* We know that $C_c(\mathbf{R}^n)$ is $L^p$ dense in $\mathscr{L}^p$. Let $g \in C_c(\mathbf{R}^n)$ be such that

$$\|f - g\|_p < \varepsilon.$$

Then we estimate

$$\|\varphi_k * f - f\|_p \leqq \|\varphi_k * f - \varphi_k * g\|_p + \|\varphi_k * g - g\|_p + \|g - f\|_p.$$

Since

$$\int \varphi_k(x)\, dx = 1,$$

we have $\|\varphi_k\|_1 = 1$. Using Theorem 1.2, we find

$$\|\varphi_k * f - \varphi_k * g\|_p = \|\varphi_k * (f - g)\|_p \leqq \|f - g\|_p < \varepsilon.$$

By Theorem 3.2 and Corollary 3.3, $\{\varphi_k * g\}$ converges uniformly to $g$, and has a support which lies close to that of $g$. This implies that

$$\|\varphi_k * g - g\|_p < \varepsilon$$

for $k$ large. The last of the three terms in our estimate above is $< \varepsilon$, thus concluding the proof.

## VIII, §4. THE SCHWARTZ SPACE AND FOURIER TRANSFORM

Let $f$ be a function on $\mathbf{R}^n$. We shall say that $f$ **tends to 0 rapidly at infinity** if for each positive integer $m$ the function

$$x \mapsto (1 + |x|)^m f(x), \qquad x \in \mathbf{R}^n,$$

is bounded for $|x|$ sufficiently large. Here as in the rest of this chapter, $|x|$ is the euclidean norm of $x$. Equivalently, the preceding condition can be formulated by saying that for every polynomial $P$ (in $n$ variables) the function $Pf$ is bounded, or that the function

$$x \mapsto |x|^m f(x)$$

is bounded, for $x$ sufficiently large (i.e. $|x|$ sufficiently large).

We define the **Schwartz space** to be the set of functions on $\mathbf{R}^n$ which are infinitely differentiable (i.e. partial derivatives of all orders exist and are continuous), and which tend to 0 rapidly at infinity, as well as their partial derivatives of all orders.

**Example of such functions.** In one variable, $e^{-x^2}$ is one, and similarly in $n$ variables if we interpret $x^2$ as the dot product $x \cdot x$, which we also write $x^2$. As a matter of notation, we shall write $xy$ instead of $x \cdot y$ if $x$, $y$ are elements of $\mathbf{R}^n$.

If $f$ is in the Schwartz space and $P$ is a polynomial, then the product $Pf$ is in the Schwartz space.

If $f$ is a $C^\infty$ function of one variable which is 0 outside some bounded interval, then $f$ is in the Schwartz space. As an example, one can take the function

$$f(x) = \begin{cases} e^{-1/(x-a)(b-x)} & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

An analogous function in $n$ variables can be obtained by taking the product

$$f(x_1) \cdots f(x_n).$$

It is clear that the Schwartz space is a vector space, which we denote by $\mathrm{Sch}(\mathbf{R}^n)$ or simply by $S$. We take all our functions to be complex valued, so $S$ is a space over $\mathbf{C}$.

We let $D_j$ be the partial derivative with respect to the $j$-th variable. For each $n$-tuple of integers $\geqq 0$, $p = (p_1, \ldots, p_n)$, we write

$$D^p = D_1^{p_1} \cdots D_n^{p_n},$$

so that $D^p$ is a partial differential operator, which maps $S$ into itself. As a matter of notation, we write

$$|p| = p_1 + \cdots + p_n.$$

It is also convenient to use the notation $M_j f$ for the function such that

$$(M_j f)(x) = x_j f(x).$$

Thus, $M_j$ is multiplication by the $j$-th variable. Also

$$M^p f = M_1^{p_1} \cdots M_n^{p_n} f,$$

so that

$$(M^p f)(x) = x_1^{p_1} \cdots x_n^{p_n} f(x).$$

In what follows, we shall take the integral of certain functions over $\mathbf{R}^n$, and we use the following notation:

$$\int f(x)\,dx = \int_{\mathbf{R}^n} f(x)\,dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\,dx_1 \cdots dx_n.$$

Since our functions will be taken from $S$, there is no convergence problem, because for $x$ sufficiently large, we have for some constant $C$:

$$|f(x)| \leqq \frac{C}{(1 + x_1^2) \cdots (1 + x_n^2)},$$

and we can view the integral as a repeated integral, the order of integration being arbitrary. The justification is at the level of elementary calculus. Furthermore, we differentiate under the integral sign, using the formula

$$\frac{\partial}{\partial y_j} \int K(x, y)\,dx = \int \frac{\partial}{\partial y_j} K(x, y)\,dx$$

for suitable functions $K$ in situations where this is obviously permissible (justification loc. cit.), namely when the partial derivatives of $K$ exist, are continuous, and are bounded by an absolutely integrable function of $x$, as in Lemma 2.2.

We shall also change variables in an integral, but nothing worse than the following cases:

$$\int f(x - y)\,dx = \int f(x)\,dx, \qquad \int f(-x)\,dx = \int f(x)\,dx.$$

$$\text{If } c > 0, \text{ then } \int f(cx)\,dx = \frac{1}{c^n} \int f(x)\,dx.$$

The general change of variables formula, of which these are but elementary cases, will be proved in detail in Chapter XXI, §2.

Finally, for normalization purposes, we shall write formally

$$d_1 x = \frac{1}{(2\pi)^{n/2}}\, dx.$$

This makes some formulas come out more symmetrically at the end.

We now define the **Fourier transform** of a function $f \in S$ by

$$\hat{f}(y) = \int f(x) e^{-ixy}\, d_1 x.$$

Remember that $xy = x \cdot y$.

Since

$$\frac{\partial}{\partial y_j} f(x) e^{-ixy} = f(x)(-i)x_j e^{-ixy},$$

we see that we can differentiate under the integral sign, and that

$$D_j \hat{f} = (-i)(M_j f)^{\wedge}.$$

By induction, we get

$$\boxed{D^p \hat{f} = (-i)^{|p|}(M^p f)^{\wedge}.}$$

The analogous formula reversing the roles of $D^p$ and $M^p$ is also true, namely:

$$\boxed{M^p \hat{f} = (-i)^{|p|}(D^p f)^{\wedge}.}$$

To see this, we consider

$$y_j \hat{f}(y) = \int f(x) y_j e^{-ixy}\, d_1 x$$

and integrate by parts with respect to the $j$-th variable first. We let

$$u = f(x) \qquad \text{and} \qquad dv = y_j e^{-ixy}.$$

Then $v = ie^{-ixy}$ and the term $uv$ between $-\infty$ and $+\infty$ gives zero contri-

bution because $f$ tends to 0 at infinity. Hence

$$M_j\hat{f}(y) = (-i)\int D_jf(x)e^{-ixy}\,d_1x = (-i)(D_jf)^\wedge(y).$$

Induction now yields our formula.

**Theorem 4.1.** *The Fourier transform $f \mapsto \hat{f}$ is a linear map of the Schwartz space into itself.*

*Proof.* If $f \in S$, then it is clear that $\hat{f}$ is bounded, in fact by

$$|\hat{f}(y)| \leqq \int |f(x)|\,d_1x.$$

The expression for $M^p\hat{f}$ in terms of the Fourier transform of $D^pf$, which is in $S$, shows that $M^p\hat{f}$ is bounded, so that $\hat{f}$ tends rapidly to zero at infinity. Similarly, one sees that $M^pD^q\hat{f}$ is bounded, because we let $g = D^qf$, $g \in S$, and

$$M^pD^q\hat{f} = (-i)^{|p|}M^p\hat{g}$$

is bounded. This proves our theorem.

For $f, g \in S$ we define the **convolution**

$$f * g(x) = \int f(t)g(x-t)\,d_1t.$$

This integral is obviously absolutely convergent, and the reader will verify at once that the map

$$(f, g) \mapsto f * g$$

is bilinear. Furthermore changing variables shows that

$$f * g = g * f.$$

**Theorem 4.2.** *If $f, g \in S$, then $f * g$ is also in $S$, and*

$$D^p(f * g) = D^pf * g = f * D^pg.$$

*Furthermore,*

$$(f * g)^\wedge = \hat{f}\hat{g}.$$

*Proof.* We can differentiate under the integral sign with respect to $x$, and thus obtain the formula for the partial derivatives $D^p(f * g)$, which

we see exist, are bounded, and are continuous. Now we write

$$\begin{aligned}|x|^m &\leqq (|x-t| + |t|)^m \\ &= \sum c_{jk}|x-t|^j|t|^k\end{aligned}$$

where $c_{jk}$ is a fixed integer depending only on $m$. Then

$$|x|^m|f*g(x)| \leqq \sum c_{jk} \int |t|^k|f(t)|\,|x-t|^j|g(x-t)|\,dt$$

is bounded, and we conclude that $f*g$ tends rapidly to zero at infinity. We can apply the same argument to $D^pf*g$ to conclude that $f*g$ lies in $S$. Finally, we have

$$\begin{aligned}(f*g)^\wedge(y) &= \int (f*g)(x)e^{-ixy}\,d_1x \\ &= \iint f(t)g(x-t)e^{-ixy}\,d_1t\,d_1x,\end{aligned}$$

and we can interchange the order of integration to get

$$= \iint f(t)g(x-t)e^{-ixy}\,d_1x\,d_1t.$$

We change variables, letting $u = x - t$, $d_1u = d_1x$ and see that our last integral is equal to

$$\iint f(t)g(u)e^{-iuy}e^{-ity}\,d_1u\,d_1t = \hat{f}(y)\hat{g}(y),$$

thus proving our theorem.

**Example 1.** We recall the value

$$\int e^{-x^2/2}\,dx = (2\pi)^{n/2}$$

which is obtained first in one variable using polar coordinates. Let

$$h(x) = e^{-x^2/2}.$$

*Then we contend that* $\hat{h} = h$. To see this, we differentiate under the inte-

gral sign to find, say in one variable, that

$$D\hat{h}(y) = -y\hat{h}(y).$$

Thus differentiating the quotient $\hat{h}(y)/e^{-y^2/2}$ yields 0, whence

$$\hat{h}(y) = Ce^{-y^2/2}$$

for some number $C$. This number is equal to 1, using the evaluation of the definite integral recalled above, and our present normalization of the Fourier transform, with $d_1x$ instead of $dx$.

**Example 2.** *Let $a$ be real, $a > 0$, and for any function $h \in S$ let*

$$g(x) = h(ax).$$

*Then*

$$\hat{g}(y) = \frac{1}{a^n}\hat{h}(y/a).$$

This is proved trivially, changing the variable in the integral defining the Fourier transform.

## VIII, §5. THE FOURIER INVERSION FORMULA

If $f$ is a function, we denote by $f^-$ the function such that $f^-(x) = f(-x)$. The reader will immediately verify that the minus operation commutes with all the other operations we have introduced so far. For instance:

$$(\hat{f})^- = (f^-)^\wedge, \qquad (f * g)^- = f^- * g^-, \qquad (fg)^- = f^-g^-.$$

Note that $(f^-)^- = f$.

**Theorem 5.1 (Fourier Inversion).** *For every function $f \in S$ we have $\hat{\hat{f}} = f^-$.*

*Proof.* Let $g$ be some function in $S$. After interchanging integrals, we find

$$\begin{aligned}\int \hat{f}(x)e^{-ixy}g(x)\,d_1x &= \iint f(t)e^{-itx}e^{-ixy}g(x)\,d_1t\,d_1x \\ &= \int f(t)\hat{g}(t+y)\,d_1t.\end{aligned}$$

Let $h \in S$ and let $g(u) = h(au)$ for $a > 0$. Then

$$\hat{g}(u) = \frac{1}{a^n}\hat{h}(u/a),$$

and hence

$$\int \hat{f}(x)e^{-ixy}h(ax)\,d_1x = \int f(t)\frac{1}{a^n}\hat{h}\left(\frac{t+y}{a}\right)d_1t$$
$$= \int f(au - y)\hat{h}(u)\,d_1u$$

after a change of variables,

$$u = \frac{(t+y)}{a}, \qquad d_1u = \frac{d_1t}{a^n}.$$

Both integrals depend on a parameter $a$, and are continuous in $a$. We let $a \to 0$ and find

$$h(0)\hat{\hat{f}}(y) = f(-y)\int \hat{h}(u)\,du = f(-y)\hat{\hat{h}}(0).$$

Let $h$ be the function of Example 1. Then Theorem 5.1 follows.

**Theorem 5.2.** *For every $f \in S$ there exists a function $\varphi \in S$ such that $f = \hat{\varphi}$. If $f, g \in S$,* then

$$(fg)^\wedge = \hat{f} * \hat{g}.$$

*Proof.* First, it is clear that applying the roof operation four times to a function $f$ gives back $f$ itself. Thus $f = \hat{\varphi}$, where $\varphi = f^{\wedge\wedge\wedge}$. Now to prove the formula, write $f = \hat{\varphi}$ and $g = \hat{\psi}$. Then $\hat{f} = \varphi^-$ and $\hat{g} = \psi^-$ by Theorem 5.1. Furthermore, using Theorem 4.2, we find

$$(fg)^\wedge = (\hat{\varphi}\hat{\psi})^\wedge = (\varphi * \psi)^{\wedge\wedge} = (\varphi * \psi)^- = \varphi^- * \psi^- = \hat{f} * \hat{g},$$

as was to be shown.

We introduce the violently convergent hermitian product

$$\langle f, g \rangle = \int f(x)\overline{g(x)}\,dx.$$

We observe that the first step of the proof in Theorem 5.1 yields

$$\int \hat{f}(x)g(x)\,dx = \int f(x)\hat{g}(x)\,dx$$

by letting $y = 0$ on both sides. Furthermore, we have directly from the definitions

$$\hat{\bar{f}} = \bar{\hat{f}}^{-}$$

where the bar means complex conjugate.

**Theorem 5.3 (Parseval Formula).** *For $f, g \in S$ we have*

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

*and hence*

$$\|f\|_2 = \|\hat{f}\|_2.$$

*Proof.* We have

$$\int f\bar{g} = \int \hat{f}^{-}\hat{\bar{g}} = \int \hat{f}^{-}\bar{\hat{g}}^{-} = \int \hat{f}\bar{\hat{g}}.$$

This proves what we wanted.

Theorem 5.3 shows that the map $f \mapsto \hat{f}$ is an automorphism of $S$, preserving the hermitian product and thus the $L^2$-norm, and thus extends to an isometry on $L^2$, since the Schwartz space is dense in $L^2$.

## VIII, §6. THE POISSON SUMMATION FORMULA

A function $g$ on $\mathbf{R}^n$ will be called **periodic** if $g(x + k) = g(x)$ for all $k \in \mathbf{Z}^n$. We let $T^n = \mathbf{R}^n/\mathbf{Z}^n$ be the $n$-torus. Let $g$ be a periodic $C^\infty$ function. We define its $k$-th Fourier coefficient for $k \in \mathbf{Z}^n$ by

$$c_k = \int_{T^n} g(x)e^{-2\pi ikx}\,dx.$$

The integral on $T^n$ is by definition the $n$-fold integral with the variables $(x_1, \ldots, x_n)$ ranging from 0 to 1. Integrating by parts $d$ times for any integer $d > 0$, and using the fact that the partial derivatives of $g$ are bounded, we conclude at once that there is some number $C = C(d, g)$ such that for all $k \in \mathbf{Z}^n$ we have $|c_k| \leqq C/\|k\|^d$, where $\|k\|$ is the sup norm. Hence the Fourier series

$$g(x) = \sum_{k \in \mathbf{Z}^n} c_k e^{2\pi ikx}$$

converges to $g$ uniformly.

If $f$ is in the Schwartz space, we normalize its Fourier transform in this section by

$$\hat{f}(y) = \int_{\mathbf{R}^n} f(x)e^{-2\pi ixy}\,dx$$

**Poisson Summation Formula.** *Let $f$ be in the Schwartz space. Then*

$$\sum_{m\in\mathbf{Z}^n} f(m) = \sum_{m\in\mathbf{Z}^n} \hat{f}(m).$$

*Proof.* Let

$$g(x) = \sum_{k\in\mathbf{Z}^n} f(x+k).$$

Then $g$ is periodic and $C^\infty$. If $c_m$ is its $m$-th Fourier coefficient, then

$$\sum_{m\in\mathbf{Z}^n} c_m = g(0) = \sum_{k\in\mathbf{Z}^n} f(k).$$

On the other hand, interchanging a sum and integral, we get

$$\begin{aligned} c_m = \int_{T^n} g(x)e^{-2\pi imx}\,dx &= \sum_{k\in\mathbf{Z}^n}\int_{T^n} f(x+k)e^{-2\pi imx}\,dx \\ &= \sum_{k\in\mathbf{Z}^n}\int_{T^n} f(x+k)e^{-2\pi im(x+k)}\,dx \\ &= \int_{\mathbf{R}^n} f(x)e^{-2\pi imx}\,dx = \hat{f}(m). \end{aligned}$$

This proves the Poisson summation formula.

## VIII, §7. AN EXAMPLE OF FOURIER TRANSFORM NOT IN THE SCHWARTZ SPACE

The Fourier transform of a function is often a complicated object, but to deal with applications, all that is frequently needed are estimates on its growth behavior. Functions in the Schwartz space provide the simplest class of functions for which the Fourier transform behaves in a particularly simple manner. We give here an example which is more complicated. Let $\varphi$ be the characteristic function of the unit disc in the plane, that is

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leqq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then $\varphi$ has compact support, and is certainly in $\mathscr{L}^1$. Its Fourier transform is therefore given by the integral

$$\hat{\varphi}(y) = \int_{|x| \leqq 1} e^{-2\pi i x \cdot y}\, dx = \int_{|x| \leqq 1} e^{2\pi i x \cdot y}\, dx.$$

This Fourier transform depends only on the distance $s = |y|$, and if we use polar coordinates, then we can rewrite the integral in the form

$$\hat{\varphi}(y) = \int_0^1 \left[ \int_0^{2\pi} e^{2\pi i r s \cos\theta}\, d\theta \right] r\, dr.$$

But the inner integral is a classical **Bessel function**, namely by definition, for any integer $n$ one lets

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ni\theta + iz\sin\theta}\, d\theta.$$

Thus

$$\hat{\varphi}(y) = 2\pi \int_0^1 J_0(2\pi r s) r\, dr.$$

As an example of concrete analysis over the reals, we shall estimate the Bessel function for $z$ real tending to infinity.

**Proposition 7.1.** *We have*

$$J_n(t) \ll t^{-1/2} \qquad \textit{for} \quad t \to \infty.$$

(*The sign* $\ll$ *means that the left-hand side is bounded in absolute value by a constant times the right-hand side, namely* $O(t^{-1/2})$.)

*Proof.* For concreteness, we deal with the case $n = 0$, and we shall just consider a typical integral contributing to $J_0(t)$, namely

$$\int_0^{\pi} e^{it\cos\theta}\, d\theta = \int_{-1}^{1} e^{itu} \frac{du}{\sqrt{1 - u^2}}.$$

Again typically, we show that

$$\int_0^1 e^{itu} \frac{1}{\sqrt{1 - u^2}}\, du = O(t^{-1/2}).$$

We may rewrite the integral in the form

$$\int_0^1 e^{itu}\frac{1}{\sqrt{1-u}}g(u)\,du,$$

where $g(u) = 1/\sqrt{1+u}$ is $C^\infty$ over the interval. Integrating by parts (cf. also Lemma 7.3), we see that the desired integral satisfies the bound:

$$\ll (\|g\| + \|g'\|) \max_{0 \leqq x \leqq 1} \left| \int_0^x e^{itu}\frac{1}{\sqrt{1-u}}\,du \right|.$$

Thus we are reduced to the following lemma.

**Lemma 7.2.** *Let $0 \leqq a \leqq b \leqq 1$. Then uniformly in $a$, $b$ we have*

$$\int_a^b e^{itu}\frac{1}{\sqrt{1-u}}\,du = O(t^{-1/2}).$$

*Proof.* Let $v = 1 - u$, and then $tv = r$. Then the integral is estimated by the absolute value of

$$t^{-1/2}\int_A^B e^{ir}\frac{1}{\sqrt{r}}\,dr,$$

where $0 \leqq A \leqq B$. But writing $e^{ir} = \cos r + i \sin r$, and noting that $1/\sqrt{r}$ is monotone decreasing, we see that the integral on the right-hand side is uniformly bounded independently of $A$, $B$. This proves the lemma, and also concludes the proof of the proposition for $n = 0$.

The integration by parts shows that the asymptotic behavior of the Fourier transform depends only on the singularity. The case just treated is typical, and we let the reader handle the proof in general by using the next lemma, which shows how the singularity affects the estimate.

**Lemma 7.3.** *Let $[a, b)$ be a half-open interval. Let $f$ be a continuous function on this interval, which is also in $\mathscr{L}^1$. Let $g$ be $C^1$ on the closed interval $[a, b]$. Then the Fourier transform satisfies the estimate*

$$\int_a^b g(u)e^{itu}f(u)\,du \ll (\|g\| + \|g'\|)\|F_t\|,$$

*where*

$$F_t(x) = \int_a^x e^{itu}f(u)\,du,$$

*and $\|F_t\|$ is the sup norm for $x \in [a, b]$.*

*Proof.* This is an immediate consequence of integration by parts.

**Theorem 7.4.** *Let $\varphi$ be the characteristic function of the unit disc in the plane. Then*

$$\hat{\varphi}(y) = O(|y|^{-3/2}).$$

*Proof.* As before, let $s = |y|$. By definition, we have

$$\frac{1}{2}\int_0^1 J_0(rs)r\,dr = \int_0^1\int_0^1 \cos(urs)(1-u^2)^{-1/2}\,du\,r\,dr$$

$$[\text{setting } ur = t,\ r\,du = dt] = \int_0^1\int_0^r \cos(ts)\big(1-(t/r)^2\big)^{-1/2}\,dt\,dr$$

$$= \int_0^1 \cos(ts)\int_t^1 \big(1-(t/r)^2\big)^{-1/2}\,dr\,dt$$

$$[\text{by direct integration}] = \int_0^1 \cos(ts)(1-t^2)^{1/2}\,dt$$

$$[\text{integration by parts}] = -\frac{1}{s}\int_0^1 \sin(ts)\frac{t}{\sqrt{1-t^2}}\,dt.$$

Estimating this last integral as in Lemmas 7.2 and 7.3 concludes the proof.

## VIII, §8. EXERCISES

1. Show that the Landau functions form a Dirac sequence.
2. In the case of Fourier series as in Example 2, show that $D_n * f$ is the partial sum of the Fourier series of $f$. Prove that $\{K_n\}$ as defined in Example 2 is a Dirac sequence. (To see this worked out, cf. my *Undergraduate Analysis*, Chapter 12, §3.)
3. Prove all the facts stated in Example 3, namely:
   (a) The Poisson family is a Dirac family.
   (b) The Poisson family satisfies the Laplace equation, so is harmonic on the disc.
   (c) For a continuous periodic function $f$ of $\theta$, the function

   $$F(r, \theta) = P_r * f(\theta)$$

   satisfies Laplace's equation, that is $\Delta F = 0$. You will need to differentiate under the integral sign, and whatnot.
4. Prove the analogous statements for the heat equation of Example 4, replacing the words "Poisson" and "Laplace" by "heat" in the preceding exercise, and

using a function $f$ of a variable $x$ on $\mathbf{R}$ (or $\mathbf{R}^n$) instead of a periodic function of $\theta$.

5. Prove the statements about the heat family for the Schroedinger operator in Example 5.

 The formula for $F(x, y, t)$ can be arrived at naturally as follows. Suppose $F$ has the form

$$F(x, y, t) = \exp\bigl(a(t)(x^2 + y^2) + 2b(t)xy + c(t)\bigr).$$

 Applying the heat operator, one sees that for $F$ to satisfy the heat equation it suffices that the unknown coefficients $a$, $b$, $c$ satisfy ordinary differential equations in the variable $t$, with a solution which is given by the elementary hyperbolic exponential functions as stated.

6. For $y > 0$ define

$$\varphi_y(x) = \frac{1}{\pi}\frac{y}{x^2 + y^2}.$$

 (a) Prove that $\{\varphi_y\}$ is a Dirac family, for $y \to 0$ (instead of $k \to \infty$).

 (b) Show that as a function of two variables $\varphi(x, y) = \varphi_y(x)$ satisfies Laplace's equation, i.e. $\varphi$ is harmonic on the upper half plane. With this construction, we get harmonic functions on the upper half plane having given bounded boundary value on the real line.

 The Dirac family of this example will be used in Chapter XX, §2 in connection with functional analysis.

7. Although Dirac families cover a lot of territory, they are not universally applicable. At the most basic level, for Fourier series, one still wants to know conditions under which the ordinary Fourier series $D_n * f$ converges to the function $f$. We give here an example of an object which satisfies the conditions of a Dirac family except for the positivity, on the circle, so for periodic functions.

 For $t > 0$ and $x \in \mathbf{R}$, let

$$\theta_t(x) = \theta(x, t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{2\pi i n x}.$$

 (a) Show that $\theta$ satisfies the conditions of a Dirac family, except for the positivity condition. Note that $\theta_t$ is periodic in the variable $x$.

 (b) For $f$ continuous periodic, show that $\theta_t * f$ converges to $f$ uniformly as $t \to 0$.

 (c) Show that $\theta$ satisfies the heat equation, and so does $\theta_t * f(x)$, as a function of $(x, t)$. The heat equation is normalized here in the form

$$\frac{1}{4\pi}\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} = 0.$$

8. Let $f \in \mathscr{L}^1(\mathbf{R}) \cap \mathscr{L}^2(\mathbf{R})$ and suppose the function $xf^\wedge(x)$ is in $\mathscr{L}^1(\mathbf{R})$. Prove that there is a $C^1$ function $g$ such that $g = f$ almost everywhere, and give a formula for $g$.

9. Let $A > 0$ and let $f \in \mathscr{L}^1(\mathbf{R})$. Define

$$f_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f^\wedge(t) e^{itx}\, dt.$$

Prove that

$$f_A(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \frac{\sin A(x-y)}{x-y}\, dy.$$

10. Prove that there is a function $h$ in the Schwartz space such that $h^\wedge$ has compact support and $h(0) \neq 0$. Show that for such a function, $h_A = h$ for all $A$ sufficiently large (notation as in Exercise 9).

11. Let $f \in \mathscr{L}^1(\mathbf{R})$. Prove that $f^\wedge$ is uniformly continuous on $\mathbf{R}$.

12. **The lattice point problem.** Let $N(R)$ be the number of lattice points (that is, elements of $\mathbf{Z}^2$) in the closed disc of radius $R$ in the plane. A famous conjecture asserts that

$$N(R) = \pi R^2 + O(R^{1/2+\varepsilon})$$

for every $\varepsilon > 0$. It is known that the error term cannot be $O\big(R^{1/2}(\log R)^k\big)$ for any positive integer $k$ (result of Hardy and Landau). Prove the following best-known result of Sierpinski–Van der Corput–Vinogradov–Hua:

$$N(R) = \pi R^2 + O(R^{2/3}).$$

*Sketch of Proof.* Let $\varphi$ be the characteristic function of the unit disc, and put

$$\varphi_R(x) = \varphi(x/R).$$

Let $\psi$ be a $C^\infty$ function with compact support, positive, and such that

$$\int_{\mathbf{R}^2} \psi(x)\, dx = 1.$$

Let

$$\psi_\varepsilon(x) = \varepsilon^{-2}\psi(x/\varepsilon).$$

Then $\{\psi_\varepsilon\}$ is a Dirac family for $\varepsilon \to 0$, and we can apply the Poisson summation formula to the convolution $\varphi_R * \psi_\varepsilon$ to get

$$\begin{aligned}\sum_{m \in \mathbf{Z}^2} \varphi_R * \psi_\varepsilon(m) &= \sum_{m \in \mathbf{Z}^2} \hat{\varphi}_R(m)\hat{\psi}_\varepsilon(m). \\ &= \pi R^2 + \sum_{m \neq 0} \pi R^2 \hat{\varphi}(Rm)\hat{\psi}(\varepsilon m).\end{aligned}$$

We shall choose $\varepsilon$ depending on $R$ to make the error term best possible.

Note that $\varphi_R * \psi_\varepsilon(x) = \varphi_R(x)$ if $\operatorname{dist}(x, S_R) > \varepsilon$, where $S_R$ is the circle of radius $R$. Therefore we get an estimate

$$|\text{left-hand side} - N(R)| \ll \varepsilon R.$$

Splitting off the term with $m = 0$ on the right-hand side, we find (using Theorem 7.4):

$$\sum_{m \neq 0} R^2 \hat{\varphi}(Rm)\hat{\psi}(\varepsilon m) \ll R^{2-3/2} \sum_{m \neq 0} |m|^{-3/2} \hat{\psi}(\varepsilon m).$$

But we can compare this last sum with the integral

$$\int_1^{\infty} r^{-3/2} \hat{\psi}(\varepsilon r) r \, dr = O(\varepsilon^{-1/2}).$$

Therefore we find

$$N(R) = \pi R^2 + O(\varepsilon R) + O(R^{1/2} \varepsilon^{-1/2}).$$

We choose $\varepsilon = R^{-1/3}$ to make the error term $O(R^{2/3})$, as desired.

For relations of the lattice point problem to the eigenvalue problem see Guillemin [Gui].

## CHAPTER IX

# Integration and Measures on Locally Compact Spaces

On a locally compact space, it is as natural to deal with continuous functions having compact support as it is natural to deal with step functions. Thus we must establish the relations which exist between functionals on the former or the latter. As we shall see, they essentially amount to the same thing.

Thus the main point of this chapter, is to see how one can associate a measure to a functional on $C_c(X)$. Applications will be given in Chapter X and in the spectral theorem of Chapter XX. The measure derived from that situation is called a spectral measure.

Specializing to euclidean spaces, we relate integration with differentiation, using the infinitely differentiable functions and partial derivatives to define distributions, generalizing the notion of measure. There is no question here of going deeply into this theory, but only of showing readers how it arises naturally, and of making it easier for them to read standard treatises devoted to the subject.

If the locally compact space is a locally compact group, then one can ask for the existence of an integral and a positive measure which are invariant under left translations. This is dealt with in Chapter XII.

Both in this chapter and in Chapter XI, we prove the existence of partitions of unity (in the locally compact and locally euclidean cases, respectively). Strictly speaking, this is a tool belonging to general topology, but we postponed dealing with it until it was needed. Such partitions are used to glue together certain maps into a vector space, given locally. They are thus used to reduce certain types of global questions to local ones.

*Throughout this chapter we let $X$ be a locally compact Hausdorff space.*

## IX, §1. POSITIVE AND BOUNDED FUNCTIONALS ON $C_c(X)$

We denote by $C_c(X)$ the vector space of continuous functions on $X$ with **compact support** (i.e. vanishing outside a compact set). We write $C_c(X, \mathbf{R})$ or $C_c(X, \mathbf{C})$ if we wish to distinguish between the real or complex valued functions.

We do not give formally a topology to $C_c(X)$, but observe that there are two natural ones. Of course, we always have the sup norm, defined on $C_c(X)$ since every function is bounded, vanishing outside a compact set.

The other topology would come from considering the subspaces $C(K)$ for each compact subset $K$ of $X$, and observing that $C_c(X)$ is the union of all $C(K)$ for all $K$. One can then give $C_c(X)$ a topology called the inductive limit of the topologies coming from the sup norms on each subspace $C(K)$. We do not go into this here, but we make additional remarks at the end of §4.

We denote by $C_K(X)$ the subspace of $C_c(X)$ consisting of those functions which vanish outside $K$. (Same notation $C_S(X)$ for those functions which are 0 outside any subset $S$ of $X$. Most of the time, the useful subsets in this context are the compact subsets $K$.)

A linear map $\lambda$ of $C_c(X)$ into the complex numbers (or into a normed vector space, for that matter) is said to be **bounded** if there exists some $C \geqq 0$ such that we have

$$|\lambda f| \leqq C\|f\|$$

for all $f \in C_c(X)$. Thus $\lambda$ is bounded if and only if $\lambda$ is continuous for the norm topology.

A linear map $\lambda$ of $C_c(X)$ into the complex numbers is said to be **positive** if we have $\lambda f \geqq 0$ whenever $f$ is real and $\geqq 0$.

**Lemma 1.1.** *Let $\lambda\colon C_c(X) \to \mathbf{C}$ be a positive linear map. Then $\lambda$ is bounded on $C_K(X)$ for any compact $K$.*

*Proof.* By the corollary of Urysohn's lemma, there exists a continuous real function $g \geqq 0$ on $X$ which is 1 on $K$ and has compact support. If $f \in C_K(X)$, let $b = |f|$. Say $f$ is real. Then $bg \pm f \geqq 0$, whence

$$\lambda(bg) \pm \lambda f \geqq 0$$

and $|\lambda f| \leqq b\lambda(g)$. Thus $\lambda g$ is our desired bound.

A complex valued linear map on $C_c(X)$ which is bounded on each subspace $C_K(X)$ for every compact $K$ will be called a **$C_c$-functional** on $C_c(X)$. In accordance with a previous definition, a functional on $C_c(X)$ which is also continuous for the sup norm will be called a **bounded** functional. It is clear that a bounded functional is also a $C_c$-functional.

**Theorem 1.2.** *Let $\lambda$ be a bounded real functional on $C_c(X, \mathbf{R})$. Then $\lambda$ is expressible as the difference of two positive bounded functionals.*

*Proof.* If $f \geqq 0$ is in $C_c(X)$, define

$$\lambda^+ f = \sup \lambda g \quad \text{for} \quad 0 \leqq g \leqq f \quad \text{and} \quad g \in C_c(X, \mathbf{R}).$$

Then $\lambda^+ f \geqq 0$ and $\lambda^+ f \leqq |\lambda| \, \|f\|$. Let $c \in \mathbf{R}$, $c > 0$. Then

$$\lambda^+(cf) = \sup \lambda(cg) \quad \text{for} \quad 0 \leqq g \leqq f,$$

whence $\lambda^+(cf) = c\lambda^+(f)$. Let $f_1$, $f_2$ be functions $\geqq 0$ in $C_c(X, \mathbf{R})$. Then taking $0 \leqq g_1 \leqq f_1$ and $0 \leqq g_2 \leqq f_2$, we have

$$\begin{aligned} \lambda^+ f_1 + \lambda^+ f_2 &= \sup \lambda g_1 + \sup \lambda g_2 \\ &= \sup(\lambda g_1 + \lambda g_2) = \sup \lambda(g_1 + g_2) \\ &\leqq \lambda^+(f_1 + f_2). \end{aligned}$$

Conversely, let $0 \leqq g \leqq f_1 + f_2$. Then $0 \leqq \inf(f_1, g) \leqq f_1$ and

$$0 \leqq g - \inf(f_1, g) \leqq f_2.$$

Hence

$$\begin{aligned} \lambda g &= \lambda(\inf(f_1, g)) + \lambda(g - \inf(f_1, g)) \\ &\leqq \lambda^+ f_1 + \lambda^+ f_2. \end{aligned}$$

Taking the sup on the left implies that $\lambda^+(f_1 + f_2) \leqq \lambda^+ f_1 + \lambda^+ f_2$, thus proving that $\lambda^+$ is additive.

We extend the definition of $\lambda^+$ to all elements of $C_c(X, \mathbf{R})$ by expressing an arbitrary $f$ as a difference

$$f = f_1 - f_2$$

where $f_1, f_2 \geqq 0$ and letting

$$\lambda^+ f = \lambda^+ f_1 - \lambda^+ f_2.$$

The additivity of $\lambda^+$ on functions $\geqq 0$ implies at once that this is well defined, i.e. independent of the expression of $f$ as a difference of positive functions. One then sees at once that this extension of $\lambda^+$ is linear. If $f \geqq 0$, then $\lambda^+ f \geqq 0$ and also $\lambda^+ f \geqq \lambda f$. We now define $\lambda^-$ by

$$\lambda^- f = \lambda^+ f - \lambda f.$$

Then $\lambda^-$ is linear, and

$$\lambda = \lambda^+ - \lambda^-.$$

Furthermore, both $\lambda^+$ and $\lambda^-$ are positive. Finally it is verified at once that $\lambda^+$ and $\lambda^-$ are bounded, thus proving our theorem.

**Note on Terminology.** When dealing exclusively with Banach spaces, as was the case until this section, we used the word functional to apply to linear maps into the scalars, continuous with respect to the given norm. In dealing with $C_c(X)$, we shall usually say functional instead of $C_c$-functional as defined above, and use an adjective (positive, bounded) to describe any additional properties that such a linear map may have.

A positive functional satisfies a strong continuity property with respect to increasing or decreasing sequences of continuous functions.

**Theorem 1.3 (Dini).** *Let $f \in C_c(X, \mathbf{R})$ be $\geqq 0$, and let $\{f_n\}$ be a sequence of positive functions in $C_c(X)$ which is increasing to $f$. Then $\{f_n\}$ converges to $f$ uniformly. More generally, let $\Phi$ be a family of positive functions in $C_c(X, \mathbf{R})$ which are $\leqq f$, and such that*

$$\sup_{\varphi \in \Phi} \varphi = f.$$

*Assume that if $\varphi, \psi \in \Phi$, then $\sup(\varphi, \psi) \in \Phi$. Given $\varepsilon$, there exists $\varphi \in \Phi$ such that $\|f - \varphi\| < \varepsilon$. If $\lambda$ is a positive functional on $C_c(X)$, then*

$$\lambda f = \sup_{\varphi \in \Phi} \lambda \varphi.$$

*Proof.* The assertion concerning the sequence is a special case of the assertion concerning the family. We prove the latter. Let $f$ vanish outside the compact set $K$. For each $x \in K$, we can find a function $\varphi_x \in \Phi$ such that

$$f(x) - \varphi_x(x) < \varepsilon.$$

Then there is some open neighborhood $V_x$ of $x$ such that

$$f(y) - \varphi_x(y) < \varepsilon \qquad \text{for all} \quad y \in V_x.$$

If we cover $K$ by a finite number of such neighborhoods $V_{x_i}$ $(i = 1, \dots, n)$, and let

$$\varphi = \sup(\varphi_{x_1}, \dots, \varphi_{x_n}),$$

then $\|f - \varphi\| < \varepsilon$. This proves the first part of the theorem. The last assertion follows by the continuity of $\lambda$ (Lemma 1.1).

## IX, §2. POSITIVE FUNCTIONALS AS INTEGRALS

The main result of the chapter is to interpret a functional on $C_c(X)$ as an integral. Let $\mathscr{M}$ be the algebra of Borel sets in $X$. If $\mu$ is a positive measure on $\mathscr{M}$ which is finite on compact sets, then $\mu$ gives rise to a positive functional, denoted by $d\mu$, and given by

$$\langle f, d\mu \rangle = \int_X f\, d\mu.$$

We shall prove the converse (Riesz' theorem), and first obtain a positive measure from a positive functional.

If $f$ is a function on $X$, we define its **support** to be the closure of the set of all $x$ such that $f(x) \neq 0$. Thus the support is a closed set. We denote it by $\operatorname{supp}(f)$.

We use the following notation as in Rudin [Ru 1], which we more or less follow for the proof of Theorem 2.3. If $V$ is open, we write

$$f \prec V$$

to mean that $f$ is real, $f \in C_c(X)$, $0 \leqq f \leqq 1$, and $\operatorname{supp}(f) \subset V$. Similarly, if $K$ is compact we write

$$K \prec f$$

to mean $\chi_K \leqq f \leqq 1$, and of course $f \in C_c(X)$.

**Lemma 2.1.** *Given $K$ compact, $K \subset V$ open, there exists some $f$ such that*

$$K \prec f \prec V.$$

*Proof.* This is an immediate consequence of Urysohn's lemma. All we have to do is choose some open set $W$ with compact closure $\overline{W}$ such that $K \subset W \subset \overline{W} \subset V$, and use the normality of $\overline{W}$ to find a function $f$ with $0 \leqq f \leqq 1$ which is 1 on $K$ and 0 on the boundary of $\overline{W}$. We then extend this function to be 0 outside $\overline{W}$. This extension is continuous on all of $X$.

**Lemma 2.2.** *If $V$ is open, then we have*

$$\chi_V = \sup f \qquad \textit{for} \quad f \prec V.$$

*Proof.* Given $x \in V$, there exists an open neighborhood $W$ of $x$ such that $x \in W \subset \overline{W} \subset V$, and such that $\overline{W}$ is compact. We can find a function $f$ with $0 \leqq f \leqq 1$ such that $f(x) = 1$ and $f$ is 0 on the complement of $W$, by the corollary of Urysohn's lemma. This proves our assertion.

**Theorem 2.3 (Riesz Theorem, Part 1).** *Let $\lambda$ be a positive functional on $C_c(X)$. There exists a unique positive Borel measure satisfying conditions* (i) *and* (ii) *below, and this measure also satisfies* (iii) *and* (iv).

(i) *If $V$ is open, then*

$$\mu(V) = \sup \lambda g \qquad \textit{for} \quad g \prec V.$$

(ii) *If $A$ is a Borel set, then*

$$\mu(A) = \inf \mu(V) \qquad \textit{for } V \textit{ open} \supset A.$$

(iii) *If $K$ is compact, then $\mu(K)$ is finite.*

(iv) *If $A$ is a Borel set and $A$ is $\sigma$-finite, or $A$ is open, then*

$$\mu(A) = \sup \mu(K) \qquad \textit{for } K \textit{ compact} \subset A.$$

**Remark 1.** From (ii) and the remarks before Theorem 2.3, we see at once that for any compact $K$ we have

$$\mu(K) = \inf \lambda f \qquad \text{for} \quad K \prec f.$$

**Remark 2.** The uniqueness of $\mu$ satisfying (i) and (ii) is obvious, because (i) determines $\mu$ on open sets, and (ii) determines $\mu$ on all Borel sets.

**Remark 3.** It is convenient to introduce a word to summarize the main properties listed in Theorem 2.2. We shall say that a positive measure $\mu$ on a locally compact space, defined on a $\sigma$-algebra $\mathscr{M}$ containing the Borel sets, is **$\sigma$-regular** if it satisfies properties (ii), (ii), and (iv) of Theorem 2.2 for the sets of $\mathscr{M}$. Even though $X$ itself need not be $\sigma$-finite, in applications only $\sigma$-finite sets arise because for instance any function in $\mathscr{L}^1$ is equivalent to a function vanishing outside a $\sigma$-finite set.

Let $\mathscr{M}$ be a $\sigma$-algebra containing the Borel sets. A positive measure $\mu$ on $\mathscr{M}$ is said to be **regular** if $\mu(K)$ is finite for all compact $K$, and if in

addition, for all $A \in \mathscr{M}$ we have

$$\mu(A) = \inf \mu(V) \qquad \text{for } V \text{ open} \supset A,$$

$$\mu(A) = \sup \mu(K) \qquad \text{for } K \text{ compact} \subset A.$$

Cases when a measure is not regular are to be regarded as pathological. Our definitions are adjusted so that the following statement is merely a rephrasing of parts of our definitions:

*If $X$ is $\sigma$-finite and $\mu$ is $\sigma$-regular, then $\mu$ is regular.*

Note that the property that $\mu(V) = \sup \mu(K)$ for compact $K \subset V$ is satisfied by open $V$. This is convenient because even in pathological situations, we are able to define the measure of Theorem 2.3 on Borel sets rather than on a more restricted algebra (e.g. that generated by the compact sets, as is sometimes done in the literature). Observe that if we know that property (iv) is satisfied by all sets $A$ of finite measure, then it follows at once for any $\sigma$-finite $A$. Indeed, let $\{A_n\}$ be a disjoint sequence of sets of finite measure, and let $K_n$ be a compact subset of $A_n$ such that $\mu(A_n - K_n) < \varepsilon/2^n$. Then $K_1 \cup \cdots \cup K_n$ is compact, and $\mu(K_1 \cup \cdots \cup K_n)$ tends to the measure of $\bigcup A_n$ as $n \to \infty$, whether this measure is finite or not.

For an example of pathology, let $I$ be a non-countable set of indices, and let $\{X_i\}$ $(i \in I)$ be disjoint copies of the interval $[0, 1]$. Let $x_i$ be a point in $X_i$, and let $S$ be the union of all $x_i$. Then $S$ is discrete, and has infinite measure, but all compact subsets of $S$ are finite and have measure 0.

*We now come to the proof of Theorem* 2.3.

We shall actually define $\mu$ on a larger algebra than that of the Borel sets, more or less the largest algebra such that the measure still satisfies our four conditions. For instance, it is clear that *the complete measure obtained from a measure satisfying our properties still satisfies these properties.*

Until the end of the proof of Theorem 2.6 we let $f$, $g$, $h$, denote elements of $C_c(X)$ which are real $\geqq 0$.

**Lemma 2.4.** *Let $\lambda$ be a positive functional on $C_c(X)$. For each open set $V$, define*

$$\mu(V) = \sup \lambda g \qquad \textit{for} \quad g \prec V.$$

*For any subset $Y$ of $X$ define*

$$\mu(Y) = \inf \mu(V) \qquad \textit{for} \quad Y \subset V.$$

*Then $\mu$ is an outer measure on the algebra of all subsets of $X$.*

*Proof.* It is clear that $\mu(\varnothing) = 0$ and that if $A \subset B$, then $\mu(A) \leqq \mu(B)$. For convenience of notation, we write

$$\sup(f, g) = f \cup g \qquad \text{and} \qquad \inf(f, g) = f \cap g.$$

We prove that if $V_1$, $V_2$ are open, then

$$\mu(V_1 \cup V_2) \leqq \mu(V_1) + \mu(V_2). \tag{1}$$

Let $h \prec V_1 \cup V_2$. Let $\Phi$ be the family of all functions $\sup(g_1, g_2)$ with $g_i \prec V_i$ for $i = 1, 2$. Then $\Phi$ is closed under the sup operation (on a finite number of elements), and we have

$$\sup_{g \in \Phi} g = \chi_{V_1 \cup V_2}.$$

Let $\Phi_h$ be the family of all functions

$$h \cap (g_1 \cup g_2), \qquad g_i \prec V_i, \quad i = 1, 2.$$

Then $h$ is the sup of all functions in $\Phi_h$, whence by Theorem 1.3

$$\begin{aligned} \lambda h &= \sup_{g_1, g_2} \lambda(h \cap (g_1 \cup g_2)) \\ &\leqq \sup_{g_1, g_2} \lambda(h \cap g_1 + h \cap g_2) \\ &\leqq \mu(V_1) + \mu(V_2). \end{aligned}$$

Taking the sup over all $h$ on the left yields our inequality (1).

Now let $\{A_n\}$ be a sequence of subsets of $X$, and $A = \bigcup A_n$. Let $V_n$ be open, $A_n \subset V_n$, and

$$\mu(V_n) \leqq \mu(A_n) + \frac{\varepsilon}{2^n}.$$

Let $V = \bigcup V_n$. Then $\bigcup A_n \subset \bigcup V_n = V$. Let $g \prec V$. Since $g$ has compact support, there is some $n$ such that

$$g \prec V_1 \cup \cdots \cup V_n,$$

whence by (1) and induction,

$$\lambda g \leqq \mu(V_1) + \cdots + \mu(V_n).$$

Taking the sup over all $g$ on the left yields

$$\mu(A) \leqq \mu(V) \leqq \sum \mu(A_n) + \varepsilon.$$

This proves our lemma.

**Remark.** The special role played by compact and open sets in constructing the algebra of measurable sets and the measure on it will stem from the following property:

*If* $K \prec f \prec V$, *then* $\mu(K) \leqq \lambda f \leqq \mu(V)$.

The right inequality is obvious. As for the left one, let $W$ be the set of $x$ such that $f(x) > 1 - \varepsilon$. Then $K \subset W$. Let $g \prec W$ be such that

$$\mu(W) \leqq \lambda g + \varepsilon.$$

We have

$$(1 - \varepsilon)g \leqq f \qquad \text{whence} \quad (1 - \varepsilon)\lambda g \leqq \lambda f.$$

Hence

$$\mu(K) \leqq \mu(W) \leqq \lambda g + \varepsilon \leqq \frac{\lambda f}{1 - \varepsilon} + \varepsilon.$$

This implies that $\mu(K) \leqq \lambda f$, as contended.

Our remark shows the main idea of what follows. We recover characteristic functions of certain sets by squeezing them between compact and open sets, and comparing them with functions $f \in C_c(X)$ on which the given functional is defined. The $K$ and $V$ allow us to use the old technique of lower and upper sums respectively. We first have to recover the measure itself, however, and we proceed to do this. For convenience of notation, the outer measure $\mu$ described in Lemma 2.4 will be called the **outer measured determined by** $\lambda$.

**Lemma 2.5.** *Let* $\mathscr{A}$ *be the collection of all subsets* $A$ *of* $X$ *such that* $\mu(A) < \infty$, *and*

$$\mu(A) = \sup \mu(K) \qquad \textit{for } K \textit{ compact} \subset A.$$

*Then* $\mathscr{A}$ *is an algebra containing all compact sets and all open sets of*

*finite measure. Furthermore, $\mu$ is a positive measure on $\mathscr{A}$. In fact, if $\{A_n\}$ is a disjoint sequence of elements of $\mathscr{A}$, and $A = \bigcup A_n$, then*

$$\mu(A) = \sum \mu(A_n).$$

*If in addition $\mu(A) < \infty$, then $A \in \mathscr{A}$.*

*Proof.* If $K$ is compact, then there exists an open $V$ containing $K$ such that $\bar{V}$ is compact. Let $\bar{V} \prec g$. For any $f \prec V$ we have $f \leqq g$, whence $\lambda f \leqq \lambda g$ and $\mu(V) \leqq \lambda g$, so that $\mu(K) \leqq \lambda g$ is finite. It is then clear that $K \in \mathscr{A}$.

Let $V$ be open. We shall prove that

$$\mu(V) = \sup \mu(K) \qquad \text{for } K \text{ compact, } K \subset V.$$

We may assume that $\mu(V) > 0$. To cover the case when $\mu(V) = \infty$, we let $r$ be a real number such that $0 < r < \mu(V)$. There exists $f$ such that

$$r < \lambda f \leqq \mu(V).$$

Let $K$ be the support of $f$. If $W$ is an open set containing $K$, then $f \prec W$, whence

$$r < \lambda f \leqq \mu(W),$$

and therefore $r \leqq \mu(K)$. This proves that $\mu(V) = \sup \mu(K)$ for $K$ compact $\subset V$. In particular, if $\mu(V)$ is finite, then $V \in \mathscr{A}$.

Before proving that $\mathscr{A}$ is an algebra, we find it convenient to have the finite additivity. Actually, it is no more troublesome to prove the countable additivity. First we prove that if $K_1$, $K_2$ are disjoint and compact, then

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2).$$

Let $V_1$, $V_2$ be disjoint open sets containing $K_1$, $K_2$ respectively. Let $W$ be open such that

$$\mu(W) \leqq \mu(K_1 \cup K_2) + \varepsilon.$$

Let $g_i \prec W \cap V_i$ be such that for $i = 1, 2$ we have

$$\mu(W \cap V_i) \leqq \lambda g_i + \varepsilon.$$

Then

$$\begin{aligned}
\mu(K_1) + \mu(K_2) &\leqq \mu(W \cap V_1) + \mu(W \cap V_2) \\
&\leqq \lambda g_1 + \lambda g_2 + 2\varepsilon \\
&= \lambda(g_1 + g_2) + 2\varepsilon \\
&\leqq \mu(W) + 2\varepsilon \leqq \mu(K_1 \cup K_2) + 3\varepsilon.
\end{aligned}$$

The reverse inequality is true because $\mu$ is an outer measure, so we get the desired equality.

Given a sequence of disjoint sets $\{A_n\}$ in $\mathscr{A}$, let $K_n \subset A_n$ be compact, such that

$$\mu(A_n) \leqq \mu(K_n) + \frac{\varepsilon}{2^n}.$$

Let $A = \bigcup A_n$. Then for all $n$,

$$\sum_{i=1}^{n} \mu(A_i) \leqq \sum_{i=1}^{n} \mu(K_i) + \varepsilon = \mu(K_1 \cup \cdots \cup K_n) + \varepsilon$$
$$\leqq \mu(A) + \varepsilon.$$

Letting $n$ tend to infinity, and then $\varepsilon$ tend to 0, together with the fact that $\mu$ is an outer measure, shows that

$$\sum_{n=1}^{\infty} \mu(A_n) = \mu(A).$$

This gives the countable additivity and also proves that $A \in \mathscr{A}$ if $\mu(A)$ is finite.

We can now prove that $\mathscr{A}$ is an algebra. Clearly the empty set is in $\mathscr{A}$. If $A_1, A_2 \in \mathscr{A}$, we can find compact sets $K_1$, $K_2$ and open sets $V_1$, $V_2$ such that

$$K_i \subset A_i \subset V_i \qquad (i = 1, 2)$$

and such that for $i = 1, 2$ we have

$$\mu(K_i) \leqq \mu(A_i) \leqq \mu(V_i) < \mu(K_i) - \varepsilon.$$

In particular, by the finite additivity of $\mu$, we have

$$\mu(V_i - K_i) < \varepsilon.$$

Since

$$(V_1 \cup V_2) - (K_1 \cup K_2) \subset (V_1 - K_1) \cup (V_2 - K_2),$$

we get

$$\mu(A_1 \cup A_2) < \mu(K_1 \cup K_2) + 2\varepsilon.$$

It follows that $A_1 \cup A_2$ lies in $\mathscr{A}$. Next we note that $K_1 - V_2$ is compact, $V_1 - K_2$ is open, and

$$(K_1 - V_2) \subset (A_1 - A_2) \subset (V_1 - K_2).$$

The difference of the two extreme sets satisfies

$$(V_1 - K_2) - (K_1 - V_2) \subset (V_1 - K_1) \cup (V_2 - K_2)$$

so that $A_1 - A_2$ lies in $\mathscr{A}$. Since we can write

$$A_1 \cap A_2 = A_2 - (A_2 - A_1)$$

it follows that $A_1 \cap A_2$ lies in $\mathscr{A}$, thus showing that $\mathscr{A}$ is an algebra, and proving our lemma.

**Theorem 2.6.** *Let $\lambda$ be a positive functional on $C_c(X)$. Let $\mu$ be the outer measure determined by $\lambda$, and let $\mathscr{A}$ be the algebra of all sets $A$ of finite measure such that*

$$\mu(A) = \sup \mu(K) \qquad \text{for } K \text{ compact} \subset A.$$

*Let $\mathscr{M}$ be the collection of all subsets $Y$ of $X$ such that $Y \cap K$ lies in $\mathscr{A}$ for all compact $K$. Then $\mathscr{M}$ is a $\sigma$-algebra containing the Borel sets, and $\mu$ is a positive measure on $\mathscr{M}$. Furthermore, $\mathscr{A}$ consists of the sets of finite measure in $\mathscr{M}$.*

*Proof.* It is clear that $\mathscr{A} \subset \mathscr{M}$. Let $\mathscr{M}_K$ as usual denote the collection of all sets $Y \cap K$ with $Y \in \mathscr{M}$. Then $\mathscr{M}_K = \mathscr{A}_K$, and is therefore a $\sigma$-algebra in $K$ for each compact $K$, by Lemma 2.5. It follows immediately that $\mathscr{M}$ itself is a $\sigma$-algebra, because the operations of countable union, intersection, and complementation in $X$ commute with the operation of intersecting with $K$. (Cf. Lemma 6.2 of Chapter VI where we met a similar situation.)

That $\mathscr{M}$ contains all closed sets is obvious because if $Y$ is closed and $K$ compact, then $Y \cap K$ is compact and so lies in $\mathscr{A}$. Therefore $\mathscr{M}$ contains the Borel sets.

Let $A$ be of finite measure in $\mathscr{M}$. Let $V$ be open containing $A$, and of finite measure. Let $K$ be compact $\subset V$ such that

$$\mu(V) < \mu(K) + \varepsilon.$$

Since $A \cap K$ lies in $\mathscr{A}$, there is some compact $K' \subset A \cap K$ such that

$$\mu(A \cap K) < \mu(K') + \varepsilon.$$

But $A \subset (A \cap K) \cup (V - K)$, so that

$$\mu(A) \leqq \mu(A \cap K) + \mu(V - K) \leqq \mu(K') + 2\varepsilon.$$

This proves that $A$ lies in $\mathscr{A}$, and therefore that $\mathscr{A}$ is precisely the algebra of sets of finite measure in $\mathscr{M}$.

Finally, let $\{A_n\}$ be a disjoint sequence in $\mathscr{M}$. If some $A_n$ has infinite measure, the countable additivity of $\mu$ on $\bigcup A_n$ is clear. If all $A_n$ have finite measure, then Lemma 2.5 applies. This proves our theorem.

The measure of Theorem 2.3 (or Theorem 2.6) will be called the **associated measure of** $\lambda$, or **the measure determined by** $\lambda$. In applications, one needs it mainly on the Borel sets (or the completion of the Borel sets).

We now wish to prove that the functional $\lambda$ is given by the integral. First we note that *if $\mu$ is a $\sigma$-regular measure and $f \in C_c(X)$, then $f$ is in $\mathscr{L}^1(\mu)$.* Indeed, $f$ being continuous implies that $f$ is measurable. Also $f$ vanishes outside a compact set (so of finite measure), and is bounded on that set, and hence $f$ is in $\mathscr{L}^1(\mu)$, say by Corollary 5.9 of the dominated convergence theorem (Chapter VI). Next we need a lemma.

**Partitions of Unity.** *Let $K$ be compact and let $\{U_1, \dots, U_n\}$ be an open covering of $K$. There exist functions $f_i$ $(i = 1, \dots, n)$ such that $f_i \prec U_i$ and such that*

$$\sum_{i=1}^{n} f_i(x) = 1, \qquad \textit{all } x \in K.$$

*Proof.* For each $x \in K$ let $W_x$ be an open neighborhood of $x$ such that $\overline{W}_i \subset U_{i(x)}$ for some index $i(x)$. We can cover $K$ by a finite number of open sets $W_{x_1}, \dots, W_{x_m}$. Let $V_i$ be the union of all open sets $W_{x_j}$ such that $\overline{W}_{x_j} \subset U_i$. Then $\{V_1, \dots, V_n\}$ is an open covering of $K$. Furthermore $\overline{V}_i \subset U_i$. Let $g_i$ be a function such that

$$\overline{V}_i \prec g_i \prec U_i.$$

Let

$$\begin{aligned} f_1 &= g_1, \\ f_2 &= g_2(1 - g_1), \\ &\vdots \\ f_n &= g_n(1 - g_1) \cdots (1 - g_{n-1}). \end{aligned}$$

Then $f_i \prec U_i$, and by induction one sees at once that

$$f_1 + \cdots + f_n = 1 - (1 - g_1) \cdots (1 - g_n).$$

From this our condition $\sum f_i(x) = 1$ for $x \in K$ follows at once.

The functions $\{f_i\}$ are said to form a **partition of unity over $K$, subordinate to the covering $\{U_1,\dots,U_n\}$**.

**Theorem 2.7 (Riesz Theorem, Part 2).** *Let $\lambda$ be a positive functional on $C_c(X)$, and let $\mu$ be the Borel measure determined by $\lambda$. For all $f \in C_c(X)$ we have*

$$\lambda f = \int_X f \, d\mu.$$

*Proof.* It suffices to prove our statement when $f$ is real, $\|f\| \neq 0$. It will also suffice to prove the inequality

$$\lambda f \leqq \int_X f \, d\mu$$

(the reverse inequality following by considering $-f$ instead of $f$). Let $K$ be the support of $f$. Given $\varepsilon$, which we may assume $\leqq \|f\|$, we can find a partition $\{A_1,\dots,A_n\}$ of $K$ by measurable sets, a step function

$$\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$$

such that $f \leqq \varphi \leqq f + \varepsilon$, and open sets $V_i \supset A_i$ such that

$$(*) \qquad \mu(V_i) \leqq \mu(A_i) + \frac{\varepsilon}{n\|f\|}$$

and also that $f \leqq c_i$ on $V_i$. [For instance, cut an interval containing the image of $f$ into $\varepsilon/2$-subintervals, say half closed to make them disjoint, and let $c_i'$ be the right end point of each subinterval. Let $A_i$ be the inverse image in $K$ of the $i$-th subinterval. Let $c_i = c_i' + \varepsilon/2$. For each $i$ let $W_i$ be open $\supset A_i$ such that $f \leqq c_i$ on $W_i$, and shrink $W_i$ to an open $V_i \supset A_i$ satisfying $(*)$.]

Let $\{h_1,\dots,h_n\}$ be a partition of unity over $K$ subordinate to $\{V_1,\dots,V_n\}$. Then $fh_i$ has support in $V_i$, and $fh_i \leqq c_i h_i$. Furthermore, $K \prec \inf(1, \sum h_i)$, so that

$$\mu(K) \leqq \lambda(\textstyle\sum h_i) = \sum \lambda h_i.$$

Let $c = \max|c_i|$. Then $c \leqq f + \varepsilon$. We have

$$\begin{aligned}\lambda f &= \sum_{i=1}^{n} \lambda(fh_i) \leqq \sum_{i=1}^{n} \lambda(c_i h_i) \\ &= \sum_{i=1}^{n} c_i \lambda h_i = \sum (c_i + c)\lambda h_i - c \sum \lambda h_i\end{aligned}$$

$$\leqq \sum_{i=1}^{n} (c_i + c)\mu(V_i) - c\mu(K)$$

$$\leqq \sum_{i=1}^{n} (c_i + c)\left[\mu(A_i) + \frac{\varepsilon}{n\|f\|}\right] - c\mu(K)$$

$$\leqq \int_K \varphi \, d\mu + 4\varepsilon + c\mu(K) - c\mu(K)$$

$$\leqq \int_K f \, d\mu + \varepsilon\mu(K) + 4\varepsilon.$$

This proves our inequality since the integral of $f$ over $K$ is the same as the integral of $f$ over $X$, and concludes the proof of our theorem.

**Corollary 2.8.** *Let $M_0$ be the set of $\sigma$-regular positive Borel measures on $X$. The map*

$$\mu \mapsto d\mu$$

*is an additive bijection between $M_0$ and the set of positive functionals on $C_c(X)$.*

*Proof.* Theorems 2.3 and 2.7 show that the map $\mu \mapsto d\mu$ is surjective. Let $\mu_1$, $\mu_2$ be positive measures satisfying conditions (ii), (iii), and (iv) of Theorem 2.3 and assume that $d\mu_1 = d\mu_2$. To show that $\mu_1 = \mu_2$, it suffices to prove that the two measures coincide on compact sets, because then (iv) shows that they coincide on open sets, and (ii) shows that they are equal on Borel sets. Let $K$ be compact, and let $V$ be an open set containing $K$ such that

$$\mu_2(V) < \mu_2(K) + \varepsilon.$$

Let $K \prec f \prec V$. Then $\chi_K \leqq f \leqq \chi_V$, whence

$$\mu_1(K) \leqq \int_X f \, d\mu_1 = \int_X f \, d\mu_2 \leqq \mu_2(V) \leqq \mu_2(K) + \varepsilon.$$

This proves one inequality, and the other follows by symmetry. Thus we get a bijection between $M_0$ and the set of positive functionals on $C_c(X)$. This bijection is obviously additive. This proves our corollary.

## IX, §3. REGULAR POSITIVE MEASURES

**Theorem 3.1.** *Let $\mu$ be a positive $\sigma$-regular Borel measure on $X$. Then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leqq p < \infty$.*

*Proof.* The step functions are dense in $L^p(\mu)$, and it thus suffices to prove that for any set $A$, of finite measure given $\varepsilon$ we can find some $f \in C_c(X)$ such that

$$\|\chi_A - f\|_p < \varepsilon.$$

We take $K$ compact, $V$ open such that $K \subset A \subset V$, and $\mu(V) < \mu(K) + \varepsilon$. Let $K \prec f \prec V$. Then

$$\mu(K) \leqq \int_X f \, d\mu \leqq \mu(V)$$

and

$$\|\chi_A - f\|_p \leqq \|\chi_A - \chi_K\|_p + \|f - \chi_K\|_p < \varepsilon^{1/p} + \varepsilon^{1/p},$$

thus proving our assertion.

**Corollary 3.2.** *If $f \in \mathscr{L}^1(\mu)$ and $\int f\varphi \, d\mu = 0$ for all $\varphi \in C_c(X)$, then $f = 0$ almost everywhere.*

*Proof.* Let $A$ be a set of finite measure. Then $\chi_A$ is the $L^1$-limit of a sequence $\{\varphi_n\}$ in $C_c(X)$ with $0 \leqq \varphi_n \leqq 1$. Taking a subsequence if necessary, we may assume, by Theorem 5.2 of Chapter VI, that $\{\varphi_n\}$ converges to $\chi_A$ almost everywhere, and thus $\{f\varphi_n\}$ converges to $f\chi_A$ almost everywhere. By the dominated convergence theorem, we conclude that $\int f\chi_A \, d\mu = 0$, and Corollary 5.16 of Chapter VI finishes the proof.

The next theorem shows that a measurable function is almost continuous, on a set of finite measure.

**Theorem 3.3 (Lusin's Theorem).** *Let $\mu$ be a positive $\sigma$-regular Borel measure on $X$. Let $f$ be a complex measurable function on $X$, and assume that there exists a set $A$ of finite measure such that $f$ is equal to $0$ outside $A$. Given $\varepsilon$, there exists $g \in C_c(X)$ and a measurable set $Z$ with $\mu(Z) < \varepsilon$ such that $f(x) = g(x)$ for $x \in X - Z$. Furthermore, we can select $g$ such that $\|g\| \leqq \|f\|$ (sup norm).*

*Proof.* Let $A_n$ be the set where $|f| \geqq n$. Since the intersection of all $A_n$ is empty, it follows that the measures $\mu(A_n)$ approach 0. Excluding a set of small measure, we suppose that $f$ is bounded.

In this case, $f$ is in $\mathscr{L}^1(\mu, \mathbf{C})$. By Theorem 3.1 there exists a sequence $\{g_n\}$ in $C_c(X)$ which is $L^1$-convergent to $f$. Taking a subsequence if necessary, and using Theorem 5.2 of Chapter VI, we may assume that there is a set $Z$ with $\mu(Z) < \varepsilon$ such that the convergence is uniform outside $Z$. By regularity, we can find a compact set $K$ contained in $A - Z$ such that $\mu(A - K) < 2\varepsilon$. The convergence of $\{g_n\}$ is uniform on

$K$, and hence the restriction of $f$ to $K$ is a continuous function $g$ on $K$. Let $V$ be open $\supset K$ such that $\bar{V}$ is compact. By Theorem 4.4 of Chapter 2 (Tietze extension theorem) we can find a continuous function $g^*$ which is equal to $g$ on $K$ and 0 on the boundary of $\bar{V}$. We extend $g^*$ to all of $X$ by giving $g^*$ the value 0 outside $\bar{V}$. Then $g^*$ is equal to $f$ on $K$ and is in $C_c(X)$.

This leaves only the last statement, that we can manage $\|g\| \leqq \|f\|$. Let $b = \|f\|$. Let $h$ be the function such that $h(z) = z$ if $|z| \leqq b$ and $h(z) = bz/|z|$ if $|z| > b$. Then $h$ is continuous, $\|h\| \leqq b$, and $h \circ g^*$ fulfills our requirements, thus proving Lusin's theorem.

## IX, §4. BOUNDED FUNCTIONALS AS INTEGRALS

Let $m$ be a complex valued measure on the Borel sets of $X$. We shall say that $m$ is **regular** if $|m|$ is regular. See Exercise 7 for examples. Recall that for complex measures, $|m|$ is always bounded, by Theorem 3.3 of Chapter VII, and that we can define the norm $\|m\| = |m|(X)$.

**Theorem 4.1.** *The complex regular Borel measures on X form a Banach space.*

*Proof.* We leave most of the proof as an exercise. We shall just prove that if $m_1, m_2$ are regular, then $m_1 + m_2$ is regular. Indeed, we have

$$|m_1 + m_2| \leqq |m_1| + |m_2|.$$

For any Borel set $A$ we select $K$ compact in $A$ such that

$$|m_1|(A - K) < \varepsilon \qquad \text{and} \qquad |m_2|(A - K) < \varepsilon.$$

Then $|m|(A - K) < 2\varepsilon$. Similarly for open sets, whence $m_1 + m_2$ is regular.

We wish to interpret regular Borel measures as bounded functionals on $C_c(X)$. The easiest way at this point is to use the Radon–Nikodym theorem, and write

$$dm = hd|m|$$

for some $h \in \mathscr{L}^1(|m|, \mathbf{C})$, with $|h| = 1$ (Theorem 3.5 of Chapter VII). Thus by definition, for $f \in C_c(X)$, we define

$$\langle f, dm \rangle = \int_X fh \, d|m|.$$

Let us denote by $M_0(X, \mathbf{C}) = M_0$ the Banach space of complex regular Borel measures on $X$. The map

$$m \mapsto dm$$

is then a linear map of $M_0$ into the dual space of $C_c(X)$ (sup norm), because we have the inequality

$$|dm| \leqq \|m\|,$$

or written out explicitly,

$$\left| \int_X f \, dm \right| \leqq \|f\| \, \|m\|.$$

In fact:

**Theorem 4.2 (Riesz Theorem, Part 3).** *The map $m \mapsto dm$ is a norm-preserving isomorphism between the space of regular complex Borel measures on $X$ and the dual space of $C_c(X)$ (with sup norm topology).*

*Proof.* Our map is obviously linear. To show that it is surjective, we view any bounded functional $\lambda$ as a functional on $C_c(X, \mathbf{R})$ and then decompose $\lambda$ into its real and imaginary parts, say $\lambda = \sigma + i\tau$, where $\sigma$, $\tau$ are then bounded functionals. We express each real bounded functional as a difference of positive functionals using Theorem 1.2, and apply part 1 of the Riesz theorem to these positive functionals to represent them by positive measures. If $\pi$ is a positive bounded functional and $\mu$ is the measure which represents $\pi$ by Theorem 2.3, then $\mu(X) < \infty$. To see this, note that by condition (iv) of this theorem, we have

$$\mu(X) = \sup \mu(K) \qquad \text{for } K \text{ compact.}$$

If $K \prec f$, we must have

$$\mu(K) \leqq \int_X f \, d\mu = \pi f \leqq C\|f\| = C$$

where $C = |\pi|$, so that in fact $\mu(X) \leqq |\pi|$. By definition and the other conditions of Theorem 2.3, we conclude that $\mu$ is regular. If $\mu_i$ with $i = 1, \ldots, 4$, are the bounded regular positive measures representing $\sigma^+$, $\sigma^-$, $\tau^+$, $\tau^-$ respectively, then the complex measure

$$m = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

is regular and represents $\lambda$, i.e. we have $\lambda = dm$, thus proving that our map is surjective.

To show that the map is injective, we have to prove that its kernel is 0. Suppose that $dm = 0$. Let $\mu = |m|$ and $dm = h\, d\mu$ with $|h| = 1$. Then $\langle f, h\rangle_\mu = 0$ for all $f \in C_c(X)$. But $C_c(X)$ is $L^1$-dense in $\mathscr{L}^1(\mu, \mathbf{C})$ by Theorem 3.1. We have the inequality

$$|\langle f, h\rangle_\mu| \leqq \|h\|\, \|f\|_1$$

for all $f \in \mathscr{L}^1(\mu, \mathbf{C})$. It follows that $\langle \varphi, h\rangle_\mu = 0$ for all step functions $\varphi$, whence $h$ is equal to 0 almost everywhere. Since $|h| = 1$ we must have $\mu(X) = 0$, thus proving $m = 0$.

Finally, write again $dm = h\, d\mu$ with $\mu = |m|$ and $|h| = 1$. Let $\lambda = dm$. We have to show that $\mu(X) \leqq |\lambda|$. By Lusin's theorem, §3, we can find a function $g \in C_c(X)$ such that $g = \bar{h}$ except on a set $Z$ of measure $< \varepsilon$, and such that $|g| \leqq 1$ on $Z$. Consequently

$$\begin{aligned} |\lambda| \geqq |\lambda g| = \left| \int_X gh\, d\mu \right| &\geqq \mu(X - Z) - \mu(Z) \\ &\geqq \mu(X) - 2\varepsilon. \end{aligned}$$

This proves the desired inequality, and concludes the proof of the theorem.

**Remark.** Let $\lambda$ be a $C_c$-functional on $C_c(X)$. For each compact subset $K$, the restriction of the functional to $C_K(X)$ is a bounded functional, which has a corresponding measure $\mu_K$ by Theorem 4.2. If $K_1 \subset K_2$ are two compact sets, then it is easily verified that the restriction of $\mu_{K_2}$ to $K_1$ is $\mu_{K_1}$. If $A$ is Borel-measurable and $A \subset K$, then we define

$$\mu_\lambda(A) = \mu_K(A),$$

which does not depend on the choice of $K$. This function $\mu_\lambda$, defined on Borel-measurable subsets $A$ of compact sets, will also be called a **measure**, and more specifically the measure associated with $\lambda$. For instance, suppose $\mu$ is a positive Borel measure on $X$, and $f$ is a measurable function, bounded on each compact set, then $\lambda = f\, d\mu$ defines such a functional, which has such an associated measure. The measure $\mu_\lambda$ could also be called the direct limit of the measures $\mu_K$, taken over all compact sets $K$.

## IX, §5. LOCALIZATION OF A MEASURE AND OF THE INTEGRAL

The introduction of partitions of unity in §2 is not as accidental as it seems. We can use them to localize a measure, or functional.

**Theorem 5.1.** *Let $\{W_\alpha\}$ be an open covering of $X$. For each index $\alpha$, let $\lambda_\alpha$ be a functional on $C_c(W_\alpha)$. Assume that for each pair of indices $\alpha$, $\beta$ the functionals $\lambda_\alpha$ and $\lambda_\beta$ are equal on $C_c(W_\alpha \cap W_\beta)$. Then there exists a unique functional $\lambda$ on $X$ whose retriction to each $C_c(W_\alpha)$ is equal to $\lambda_\alpha$. If each $\lambda_\alpha$ is positive, then so is $\lambda$.*

*Proof.* Let $f \in C_c(X)$ and let $K$ be the support of $f$. Let $\{h_i\}$ be a partition of unity over $K$ subordinated to a covering of $K$ by a finite number of the open sets $W_\alpha$. Then each $h_i f$ has support in some $W_{\alpha(i)}$ and we define

$$\lambda f = \sum_i \lambda_{\alpha(i)}(h_i f).$$

We contend that this sum is independent of the choice of $\alpha(i)$, and also of the choice of partition of unity. Once this is proved, it is then obvious (see Exercise 10) that $\lambda$ is a functional which satisfies our requirements. We now prove this independence. First note that if $W_{\alpha'(i)}$ is another one of the open sets $W_\alpha$ in which the support of $h_i f$ is contained, then $h_i f$ has support in the intersection $W_{\alpha(i)} \cap W_{\alpha'(i)}$, and our assumption concerning our functionals $\lambda_\alpha$ shows that the corresponding term in the sum does not depend on the choice of index $\alpha(i)$. Next, let $\{g_k\}$ be another partition of unity over $K$ subordinated to some covering of $K$ by a finite number of the open sets $W_\alpha$. Then for each $i$,

$$h_i f = \sum_k g_k h_i f,$$

whence

$$\sum_i \lambda_{\alpha(i)}(h_i f) = \sum_i \sum_k \lambda_{\alpha(i)}(g_k h_i f).$$

If the support of $g_k h_i f$ is in some $W_\alpha$, then the value $\lambda_\alpha(g_k h_i f)$ is independent of the choice of index $\alpha$. The expression on the right is then symmetric with respect to our two partitions of unity, whence our theorem follows.

**Corollary 5.2.** *Let $\{W_\alpha\}$ be an open covering of $X$. For each index $\alpha$, let $\mu_\alpha$ be a positive $\sigma$-regular measure on $W_\alpha$. Assume that for each pair of indices $\alpha$, $\beta$ the measures $\mu_\alpha$ and $\mu_\beta$ induce equal measures on $W_\alpha \cap W_\beta$. Then there exists a unique $\sigma$-regular positive measure $\mu$ on $X$ whose restriction to each $W_\alpha$ is equal to $\mu_\alpha$.*

*Proof.* This is merely a rewording of the theorem, in view of the correspondence between $\sigma$-regular measures and positive functionals.

Theorem 5.1 will be used only in the proof of Stokes' theorem in Chapter XXIII.

In §2 and in Theorem 5.1, we dealt with partitions of unity over a compact subset of $X$. We shall now discuss partitions of unity over all of $X$.

Let $\mathcal{U}$ be a covering of $X$, say by open sets. We say that $\mathcal{U}$ is **locally finite** if every point of $x$ has a neighborhood which intersects only finitely many elements of the covering. A **refinement** $\{V_j\}$ of a covering $\{U_i\}$ of $X$ is a covering such that each $V_j$ is contained in some $U_i$. We also say that the covering $\{V_j\}$ is **subordinated to the covering** $\{U_i\}$.

A (continuous) **partition of unity** on $X$ consists of an open covering $\{V_i\}$ of $X$ and a family of real continuous functions

$$\psi_i\colon X \to \mathbf{R}$$

satisfying the following conditions.

**PU 1.** *For all $x \in X$, we have $\psi_i(x) \geqq 0$.*

**PU 2.** *The support of $\psi_i$ is contained in $V_i$.*

**PU 3.** *The covering $\{V_i\}$ is locally finite.*

**PU 4.** *For each point $x \in X$, we have*

$$\sum \psi_i(x) = 1.$$

(The sum is taken over all $i$, but is in fact finite for any given $x$ in view of **PU 3**.) As a matter of notation, we often write that $\{(V_i, \psi_i)\}$ or simply $\{\psi_i\}$ is a partition of unity if it satisfies the previous four conditions.

In the proof of the next theorem, we use the facts (trivially proved) that if a space $X$ has a countable base, then any open covering has a countable subcovering, and any base contains a countable base.

**Theorem 5.3.** *Let $X$ be locally compact Hausdorff, and assume that the topology of $X$ has a countable base. Then $X$ admits continuous partitions of unity, subordinated to a given open covering $\mathcal{U}$.*

*Proof.* Let $U_1, U_2, \ldots, \ldots$ be a base for the open sets, such that each $\overline{U}_i$ is compact. We construct first inductively a sequence $A_1, A_2, \ldots$ of compact sets whose union is $X$ and such that $A_i$ is contained in the interior of $A_{i+1}$. We let $A_1 = \overline{U}_1$. If we have constructed $A_i$ inductively, then we let $j$ be the smallest integer such that $A_i$ is contained in

$$U_1 \cup \cdots \cup U_j,$$

and we let $A_{i+1}$ be the compact set

$$\overline{U}_1 \cup \cdots \cup \overline{U}_j \cup \overline{U}_{j+1}.$$

Let Int abbreviate interior. For each point $x$ of $A_{i+1} - \mathrm{Int}(A_i)$ we can find a pair $(W_x, V_x)$ of open sets containing $x$ such that $W_x \subset \overline{W}_x \subset V_x$, such that $V_x$ is contained in $\mathrm{Int}(A_{i+2}) - A_{i-1}$, and such that $V_x$ is contained in one of the open sets of the given covering $U$. There is a finite number of pairs such that already the open sets $W_x$ cover the compact set $A_{i+1} - \mathrm{Int}(A_i)$. Taking all such finite collections of pairs for $i = 1, 2, \ldots$, we obtain a countable collection of pairs $\{(W_k, V_k)\}$ such that the $\{V_k\}$ form a locally finite covering of $X$, the $\{W_k\}$ is also an open covering, and $\overline{W}_k \subset V_k$. Let $h_k$ be such that $\overline{W}_k \prec h_k \prec V_k$ (see the beginning of §2 for the notation $\prec$.) Let

$$h = \sum_{k=1}^{\infty} h_k.$$

Let

$$\psi_k = h_k/h.$$

Then $\{\psi_k\}$ is the desired partition of unity.

**Theorem 5.4.** *Let $\{h_i\}$ $(i = 1, 2, \ldots)$ be a countable partition of unity on $X$. Let $\mu$ be a regular positive Borel measure on $X$, and let*

$$f \in \mathscr{L}^1(\mu).$$

*Then for each $i$, $h_i f$ is in $\mathscr{L}^1(\mu)$, and*

$$\sum \int_X h_i f \, d\mu = \int_X f \, d\mu,$$

*in the sense that the sum is absolutely convergent, and is equal to the integral on the right.*

*Proof.* Let

$$f_n = \sum_{i=1}^{n} h_i f.$$

Then $|f_n| \leqq |f|$, and the sequence $\{f_n\}$ is pointwise convergent to $f$. We can therefore apply the dominated convergence theorem to conclude the proof.

## IX, §6. PRODUCT MEASURES ON LOCALLY COMPACT SPACES

Let $X$, $Y$ be locally compact Hausdorff spaces, and let $\mu$, $\nu$ be positive $\sigma$-regular Borel measures on $X$ and $Y$, respectively. We let $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ denote the $\sigma$-algebras of Borel sets in $X$ and $Y$, respectively. If $X$,

$Y$ are $\sigma$-finite with respect to these measures, then Fubini's theorem applies. However, we warn the reader that in general, one does *not* have

$$\mathscr{B}(X \times Y) = \mathscr{B}(X) \otimes \mathscr{B}(Y).$$

(Even if $X$ is compact, $Y = X$. Examples are obtained by taking $X$ with abnormally many open sets.) However, we can still integrate functions in $C_c(X \times Y)$, as shown by the following results, which are nothing but corollaries of the Stone–Weierstrass theorem, expressed as lemmas.

**Lemma 6.1.** *Let $X$, $Y$ be locally compact Hausdorff spaces. Every function in $C_c(X \times Y)$ can be uniformly approximated by functions which are finite sums of type*

$$(x, y) \mapsto \sum \varphi_i(x)\psi_i(y),$$

*with $\varphi_i \in C_c(X)$ and $\psi_i \in C_c(Y)$.*

*Proof.* We may restrict ourselves to the real case. We note that functions of the above type form an algebra $A$ which separates points, and this algebra is such that if $K$ is compact in $X \times Y$, then there exists some $g \in A$ such that $g$ is equal to 1 on $K$. (For instance, if $C$, $D$ are the projections of $K$ on $X$ and $Y$, respectively, then $K \subset C \times D$, and we can write

$$g(x, y) = \varphi(x)\psi(y)$$

where $\varphi$ is 1 on $C$ and $\psi$ is 1 on $D$.) We are therefore reduced to proving a second lemma.

**Lemma 6.2.** *Let $X$ be locally compact Hausdorff, and let $A$ be an algebra of real valued functions in $C_c(X)$, which separates points, and is such that if $K$ is compact in $X$, then there exists $\alpha \in A$ which is 1 on $K$. Then $A$ is dense in $C_c(X)$ for the sup norm.*

*Proof.* Let $f \in C_c(X)$ and let $K$ be the compact support of $f$. Let $\alpha \in A$ be 1 on $K$. Let $U$ be an open set containing the support of $\alpha$, and having compact closure $\overline{U}$. The restrictions to $\overline{U}$ of elements of $A$ form an algebra, which clearly satisfies the hypotheses of the Stone–Weierstrass theorem. Therefore the restriction $f|\overline{U}$ can be uniformly approximated by elements of $A|\overline{U}$. Denote by $\| \ \|_{\overline{U}}$ the sup norm over $\overline{U}$. If we can approximate $f$ by an element $\beta \in A$ over $\overline{U}$, say

$$\|f - \beta\|_{\overline{U}} < \varepsilon,$$

then

$$\|\alpha f - \alpha\beta\|_{\overline{U}} < \varepsilon\|\alpha\|,$$

and thus we have a uniform approximation of $\alpha f$ by $\alpha\beta$ over $\bar{U}$. But $\alpha f = f$, and $\alpha\beta$ is equal to 0 outside $\bar{U}$. Thus the uniform approximation holds over all of $X$, as was to be shown.

As a matter of notation, if $\varphi$ is a function on $X$ and $\psi$ is a function on $Y$, then we denote by $\varphi \otimes \psi$ the function

$$(x, y) \mapsto \varphi(x)\psi(y),$$

and call it the product function. The set of finite sums of product functions is an algebra, which we shall call the algebra generated by the product functions.

**Theorem 6.3.** *Let $X$, $Y$ be locally compact Hausdorff spaces and let $\mu$, $\nu$ be positive $\sigma$-regular Borel measures on $X$ and $Y$, respectively. Assume that $X$, $Y$ are $\sigma$-finite with respect to these measures. Then all functions in $C_c(X \times Y)$ are in $\mathscr{L}^1(\mu \otimes \nu)$, and there exists a unique $\sigma$-regular Borel measure on $X \times Y$ which restricts to $\mu \otimes \nu$ on $\mathscr{B}(X) \otimes \mathscr{B}(Y)$.*

*Proof.* Lemma 6.1 shows that functions in $C_c(X \times Y)$ are $(\mu \otimes \nu)$-measurable, and combined with Fubini's theorem shows that these functions are in $\mathscr{L}^1(\mu \otimes \nu)$. The map

$$f \mapsto \int_{X \times Y} f \, d(\mu \otimes \nu)$$

then is obviously a positive functional on $C_c(X \times Y)$, and we can therefore apply Theorem 2.3 to get a $\sigma$-regular Borel measure having the desired properties. The Corollary 2.8 gives the uniqueness, thus proving our theorem.

## IX, §7. EXERCISES

*We assume throughout that $X$ is locally compact Hausdorff.*

1. Let $X$ be compact, and let $C(X)$ be the algebra of real continuous functions on $X$. If $\lambda$ is a functional on $C(X)$, such that $\lambda(1) = |\lambda|$, show that $\lambda$ is positive.
2. Assume that $X$ is separable. Show that every open set is $\sigma$-compact.
3. Show that the complex regular Borel measures form a Banach space.
4. Assume that $X$, $Y$ are locally compact Hausdorff and $\sigma$-compact. If $\mu$, $\nu$ are regular Borel measures on $X$, $Y$ respectively, show that $\mu \otimes \nu$ is regular.

5. Let $\mu$, $\nu$ be regular Borel measures on $\mathbf{R}^n$. Define the **convolution** $\mu * \nu$ by

$$(\mu * \nu)(A) = (\mu \otimes \nu)(\sigma^{-1}(A)),$$

where $\sigma: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is the sum. Show that $\mu * \nu$ is regular.

6. Assume that $X$ is $\sigma$-compact. Let $\mu$ be a regular Borel measure on $X$. If $A$ is measurable, show that there exists a closed set $B \subset A$ and an open set $V \supset A$ such that $\mu(V - B) < \varepsilon$.

7. Assume that every open set in $X$ is $\sigma$-compact. If $\nu$ is a positive Borel measure which is finite on compact sets, show that $\nu$ is regular. [*Hint*: Show that $\nu = \mu$ if $\mu$ is the regular measure associated with $d\nu$ as in the text. Do it first for open sets.]

8. (a) Let $M$ denote the Banach space of complex regular Borel measures on $\mathbf{R}^n$. If $m$, $m'$ are in $M$, show that for $f \in C_c(\mathbf{R}^n)$ the integral

$$\iint f(x + y)\, dm(x)\, dm'(y)$$

exists, and defines a bounded functional on $C_c(\mathbf{R}^n)$, whose measure is denoted by $m * m'$, and is called the **convolution** of $m$ and $m'$. Prove that convolution of elements of $M$ is associative, bilinear, commutative, and has a unit element. Thus $M$ is a Banach algebra.

(b) Let $\mu$ be Lebesgue measure and let $f \in \mathscr{L}^1(\mu, \mathbf{C})$. Show that for any $m \in M$ we have

$$m * \mu_f = \mu_g$$

for some $g \in \mathscr{L}^1(\mu, \mathbf{C})$. In algebraic terminology, this means that the absolutely continuous elements of $M$ (with respect to Lebesgue measure) form an ideal in $M$.

9. (a) Let $\lambda$ be a bounded functional on $C_c(X)$, and let $m$ be the regular complex Borel measure such that $dm = \lambda$. Show that $\lambda$ extends uniquely to a functional on $\mathscr{L}^1(|m|)$ by continuity, and that this follows at once from the remarks preceding Theorem 4.2.

(b) Let $\{h_i\}$ $(i = 1, 2, \ldots)$ be a countable partition of unity on $X$. Let $f \in \mathscr{L}^1(|m|)$. Show that

$$\sum \lambda(h_i f) = \lambda(f).$$

[*Note:* This obvious extension of the text, and of Theorem 5.4 in particular, is useful when dealing with manifolds. Cf. for instance Chapter XXIII, §3, §5, and §6.]

10. Verify in detail the "obvious" fact in the proof of Theorem 5.1 that $\lambda$ is a functional, in particular that for each compact set $K$ there is a number $A_K$ such that for any $f \in C_c(X)$ with support in $K$ we have $|\lambda f| \leqq A_K \|f\|$.

11. Let $\mu$ be a regular positive measure on $\mathbf{R}$. (a) Show that the functions of type $e^{-x}g(x)$ (where $g$ is a polynomial) are dense in $\mathscr{L}^1(\mathbf{R}^+, \mu)$. (b) Show that the

functions of type $e^{-x^2}g(x)$ (where $g$ is a polynomial) are dense in $\mathscr{L}^1(\mathbf{R}, \mu)$. (c) Same thing for $\mathscr{L}^p$ with $1 < p < \infty$. [*Hint*: Cf. Exercises 19 and 20 of Chapter III.]

12. Let $\mu$ be a regular positive Borel measure on $\mathbf{R}$. If $f \in \mathscr{L}^1(\mu)$ and

$$\int_{\mathbf{R}} f(x)e^{itx}\,d\mu(x) = 0$$

for all real $t$, show that $f(x) = 0$ for $\mu$-almost all $x$. [*Hint:* By a Fourier series argument, show that

$$\int_{R} fg\,d\mu = 0$$

if $g$ is $C^\infty$ of period $2N$ with large $N$, and then, also if $g \in C_c(\mathbf{R})$.]

13. Let $\mu$ be a regular positive Borel measure on $\mathbf{R}$.
(a) Assume that there exists $c > 0$ such that the function $x \mapsto e^{c|x|}$ is in $\mathscr{L}^1(\mu)$. Let $f \in \mathscr{L}^p(\mu)$, $1 < p < \infty$. If $f$ is orthogonal to all functions $\{x^n\}$ ($n \geqq 0$), i.e.

$$\int_{\mathbf{R}} f(x)x^n\,d\mu(x) = 0$$

for all $n \geqq 0$, then $f(x) = 0$ for $\mu$-almost all $x$.
(b) Let $f \in \mathscr{L}^1(\mu)$. If there exists $c > 0$ such that

$$\int_{\mathbf{R}} f(x)x^n e^{-c|x|}\,d\mu(x) = 0$$

for all $n \geqq 0$, then $f(x) = 0$ for $\mu$-almost all $x$. *Note*: Actually, (b) implies (a). [*Hint for* (a): Show that the integral in Exercise 12 is analytic in $t$ for $t$ at a distance $\leqq c/q$ from the real line, and 0 near the origin. You can also use the exercises at the end of Chapter III.]

*Examples.* Taken $d\mu(x) = e^{-x^2}\,dx$. We get the completeness of the Hermite polynomials. For the Laguerre polynomials, one takes $d\mu(x) = h(x)\,dx$, where $h(x) = 0$ if $x < 0$ and $h(x) = e^{-x}$ if $x \geqq 0$. And similarly for the other classical polynomials, which are obtained by applying the orthogonalization process to $\{x^n\}$.

14. Let $X$ be compact and let $\mu$ be a regular measure on $X$. Let $A$ be a subset of $X$ whose boundary has measure 0. Let $\{x_n\}$ be a sequence of points of $X$ having the following property. For every continuous function $f$ on $X$ we have

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int f\,d\mu.$$

Let $N(A, n)$ be the number of indices $i \leqq n$ such that $x_i \in A$. Prove that

$$\lim_{\mu\to\infty} \frac{N(A, n)}{n} = \mu(A).$$

This is the equidistribution of the sequence $\{x_n\}$ in $X$. In some applications, instead of using condition (*) for all continuous functions $f$ one uses it only for a vector space of more special functions, dense in $C(X)$ (e.g. the space generated by characters on a compact group).

15. **(Karamata's theorem.)** Let $\mu$ be a regular positive Borel measure on $\mathbf{R}^+$ such that the integral

$$\int_0^\infty e^{-tx}\,d\mu(x)$$

converges for $t > 0$, and such that

$$\lim_{t\to 0} t^r \int_0^\infty e^{-tx}\,d\mu(x) = C$$

for some positive constants $r$ and $C$. If $f$ is a continuous function on $[0, 1]$, then

$$\lim_{t\to 0} t^r \int_0^\infty f(e^{-tx})e^{-tx}\,d\mu(x) = \frac{C}{\Gamma(r)} \int_0^\infty f(e^{-t})t^{r-1}e^{-t}\,dt.$$

[*Hint*: By Weierstrass' approximation, it suffices to prove the theorem when $f$ is a polynomial, and hence when $f(x) = x^k$. This is done by direct computation.]

For an application of Karamata's theorem, see [BGV, p. 95].

16. Let $\mu$ be a finite positive regular Borel measure on $\mathbf{R}$. Assume that

$$\int_{\mathbf{R}} e^{inx}\,d\mu(x) = 0 \qquad \text{for all} \quad n \in \mathbf{Z}.$$

Prove that $\mu = 0$.

CHAPTER X

# Riemann–Stieltjes Integral and Measure

This chapter gives an example of a measure which arises from a functional, defined essentially by generalizations of Riemann sums. We get here into special aspects of the real line, as distinguished from the general theory of integration on general spaces.

## X, §1. FUNCTIONS OF BOUNDED VARIATION AND THE STIELTJES INTEGRAL

Let us start with a finite interval $[a, b]$ on the real line. To each partition

$$P = [a = x_0, x_1, \ldots, x_n = b]$$

we associate its **size**,

$$\sigma(P) = \text{size } P = \max_k (x_{k+1} - x_k).$$

Let

$$f\colon [a, b] \to E$$

be a mapping of the interval into a Banach space. Let $P$ be a partition of $[a, b]$ as above. We define the **variation** $V_P(f)$ to be

$$V_P(f) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

We define the **variation**

$$V(f) = \sup_P V_P(f),$$

where the sup (least upper bound if it exists, otherwise $\infty$) is taken over all partitions. If $V(f)$ is finite, then $f$ is called of **bounded variation**, and $f$ is bounded.

**Examples.** If $f$ is real valued, increasing, and bounded on $[a, b]$, then $f$ is obviously of bounded variation, in fact bounded by $f(b) - f(a)$.

If $f$ is differentiable on $[a, b]$ and $f'$ is bounded, then $f$ is of bounded variation (mean value theorem). This is so in particular if $f$ is of class $C^1$.

The mappings of bounded variation form a vector space. In fact, if $f$ and $g$ are of bounded variation, then for $\alpha \in \mathbf{C}$,

$$V(f + g) \leqq V(f) + V(g) \qquad \text{and} \qquad V(\alpha f) = |\alpha| V(f).$$

In other words, denoting by BV($[a, b]$) the space of mappings of bounded variation, we see that the variation $V$ is a seminorm on BV($[a, b]$).

Let $f$, $g$ have values in Banach spaces $E$, $F$ respectively, and suppose given a product (bilinear map) $E \times F \to G$ into a Banach space, denoted $(u, v) \mapsto uv$, and satisfying $|uv| \leqq |u||v|$. If $f$, $g$ are of bounded variation, so is $fg$, as one verifies easily. A special case of a product occurs of course when $E = \mathbf{C}$ itself, and multiplication is just multiplication by scalars. One then obtains a bound for the variation of a product, namely

$$V(fg) \leqq \|f\| V(g) + \|g\| V(f).$$

This estimate is an immediate consequence of estimating the sums for the variation and using the triangle inequality.

The notation for the variation really should include the interval, and we should write

$$V(f, a, b).$$

**Define**

$$V_f(x) = V(f, a, x),$$

so $V_f$ is now a function of $x$, called the **variation function of** $f$.

**Proposition 1.1.** *Let $f \in BV([a, b])$.*

(i) *The function $V_f$ is increasing.*

(ii) *If $a \leqq x \leqq y \leqq b$, then*

$$V(f, a, y) = V(f, a, x) + V(f, x, y).$$

(iii) *If $f$ is continuous, then $V_f$ is continuous.*

*Proof.* For (i), we note that if $x < y$, then we can always refine a partition of $[a, y]$ to include the number $x$. Furthermore, if $P'$ is a partition refining $P$, then

$$V_P(f) \leqq V_{P'}(f).$$

Then (i) follows at once. For (ii), we again use the fact that a partition of $[a, y]$ can be refined to contain $x$. Finally, suppose that $f$ is continuous. By (ii), the continuity from the right of $V_f$ amounts to proving that

$$\lim_{y \to x} V(f, x, y) = 0.$$

Suppose that the limit is not 0. Then there exists a number $\delta > 0$ such that

$$V(f, x, t) > \delta \qquad \text{for all} \quad x < t \leqq y.$$

Let $x = x_0 < x_1 < \cdots < x_n = y$ be a partition of $[x, y]$ such that

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| > \delta.$$

By the continuity of $f$ at $x$, we can select $y_1$ very close to $x$ and in particular $x < y_1 < x_1$ such that the inequality remains valid if we replace the term

$$|f(x_1) - f(x)| \qquad \text{by} \qquad |f(x_1) - f(y_1)|.$$

Thus we have proved:

*There exists $y_1$ with $x < y_1 < y$ such that*

$$V(f, y_1, y) > \delta.$$

Now we repeat this procedure with $y$ replaced by $y_1$, and find $y_2$ with $x < y_2 < y_1$ such that

$$V(f, y_2, y_1) > \delta.$$

After $N$ steps, we get

$$V(f, y_n, y) > N\delta.$$

Since $V(f, y_n, y) \leqq V(f, x, y)$, this gives a contradiction, concluding the proof.

**Theorem 1.2.** *Let $f$ be a real valued function on $[a, b]$ of bounded variation. Then there exist increasing functions $g$, $h$ on $[a, b]$ such that*

*$g(a) = h(a) = 0$, and*

$$f(x) - f(a) = g(x) - h(x),$$
$$V_f(x) = g(x) + h(x).$$

*If $f$ is continuous, so are $g$ and $h$.*

*Proof.* Define $g$, $h$ by the formulas

$$2g = V_f + f - f(a) \qquad \text{and} \qquad 2h = V_f - f + f(a).$$

If $f$ is continuous, so are $g$ and $h$ by Theorem 1.1(iii). In any case, $g(a) = h(a) = 0$ and the two formulas of the theorem are valid. There remains only to prove that $g$, $h$ are increasing. Let $a \leqq x \leqq y \leqq b$. Then by additivity of Proposition 7.1(ii),

$$2g(y) - 2g(x) = V(f, x, y) + f(y) - f(x) \geqq 0,$$

so $g$ is increasing, and similarly $h$ is increasing, thus concluding the proof.

We now generalize the notion of Riemann integral to that of Stieltjes. Let $f$, $g$ be bounded maps of $[a, b]$ into Banach spaces $E$, $F$ respectively, and assume given a product $E \times F \to G$ denoted by $(u, v) \mapsto uv$ such that $|uv| \leqq |u||v|$. Given a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ and numbers $c_k$ with $x_k \leqq c_k \leqq x_{k+1}$, we define the **Riemann–Stieltjes sum** (abbreviated RS-sum) as before, namely

$$S(P, c, f, g) = S(P, c) = \sum_{k=0}^{n-1} f(c_k)[g(x_{k+1}) - g(x_k)].$$

Denote by $\sigma(P)$ the size of the partition $P$. We say that the **limit**

$$\lim_{\sigma(P) \to 0} S(P, c)$$

exists if there exists $L \in G$ such that given $\varepsilon$ there exists $\delta$ such that whenever $\sigma(P) < \delta$ then $|S(P, c) - L| < \varepsilon$. If

$$\lim_{\sigma(P) \to 0} S(P, c, f, g)$$

exists, we say that $f$ is **RS(g)-integrable**, and we denote the limit by the integral called the **Riemann–Stieltjes integral**

$$\int_a^b f\, dg.$$

When $g(x) = x$, then the integral is just the **Riemann integral**, and we call the function **Riemann integrable**.

It is immediate that the set of RS($g$)-integrable maps is a vector space, and that the RS-integral is a functional on this space. The space itself will also be denoted by **RS(*g*)**, or RS($g$, $[a, b]$) if we want to specify $[a, b]$.

Let $a \leqq b \leqq c$. It is immediately verified that if $f$ is in RS($g$, $[a, c]$), then $f$ is in RS($g$, $[a, b]$) and RS($g$, $[b, c]$) and we have the usual property

$$\int_a^c f\,dg = \int_a^b f\,dg + \int_b^c f\,dg.$$

(*Warning:* The converse may not hold!)

Observe also that the RS-integral is linear in $g$. In other words if $f \in \mathrm{RS}(g_1)$ and $f \in \mathrm{RS}(g_2)$, then $f \in \mathrm{RS}(g_1 + g_2)$; if $\alpha \in \mathbf{C}$ then $f \in \mathrm{RS}(\alpha g)$; and we have linearity of the integral keeping $f$ fixed, viewing the integral as a function of $g$, that is

$$\int_a^b f\,d(g_1 + g_2) = \int_a^b f\,dg_1 + \int_a^b f\,dg_2, \qquad \int_a^b f\,d(\alpha g) = \alpha \int_a^b f\,dg.$$

These are immediate, using the triangle inequality.

We have the usual estimate with the sup norm.

**Proposition 1.3.** *Assume $f \in \mathrm{RS}(g)$, and $g$ of bounded variation. Then*

$$\left| \int_a^b f\,dg \right| \leqq \|f\|\,V(g),$$

*where $\|f\|$ is the sup norm of $f$ on $[a, b]$.*

*Proof.* Immediate by estimating an approximating RS-sum and using the triangle inequality.

**Proposition 1.4.** *We have $f \in \mathrm{RS}(g)$ if and only if $g \in \mathrm{RS}(f)$, and in that case, the formula for integration by parts holds, namely*

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a).$$

*Proof.* Suppose that $\int_a^b g\,df$ exists. Consider a RS-sum which we unwind using summation by parts:

$$\begin{aligned} S(P, c, f, g) &= \sum_{k=0}^{n-1} f(c_k)[g(x_{k+1}) - g(x_k)] \\ &= -\sum_{k=1}^{n-1} [f(c_k) - f(c_{k-1})]g(x_k) + f(c_{n-1})g(b) - f(a)g(a) \\ &= f(b)g(b) - f(a)g(a) - S(Q, g, f) \end{aligned}$$

where

$$S(Q, g, f) = g(a)[f(c_0) - f(a)] + \sum_{k=1}^{n-1} g(x_k)[f(c_k) - f(c_{k-1})]$$
$$- g(b)[f(b) - f(c_{n-1})].$$

Here we have written the product in reverse direction.

Then $S(Q, g, f)$ is a RS-sum with respect to the partition

$$Q: [a = c_0, c_1, \ldots, c_{n-1}, b]$$

with the intermediate points $a$, $x_1$, $\ldots, x_{n-1}$, $b$. When the size of $P$ approaches 0, so does the size of $Q$, and by hypothesis, the sum $S(Q, g, f)$ approaches the integral $\int_a^b g\, df$, thereby completing the proof of the proposition.

Finally we give a criterion for RS-integrability.

**Proposition 1.5.** *Assume that $f$ is continuous and $g$ of bounded variation on $[a, b]$. Then $f \in \mathrm{RS}(g)$.*

*Proof.* Given $\varepsilon$ let $\delta$ be such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Let $P$, $P'$ be partitions of size $< \delta$. To estimate $|S(P, c) - S(P'. c')|$, we may assume without loss of generality that $P'$ is a refinement of $P$. Thus it suffices to prove two estimates: if $P' = P$ but we change the choice of intermediate points $c$ to $c'$, then the difference of the sums is small; and if $P'$ is obtained from $P$ by inserting one more point in the partition, then again the difference of the sums is small. As to the first step, letting $P = P'$, we have

$$|S(P, c) - S(P, c')| \leqq \sum |f(c_i) - f(c_i')||g(x_{i+1}) - g(x_i)| \leqq \varepsilon V(g),$$

which gives us the desired estimate. Secondly, suppose that $P'$ is obtained from $P$ by inserting one point, say $x_j'$ with $x_j \leqq x_j' \leqq x_{j+1}$. Then the size of $P'$ is still $< \delta$. By the first step, to get the desired estimate for $|S(P, c) - S(P', c')|$ we may assume without loss of generality that for $i \neq j$ we have $x_i' = x_i$, that $c_j = x_j'$, and $x_j'$ is also selected as the intermediate point for the two intervals $[x_j, x_j']$ and $[x_j', x_{j+1}]$ of the partition $P'$. Then

$$S(P, c) - S(P', c') = 0,$$

and the proposition is proved.

Often the RS-integral can be computed as a Riemann integral, as in the next proposition.

**Proposition 1.6.** *Let $f$ be continuous and suppose that $g$ is real differentiable on $[a, b]$ with Riemann-integrable $g'$. Then $f \in \mathrm{RS}(g)$, and*

$$\int_a^b f\,dg = \int_a^b f(x)g'(x)\,dx,$$

*where the integral on the left is the* RS-*integral, and the integral on the right is the usual Riemann integral.*

*Proof.* This is immediate by using the mean value theorem on $g$ and estimating the RS-sum by the triangle inequality and the hypothesis that $g'$ is Riemann integrable.

Finally we consider the special case when $f$, $g$ are complex valued or even real valued. Suppose $g$ is of bounded variation. Then on $C([a, b])$ we obtain a bounded functional

$$f \mapsto \int_a^b f\,dg.$$

By Proposition 1.1, and the linearity of the integral in $f$ and $g$, we may decompose the integral into a sum of four terms, each term involving only real valued functions. Furthermore, if $f \in C_c(\mathbf{R})$, then the support of $f$ lies in some bounded interval $[a, b]$ and we may define

$$\int_{\mathbf{R}} f\,dg = \int_a^b f\,dg,$$

the right side being independent of the choice of $[a, b]$. Suppose that there exists a number $B > 0$ such that

$$V(f, a, b) \leqq B \qquad \text{for all} \quad [a, b],$$

so the variations are uniformly bounded on finite intervals. In this latter case, the space of such functions is denoted by $\mathrm{BV}(\mathbf{R})$, and called the space of functions of **bounded variation on R**. On this space, we define

$$V_{\mathbf{R}}(f) = \sup_{[a,b]} V(f, a, b).$$

Then $V_{\mathbf{R}}(f)$ is a seminorm on $\mathrm{BV}(\mathbf{R})$, and results of this section extend to $\mathrm{BV}(\mathbf{R})$. In particular, the estimate of Proposition 1.2 holds over $\mathbf{R}$, that is

$$\left| \int_{\mathbf{R}} f\,dg \right| \leqq \|f\|\, V_{\mathbf{R}}(g)$$

for $f \in C_c(\mathbf{R})$ and $g \in BV(\mathbf{R})$. By the general representation theorem, there exists a unique complex valued measure $\mu_g$ such that

$$\int_{\mathbf{R}} f\, dg = \int_{\mathbf{R}} f\, d\mu_g \qquad \text{for all} \quad f \in C_c(\mathbf{R}).$$

The variation $V_{\mathbf{R}}(g)$ simply provides an explicit concrete example of the general notion of the norm $\|\mu_g\|$ of the measure. We call $\mu_g$ the **Riemann–Stieltjes measure associated with** $g$.

If $h$ is a bounded increasing function, then $\mu_h$ is a finite positive measure on $\mathbf{R}$. It is immediate from the proof of Theorem 1.2 that the decomposition of a real valued function $g$ of bounded variation on a finite closed interval is also valid on $\mathbf{R}$, by the same formula using the total variation. Then the decomposition of Theorem 1.2 gives an example when a real valued measure is expressed as a difference of two positive measures.

In some applications of the next section, we shall mix conditions when a function is $L^1(\mathbf{R})$ and also conditions of bounded variation. The following observation is sometimes useful.

**Lemma 1.7.** *Suppose $f \in L^1(\mathbf{R}) \cap BV(\mathbf{R})$. Then $f$ vanishes at infinity, in the sense that*

$$\lim_{x \to \infty} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 0.$$

Indeed, suppose $f(x) \geqq c > 0$ for infinitely many $x$. Then $f(x) \geqq c/2$ (say) for all $x$ sufficiently large, otherwise $f$ would not be in $L^1(\mathbf{R}) \cap BV(\mathbf{R})$. Under these circumstances, in a situation when integration by parts is valid for finite intervals as in Proposition 1.4, the extra terms

$$f(x)g(x)\Big|_{-\infty}^{\infty} = \lim_{a,b \to \infty} [f(b)g(b) - f(a)g(a)]$$

vanish.

**Proposition 1.8.**

(a) *Let $f$ be of bounded variation. Then the set of points of discontinuity of $f$ is countable.*
(b) *Assume $f$ continuous at $a$ and $b$ with $a \leqq b$. Then*

$$f(b) - f(a) = \int_a^b d\mu_f(x).$$

*In particular, the set consisting of the single point $a$ has $\mu_f$-measure* 0.

*Proof.* We leave part (a) as an easy exercise (see Exercise 4), done by a routine estimate. As for part (b), we note that the constant function 1 is continuous on $[a, b]$, and all Riemann–Stieltjes sums give the same value $f(b) - f(a)$, as desired.

I have given above enough results on functions of bounded variations to indicate the flavor of the theory, and to suffice for some applications: to the spectral theorem of Chapter XX, §5, and to Fourier analysis in the next section. For more on functions of bounded variations, see Natanson [Nat].

We end this section with a useful mean value theorem, which did not fit anywhere else, and illustrates once more the technique of summation by parts.

**Proposition 1.9 (Integral Mean Value Theorem (Bonnet, 1849)).** *Let $f$ and $g$ be Riemann integrable on $[a, b]$. Assume that $f$ is positive increasing. Then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)g(x)\,dx = f(b)\int_c^b g(x)\,dx.$$

*Proof.* Let $P = [a = x_0, x_1, \ldots, x_n]$ be a partition of $[a, b]$ such that the Riemann sum

$$S = \sum_{i=1}^{n} f(x_i)g(x_i)(x_i - x_{i-1}) \qquad \text{approximates} \int_a^b f(x)g(x)\,dx.$$

Let $a_i = f(x_i)$ and $b_i = g(x_i)(x_i - x_{i-1})$. Summing by parts, and letting

$$B_i = \sum_{k=i}^{n} g(x_k)(x_k - x_{k-1}),$$

we find (putting $B_{n+1} = 0$) that

$$S = f(x_0)B_1 + \sum_{i=1}^{n} B_i\big(f(x_i) - f(x_{i-1})\big).$$

Since $f(x_n) = f(b)$, we find

$$f(b)\min_k B_k \leqq S \leqq f(b)\max_k B_k.$$

But $B_k$ is a Riemann sum itself for $g$ over the interval $[x_{k-1}, x_n] = [x_{k-1}, b]$.

Varying $x_{k-1}$, we see that given $\varepsilon$, we obtain

$$\inf_c f(b) \int_c^b g(x)\,dx - 2\varepsilon \leqq \int_a^b f(x)g(x)\,dx \leqq \sup_c f(b) \int_c^b g(x)\,dx + 2\varepsilon,$$

whence the same inequality without the $\varepsilon$. But

$$x \mapsto \int_x^b g(t)\,dt$$

is continuous, so by the intermediate value theorem for continuous functions, the proposition follows.

## X, §2. APPLICATIONS TO FOURIER ANALYSIS

In this section we give applications of integration to a more refined Fourier analysis, especially using the results on functions of bounded variation from §1. We start however with the simplest situation containing no delicate estimates. The next result is a routine version of the **Riemann–Lebesgue Lemma**.

**Proposition 2.1.** *Let $f \in C_c^\infty(\mathbf{R})$. Given a positive integer $k$, there exists a constant $C = C(f, k)$ such that for $A \geqq 1$ we have*

$$\left| \int_{-\infty}^{\infty} f(x)e^{iAx}\,dx \right| \leqq \frac{C}{A^k}.$$

*Proof.* We integrate by parts $k$ times, so that a power $A^k$ comes into the denominator. The constant $C$ is the sup norm of the $k$-th derivative of $f$. Since $f$ is assumed to have compact support, the term $uv$ in the integration by parts disappears since $f(x) = 0$ for $x$ large positive and large negative. So the proposition is clear.

In particular, the proposition gives a quantitative estimate for the limit

$$\lim_{A \to \infty} \int_{-\infty}^{\infty} f(x)e^{iAx}\,dx = 0.$$

We may consider next the variation when we must take an end point into consideration.

We now introduce the condition of bounded variation. According to Zygmund [Zy], Dirichlet was the first who proved Fourier series

convergence theorems under the conditions of normalization and of being increasing, which amounts to bounded variation. See Zygmund's comments, Chapter II, 2.6.

We recall that for a function $f \in L^1(\mathbf{R})$, the Fourier transform $f^\wedge$ is defined by

$$f^\wedge(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx.$$

The function $f^\wedge$ is obviously bounded, by $\|f\|_1$.

**Proposition 2.2.** *Let $f \in \mathrm{BV}(\mathbf{R}) \cap L^1(\mathbf{R})$. Then there is a constant $C = C(f)$ such that for all $t \neq 0$ we have*

$$|f^\wedge(t)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x)e^{-itx}\,dx \right| \leqq \frac{C}{|t|}.$$

*Proof.* We integrate by parts, obtaining for the integral the value

$$-f(x)\frac{e^{-itx}}{it}\Big]_{-\infty}^{\infty} + \frac{1}{it}\int_{-\infty}^{\infty} e^{-itx}\,df(x).$$

Since $f \in \mathrm{BV}(\mathbf{R}) \cap L^1(\mathbf{R})$, we remarked in Lemma 1.7 that $f(x) \to 0$ as $x \to \pm\infty$, so the first term vanishes. The integral of the second term is finite since $f$ has bound variation, so the estimate $\leqq C/|t|$ is evident.

**Proposition 2.3.** *Let $f \in L^1(\mathbf{R})$. Then*

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} f(x)e^{iAx}\,dx = 0.$$

*Proof.* Let $g \in C_c^\infty(\mathbf{R})$ be such that $\|f - g\|_1 < \varepsilon$. Then the integral can be written and estimated in the form

$$\int_{-\infty}^{\infty} (f(x) - g(x))e^{iAx}\,dx + \int_{-\infty}^{\infty} g(x)e^{iAx}\,dx.$$

The first integral is bounded in absolute value by $\varepsilon$. The second integral is $< \varepsilon$ for large $A$ by Proposition 2.1. This proves the proposition.

Next we look at Fourier inversion under more delicate conditions than those of the Schwartz space.

Let $A > 0$, and let $f \in L^1(\mathbf{R})$. We define

$$f_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f^\wedge(t)e^{itx}\,dt.$$

We are interested in conditions under which we have $f^{\wedge\wedge} = f^-$, in the

sense that

$$f^{\wedge\wedge}(x) = \lim_{A\to\infty} f_A(x).$$

We begin by a lemma which gives another expression for $f_A$.

**Lemma 2.4.** *Let $f \in L^1(\mathbf{R})$ and let $A > 0$. Then*

$$f_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f^{\wedge}(t)e^{itx}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{\sin A(x-y)}{x-y}\,dy.$$

*If $f$ is also in* $\mathbf{BV}(\mathbf{R})$, *then $f_A$ is bounded independently of $A$.*

*Proof.* We have

$$\begin{aligned}\sqrt{2\pi}\int_{-A}^{A} f^{\wedge}(t)e^{itx}\,dt &= \int_{-A}^{A}\int_{-\infty}^{\infty} f(y)e^{-ity+itx}\,dy\,dt \\ &= \int_{-\infty}^{\infty} f(y)\int_{-A}^{A} e^{it(x-y)}\,dt\,dy \qquad \text{[by Fubini's theorem]} \\ &= \int_{-\infty}^{\infty} 2f(y)\frac{\sin A(x-y)}{x-y}\,dy,\end{aligned}$$

which proves the formula. The boundedness of $f_A$ if $f \in \mathbf{BV}(\mathbf{R})$ is left as Exercise 1 (Stieltjes integration by parts).

The next theorem gives conditions for $f^{\wedge\wedge} = f^{-}$. It involves an application of the Bonnet mean value theorem. The theorem is in Titchmarsh [Ti], who attributes the result to Prasad, Pringsheim, and Hobson.

**Theorem 2.5.** *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function satisfying:*

(a) *$f$ is of bounded variation on every finite closed interval.*
(b) *$f$ is in $L^1(\mathbf{R})$.*
(c) *$f$ is normalized so that for all $x \in \mathbf{R}$,*

$$f(x) = \tfrac{1}{2}(f(x+) + f(x-)].$$

*Then*

$$f(x) = \lim_{A\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{\sin A(x-y)}{x-y}\,dy = \lim_{A\to\infty} f_A(x).$$

*Proof.* After a translation we may assume that $x = 0$. Since $(\sin Ay)/y$ is an even function of $y$, we may assume that $f$ is even, and we are reduced to proving

$$\frac{\pi}{2}f(0) = \lim_{A\to\infty}\int_{0}^{\infty} f(y)\frac{\sin Ay}{y}\,dy,$$

under the additional assumption that $f$ is continuous at 0. We shall now see that the result depends only on the local behavior of $f$ near 0. Pick some $b > 0$. Then

$$\lim_{A\to\infty}\int_b^\infty f(y)\frac{\sin Ay}{y}\,dy = 0$$

by Proposition 2.3. Therefore we are reduced to consider the integral in the finite interval $[0, b]$, where $f$ is bounded. By additivity, we may also assume without loss of generality that $f$ is increasing, since a function of bounded variation is the difference of two increasing functions.

We observe that the theorem is valid when $f$ is constant on $[0, b]$. Indeed, in that case, we note from the change of variables $u = Ay$, $du = Ady$, that

$$\int_0^b \frac{\sin Ay}{y}\,dy = \int_0^A \frac{\sin u}{u}\,du \to \frac{\pi}{2} \quad \text{as} \quad A \to \infty.$$

Since the integral is linear in $f$, we may assume without loss of generality that $f(0) = 0$. Given $\varepsilon$ there exists $\delta$ such that for $0 \leqq y \leqq \delta$ we have

$$|f(y) - f(0)| = |f(y)| < \varepsilon.$$

Then

$$\int_0^b f(y)\frac{\sin Ay}{y}\,dy = \int_0^\delta + \int_\delta^b f(y)\frac{\sin Ay}{y}\,dy.$$

The second integral on the right approaches 0 as $A \to \infty$ by Proposition 2.3 because $(\sin Ay)/y$ is bounded for $y \geqq \delta$. So what matters is the integral over $[0, \delta]$. By the integral mean value theorem, we now find

$$\int_0^\delta f(y)\frac{\sin Ay}{y}\,dy = f(\delta)\int_c^\delta \frac{\sin Ay}{y}\,dy = f(\delta)\int_{Ac}^{A\delta}\frac{\sin u}{u}\,du.$$

Since $|f(\delta)| < \varepsilon$, and since the integral over $[Ac, A\delta]$ is bounded, we have concluded the proof of the theorem.

In the next theorem, we deal still with another situation, when we do not assume an $L^1(\mathbf{R})$-condition, but deal with oscillatory integrals to guarantee convergence in a quantitative form of the **Riemann–Lebesgue Lemma**. The result concerns a situation intermediate between Proposition 2.2 and Proposition 2.3, and in the precise given form is due to Barner [Ba 90].

**Theorem 2.6.** *Assume:*

(a) $f \in \mathrm{BV}(\mathbf{R})$.
(b) $f(x) = O(|x|^\varepsilon)$ *for some* $\varepsilon > 0$ *and* $x \to 0$.

*Then the improper integral that follows exists for* $A > 0$, *and satisfies the bound*

$$\int_0^\infty f(y)e^{iAy}\frac{dy}{y} = O_f(A^{-\varepsilon/(1+\varepsilon)}) \qquad \textit{for} \quad A \to \infty.$$

*Proof.* We can split $e^{iAy} = \cos(Ay) + i\sin(Ay)$ and consider separately the cosine and sine. The cosine is harder if anything because one has to rely on assumption (b) on $f$ to make the integrand integrable near 0. Hence we deal with the cosine for concreteness to fix ideas. We shall first prove that the integral exists (the problem is at infinity), and then we shall prove that it satisfies the stated bound as $A \to \infty$.

*Existence.* There will be some integrations by part, so we need the two auxiliary functions defined respectively for $x > 0$ and all $x \in \mathbf{R}$:

$$\mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt \qquad \text{and} \qquad \mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt.$$

**Lemma 2.7.** *We have the estimates for* $x > 0$:

$$|\mathrm{ci}(x)| \leqq \frac{2}{x} \qquad \textit{and} \qquad |\mathrm{si}(x)| \leqq \frac{2}{x}.$$

*Proof.* This is immediate after integrating by parts.

To prove the existence of the improper integral in the theorem, we define

$$g(y) = \frac{\cos Ay}{y} \qquad \text{and} \qquad G(y) = -\int_y^\infty \frac{\cos At}{t}\,dt = \mathrm{ci}(Ay)$$

for $y > 0$. Let $\delta > 0$. By Proposition 1.4 justifying integration by parts, for $\delta < a < \infty$, we get:

$$\int_\delta^a f(y)\,dG(y) = f(a)G(a) - f(\delta)G(\delta) - \int_\delta^a G(y)\,df(y).$$

We now estimate each one of the three terms on the right side.

Since $f$ is bounded by hypothesis, and $G(a) \to 0$ as $a \to \infty$, we see that $f(a)G(a) \to 0$ as $a \to \infty$, so the first term approaches 0 as $a \to \infty$.

We claim that $f(\delta)G(\delta) \to 0$ as $\delta \to 0$. To see this, we split the integral:

$$\begin{aligned}
\operatorname{ci}(x) &= -\int_x^1 \frac{\cos t}{t}\,dt - \int_1^\infty \frac{\cos t}{t}\,dt \\
&= \int_1^x \frac{\cos t - 1}{t}\,dt + \int_1^x \frac{dt}{t} - \operatorname{ci}(1) \\
&= \text{convergent integral}(x) + \log x + \text{constant as } x \to 0.
\end{aligned}$$

Let $x = Ay$ so $\log x = \log A + \log y$. Consider $f(x)\operatorname{ci}(Ax) = f(x)G(x)$. The hypothesis that $f(x) = O(|x|^\varepsilon)$ immediately implies that $f(x)\operatorname{ci}(Ax) \to 0$ as $x \to 0$, thus proving the claim that $f(\delta)G(\delta) \to 0$ as $\delta \to 0$.

Now to the third term. The limit

$$\lim_{\delta \to 0} \int_\delta^a f(y)\,dG(y)$$

exists by the hypothesis on $f$, so the integral

$$\int_0^a G(y)\,df(y)$$

exists. There remains only to prove that the tail end

$$\int_a^b G(y)\,df(y) = \int_a^b \operatorname{ci}(Ay)\,df(y)$$

approaches 0 as $a \to \infty$ and $a < b$. But the routine estimate of Proposition 1.3 gives for the absolute value of the integral a bound $V(f)2/Aa$, using also Lemma 2.7. Hence

$$\lim_{a \to \infty} \int_0^a G(y)\,df(y) \quad \text{exists.}$$

This concludes the proof of existence for the improper integral of Theorem 2.6.

*Estimate of the integral.* We shall now prove the stated estimate. We let $\lambda = 1/(1 + \varepsilon)$. We decompose the integral:

$$\begin{aligned}
\int_0^\infty f(y)\frac{\cos Ay}{y}\,dy &= \int_0^{1/A^\lambda} + \int_{1/A^\lambda}^\infty f(y)\frac{\cos Ay}{y}\,dy \\
&= \int_0^{1/A^\lambda} + \big[f(y)\operatorname{ci}(Ay)\big]_{1/A^\lambda}^\infty - \int_{1/A^\lambda}^\infty \operatorname{ci}(Ay)\,df(y)
\end{aligned}$$

$$= \int_0^{1/A^\lambda} \frac{f(y)}{y} \cos(Ay)\,dy - f(1/A^\lambda)\,\mathrm{ci}(A^{1-\lambda})$$
$$- \int_{1/A^\lambda}^{\infty} \mathrm{ci}(Ay)\,df(y).$$

We shall estimate each one of the three terms on the right side of the equality.

For the first term, the integral is taken near zero where

$$|f(y)/y| = O(y^{\varepsilon-1})$$

and cos $Ay$ is bounded. Hence the first integral in absolute value is

$$\ll \frac{1}{\varepsilon} \frac{1}{A^{\varepsilon/\lambda}} \qquad \text{for} \quad A \to \infty,$$

which is of the desired type.

For the second expression, using Lemma 2.7, we have the bound

$$f(1/A^\lambda)\,\mathrm{ci}(A^{1-\lambda}) \ll \frac{1}{A^{\varepsilon\lambda}} \frac{2}{A^{1-\lambda}} \leqq \frac{2}{A^{2\varepsilon/(1+\varepsilon)}}$$

which is also of the desired type.

For the third expression, we have the usual sup bound for the integral,

$$\left| \int_{1/A^\lambda}^{b} \mathrm{ci}(Ay)\,df(y) \right| \leqq \frac{2}{A^{1-\lambda}} V_{\mathbf{R}}(f)$$

for every $b$, and hence for the integral to $\infty$. This concludes the proof of the theorem.

**Remark.** The previous theorem is similar to many of the same kind, giving various conditions under which the Fourier transform tends to 0, which is a form of the Riemann–Lebesgue Lemma. The method of proof also follows a standard pattern, whereby one splits the integral over various intervals, near 0 and near $\infty$, with a cut-off point chosen judiciously. In determining the cut-off point, one may think of $f(x)$ as actually being equal to $x^\varepsilon$, and one first determines the cut-off as $1/A^\lambda$, or a small perturbation thereof. I don't know of a single theorem which covers all examples of that kind. Barner's conditions have shown to be useful in the applications he had in mind, and were motivated by those applications.

For a Parseval formula in a context similar to that of the present section, see [JoL].

## X, §3. EXERCISES

1. Let $f \in \mathrm{BV}(\mathbf{R})$ and also $f \in L^1(\mathbf{R})$. Define

$$S(x) = \int_0^x \frac{\sin t}{t} dt.$$

Show that for $A > 0$:

$$\left| \int_{-\infty}^{\infty} f(y) \frac{\sin A(x-y)}{x-y} dy \right| \leqq 2\|S\| V_{\mathbf{R}}(f),$$

or better,

$$|f_A| \leqq \frac{S(\pi)}{\pi} V_{\mathbf{R}}(f).$$

2. Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$, converging uniformly to $f$. Let $g$ be of bounded variation on $[a, b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n \, dg = \int_a^b f \, dg.$$

3. Let $f$ be continuous on $[a, b]$ and let $\{g_n\}$ be a sequence of functions of bounded variation, such that the variations are uniformly bounded, that is there exists a constant $B$ such that

$$V(g_n, [a, b]) \leqq B.$$

Assume that $g_n(x)$ converges to $g(x)$ for some bounded function $g$ and all $x \in [a, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f \, dg_n = \int_a^b f \, dg.$$

4. Let $f$ be of bounded variation on $[a, b]$. For each $x \in [a, b]$ define the **jump**

$$J_f(x) = \lim_{r\to 0} \sup |f(y) - f(x)|$$

where the sup is taken for $y \in [x - r, x + r]$. Prove:
(a) Given $\varepsilon$, there is only a finite number of points $x \in [a, b]$ such that $J_f(x) \geqq \varepsilon$.
(b) The set of points where $f$ is not continuous is countable. [Show that $f$ is continuous at $x$ if and only if $J_f(x) = 0$.]

5. Let $f$ be a real valued function on $[a, b]$. If the set of discontinuities of $f$ has measure 0, show that $f$ is Riemann-integrable. The converse is also true, and you can prove it as an exercise if you are sufficiently interested.

# CHAPTER XI

# Distributions

In Chapter IX, we saw how certain functionals on $C_c(X)$ gave rise to a measure. Here we consider the case when $X = \mathbf{R}^n$ and the functionals satisfy additional continuity conditions with respect to differentiation.

## XI, §1. DEFINITION AND EXAMPLES

We let $D_i = \partial/\partial x_i$ be the $i$-th partial derivative applied to functions on $\mathbf{R}^n$. For a $p$-tuple $(p_1, \ldots, p_n) = p$ of integers $\geqq 0$, we let

$$D^p = D_1^{p_1} \cdots D_n^{p_n}$$

and $|p| = p_1 + \cdots + p_n$. Then each differential operator $D^p$ operates on functions in $C^\infty(\mathbf{R}^n)$. Actually, we shall deal with the subspace of functions $C_c^\infty(\mathbf{R}^n)$ which have compact support. If $f$ is a function, we let $M_f$ be the operator which consists in multiplying by $f$, so that we have $M_f(g) = fg$, for any function $g$. We have the formula

$$D_i \circ M_f = M_f \circ D_i + M_{D_i f}$$

for any $f \in C^\infty(\mathbf{R}^n)$. We shall consider general differential operators

$$D = \sum_{|p| \leqq m} \alpha_p D^p$$

with coefficients $\alpha_p \in C^\infty(\mathbf{R}^n)$. Because of the preceding formula, we see that such differential operators, viewed as linear maps on $C_c^\infty(\mathbf{R}^n)$, form

an algebra under composition. The differential operator being written as above, we say that it has **order** $\leqq m$. It is easy to verify that its expression as above determines the coefficients $\alpha_p$ uniquely. Indeed, suppose that $D = 0$. To prove that $\alpha_p = 0$ it suffices to prove that for any $\alpha \in \mathbf{R}^n$ we have $\alpha_p(a) = 0$. For a given $p$ we consider a function given locally near $a$ by

$$f(x) = (x - a)^p = (x_1 - a_1)^{p_1} \cdots (x_n - a_n)^{p_n}.$$

Then

$$D^q f(a) = \begin{cases} 0 & \text{if } p \neq q, \\ p! & \text{if } p = q, \end{cases}$$

where $p! = p_1! \cdots p_n!$. Hence $(Df)(a) = p!\, \alpha_p(a) = 0$, whence

$$\alpha_p(a) = 0.$$

We define seminorms on $C_c^\infty(\mathbf{R}^n)$ as follows. For each differential operator $D$ and $f \in C_c^\infty(\mathbf{R}^n)$ we let

$$\pi_D(f) = \|Df\|$$

where $\| \ \|$ is the sup norm, and for each integer $m \geqq 0$ we let

$$\pi_m(f) = \sup_{|p| \leqq m} \|D^p f\|.$$

It is clear that $\pi_D$ and $\pi_m$ are seminorms on $C_c^\infty(\mathbf{R}^n)$.

For any subset $K$ of $\mathbf{R}^n$ we denote by $C_c^\infty(K)$ the space of those functions in $C_c^\infty(\mathbf{R}^n)$ whose support lies in $K$. We define a **distribution** on an open set $U$ of $\mathbf{R}^n$ to be a linear map

$$T\colon C_c^\infty(U) \to \mathbf{C}$$

such that, for every compact set $K$ contained in $U$, there exists a constant $A_K$ and an integer $m$ for which

$$|T\varphi| \leqq A_K \pi_m(\varphi), \qquad \text{all } \varphi \in C_c^\infty(K).$$

Just as it is useful to have a criterion for continuity of a linear map in terms of sequences when dealing with normed vector spaces, we have a similar criterion under the present circumstances, namely:

**Theorem 1.1.** *A linear map $T\colon C_c^\infty(U) \to \mathbf{C}$ is a distribution if and only if it satisfies the following property*:

*Let $\{\varphi_j\}$ be a sequence in $C_c^\infty(U)$, such that all $\varphi_j$ have support in a compact set $K$, and such that for every $p$, $\{D^p\varphi_j\}$ converges to 0 uniformly on $K$. Then $T\varphi_j \to 0$.*

*Proof.* It is clear that if $T$ is a distribution, then it satisfies the stated property. Conversely, assume that it satisfies this property, and let $K$ be a compact subset of $U$. For each integer $m \geqq 0$ let

$$a_m = \sup|Tf|.$$

the sup being taken for those $f \in C_c^\infty(K)$ such that $\pi_m(f) \leqq 1$. It will suffice to show that for some $m$, we have $a_m \neq \infty$. Suppose that $a_m = \infty$ for all $m$. Choose $f_m \in C_c^\infty(K)$ such that $\pi_m(f_m) \leqq 1$, but $|Tf_m| \geqq m$. Let $g_m = f_m/m$. Then

$$\pi_m(g_m) \leqq 1/m$$

and if $k \leqq m$, then

$$\pi_k(g_m) \leqq \pi_m(g_m) \leqq 1/m,$$

so

$$\lim_{m\to\infty} \pi_k(g_m) = 0.$$

But

$$\|D^p g_m\| \leqq \pi_{|p|}(g_m)$$

tends to 0 as $m \to \infty$, and $g_m$ has support in $K$. Thus $D^p g_m$ tends to 0 uniformly on $K$, and $Tg_m \to 0$ by hypothesis, a contradiction which proves the theorem.

We say that a distribution $T$ on an open set $U$ is of **order** $\leqq m$ if for each compact set $K \subset U$ there exists a number $A_K$ such that

$$|T\varphi| \leqq A_K \pi_m(\varphi)$$

for all $\varphi \in C_c^\infty(K)$. A distribution of order 0 is therefore a $C_c$-functional as in Chapter IX, §1 and the Remark at the end of §4.

We shall now give examples of distributions.

**Functions.** Let $f$ be a locally integrable function on an open set $U$ of $\mathbf{R}^n$. (We recall that this means: $f$ is $\mu$-measurable for Lebesgue measure $\mu$, and $f$ is integrable on every compact subset $K$ of $U$.) We associate with $f$ the map $T_f$ whose value on $\varphi \in C_c^\infty(U)$ is given by

$$T_f(\varphi) = \int_U f(x)\varphi(x)\,dx = \int_U f\varphi\,d\mu.$$

Then it is clear that $T_f$ is a distribution of order 0 on each compact subset $K$ of $U$. In fact, if we use the obvious notation

$$\|f\|_{1,K} = \int_K |f| \, d\mu,$$

then

$$|T_f(\varphi)| \leqq \|f\|_{1,K} \|\varphi\| \qquad \text{if} \quad \varphi \in C_c^\infty(K).$$

Furthermore, the map $f \mapsto T_f$ induces an injective linear map of $L^1(U)$ into the space of distributions on $U$, because we know from Corollary 9.5 of Chapter VI that if $T_f = T_g$ for two locally integrable functions $f$, $g$, then $f$ is equal to $g$ almost everywhere. Thus from now on, we can interpret locally integrable functions as distributions.

**Measures.** Similarly, let $\mu$ be a positive $\sigma$-regular Borel measure on the open set $U$ of $\mathbf{R}^n$. We know that $d\mu$ is a functional on $C_c(U)$, and since $C_c^\infty(U)$ is a subspace of $C_c(U)$, we can view $d\mu$ as a linear map on $C_c^\infty(U)$. Thus if $\varphi$ has support in $K$, we have

$$|\langle \varphi, d\mu \rangle| \leqq \mu(K) \|\varphi\|,$$

and we see again that $d\mu$ is a distribution of order 0 on every compact subset of $U$. We could use the notation $T_\mu$ instead of $d\mu$ for the preceding distribution. As with functions, we have an injective map $\mu \mapsto d\mu$ from the set of positive $\sigma$-regular Borel measures on $U$ into the set of distributions.

The **Dirac** distribution $\delta$ given by

$$\delta(\varphi) = \varphi(0)$$

is a special case of a distribution obtained from a measure, namely the Dirac measure. For each $a \in \mathbf{R}^n$, we also have the translate $\delta_a$ of $\delta$, given by

$$\delta_a(\varphi) = \varphi(a).$$

**Multiplication by a $C^\infty$ Function.** Let $T$ be a distribution on $U$ and let $\alpha \in C^\infty(U)$. We define $\alpha T$ to be $T \circ M_\alpha$, so that

$$(\alpha T)(\varphi) = T(\alpha\varphi).$$

It is immediately verified that $\alpha T$ is a distribution.

**Composition with Differential Operators.** Let $T$ be a distribution and $D$ a differential operator on $U$. Then $T \circ D$ (also written $TD$) is a distri-

bution $U$. Its value at $\varphi$ is

$$(TD)(\varphi) = T(D\varphi).$$

In particular, if $T$ can be represented by a locally integrable function $f$, then

$$T_f D(\varphi) = \int_U f(x) D\varphi(x)\, dx.$$

The verification that $TD$ is a distribution is immediate from the definitions.

## XI, §2. SUPPORT AND LOCALIZATION

Let $D$ be a differential operator on an open set $U$. For every open subset $V$ of $U$, we can view $D$ as a differential operator on $V$, namely considering the restriction of $D$ to those functions having support in $V$. We say that $D$ is equal to zero on $U$ if $D\varphi = 0$ for every $\varphi \in C^\infty(U)$. We say that $D$ is **locally zero** at a point $a \in U$ if $D$ is equal to zero on some open neighborhood of $a$, i.e. if there exists an open subset $V$ of $U$, containing $a$, such that $D\varphi = 0$ for every $\varphi \in C^\infty(V)$. We define the **support** of $D$ by describing its complement, namely:

$a \notin \operatorname{supp}(D)$ *if and only if* $D$ *is locally zero at* $a$.

Note that if $D$ is locally zero at $a$, then $D$ is locally zero at every point close to $a$, so that the support of $D$ is closed in $U$.

Let $T$ be a distribution on an open set $U$. We say that $T$ is zero on $U$ if $T\varphi = 0$ for each $\varphi \in C_c^\infty(U)$. If $T$ is a distribution on $U$, and $V$ is an open subset of $U$, then we can restrict $T$ from $C_c^\infty(U)$ to $C_c^\infty(V)$, and this restriction, denoted by $T|V$, is a distribution on $V$. We shall say that a distribution $T$ on $U$ is **locally zero** at a point $a \in U$ if there exists an open neighborhood $V$ of $a$ in $U$ such that the restriction of $T$ to $V$ is zero on $V$. We can thus define the **support** of $T$ by the condition:

$a \notin \operatorname{supp}(T)$ *if and only if* $T$ *is locally* 0 *at* $a$.

As before, we see that the support of $T$ is closed in $U$.

We can localize distributions just as we localized measures, by means of partitions of unity, which must now be taken to be of class $C^\infty$, not merely continuous. We restate this as a separate result.

**$C^\infty$ Partitions of Unity.** *Let $K$ be a compact set in $\mathbf{R}^n$ and let $\{U_j\}$ $(j = 1, \ldots, m)$ be an open covering of $K$. Then there exist functions $\varphi_j$*

*in* $C_c^\infty(U_j)$ *such that* $\varphi_j \geqq 0$,

$$\sum_{j=1}^{m} \varphi_j \leqq 1 \qquad \text{and} \qquad \sum_{j=1}^{m} \varphi_j = 1 \text{ on } K.$$

*Proof.* For each $x \in K$ we can find an open ball centered at $x$, of radius $r(x)$, such that the ball of twice this radius centered at $x$ is contained in some $U_j$. We cover $K$ by a finite number of such balls, say $B_1, \ldots, B_s$. For each $k = 1, \ldots, s$ we find a function $\psi_k$ which is $C^\infty$, which is equal to 1 on $B_k$, $0 \leqq \psi_k \leqq 1$, and such that $\psi_k$ vanishes outside a ball $B_k'$ centered at the same point as $B_k$ and having a slightly bigger radius. This is done by routine calculus technique, cf. Chapter VI, §9. [We recall briefly below how to do this.] Inductively, one then sees that if we let

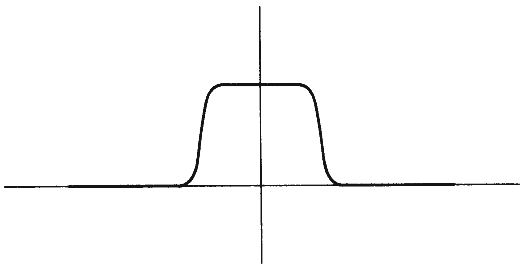
$$\alpha_1 = \psi_1,\ \alpha_2 = \psi_2(1 - \psi_1),\ \ldots, \alpha_s = \psi_s(1 - \psi_1) \cdots (1 - \psi_{s-1}),$$

then on $B_1 \cup \cdots \cup B_s$ we have

$$\alpha_1 + \cdots + \alpha_s = 1 - (1 - \psi_1) \cdots (1 - \psi_s).$$

This yields what we want, except for the fact that the indices may not be $j = 1, \ldots, m$. But it is trivial to adjust this as desired. All we have to do is to find for each $k$ and index $j(k)$ such that $B_k$ is contained in $U_j$, and then for each $j = 1, \ldots, m$ take the sum of those $\alpha_k$ such that $j(k) = j$, to obtain $\varphi_j$.

To get the function $\psi_k$ as in the preceding proof, we combine a function whose graph is indicated below with the square of the euclidean norm to get a $C^\infty$ function which is 1 on a ball, and 0 outside another ball of slightly bigger radius.



If the ball is not centered at the origin, we combine this with a translation.

**Theorem 2.1.** *Let $T$ be a distribution on an open set $U$ in* $\mathbf{R}^n$. *If $T$ is locally zero at every point, then $T = 0$ on $U$.*

*Proof.* Let $\varphi \in C_c^\infty(U)$, and let $K$ be the support of $\varphi$. For each $a \in K$ we can find an open set $U_a$ such that $T$ is zero on $U_a$. We can cover $K$ with a finite number of such open sets, say $U_1, \ldots, U_m$. Let $\{\varphi_j\}$ be a $C^\infty$ partition of unity over $K$ with $j = 1, \ldots, m$, such that supp $\varphi_j$ is contained in $U_j$. Then

$$\varphi = \sum_{j=1}^{m} \varphi_j \varphi \qquad \text{and} \qquad T\varphi = \sum_{j=1}^{m} T(\varphi_j \varphi) = 0,$$

thus proving our assertion.

**Corollary 2.2.** *Two distributions which are locally equal everywhere are equal.*

**Corollary 2.3.** *Let $T$ be a distribution on the open set $U$, and let $\varphi \in C_c^\infty(U)$. If* $\operatorname{supp}(T) \cap \operatorname{supp}(\varphi)$ *is empty, then $T\varphi = 0$.*

*Proof.* Let $K = \operatorname{supp} \varphi$, and $Q = \operatorname{supp} T$. There exists an open neighborhood $V$ of $K$ which does not intersect $Q$ and is contained in $U$. Let $\alpha \in C_c^\infty(V)$ be such that $\alpha = 1$ on $K$ and the support of $\alpha$ is contained in $V$. Then $\varphi = \alpha\varphi$ and

$$T(\varphi) = T(\alpha\varphi).$$

It is immediately verified that $\alpha T$ is locally zero everywhere, and hence that $\alpha T = 0$, so that $T\varphi = 0$, as was to be proved.

**Corollary 2.4.** *Let $T$ be a distribution on an open set $U$, and assume that $T$ has compact support $K$. Let $\varphi$, $\psi \in C_c^\infty(U)$ and assume that $\varphi = \psi$ on an open neighborhood of $K$. Then $T\varphi = T\psi$.*

*Proof.* Since $\varphi - \psi$ is equal to 0 on an open neighborhood of $K$, it follows that the supports of $\varphi - \psi$ and $T$ are disjoint, whence we can apply Corollary 2.3 to conclude the proof.

As an application of Corollary 2.4, we can extend the domain of definition of a distribution $T$ with compact support $K$ to the whole space $C^\infty(U)$. Indeed, if $\alpha$ is a function in $C_c^\infty(U)$ which is equal to 1 on an open neighborhood of $K$, and if $f \in C^\infty(U)$, then we define

$$Tf = T(\alpha f).$$

The preceding corollary shows that this value is independent of the choice of $\alpha$ subject to the condition that $\alpha = 1$ on an open neighborhood of $K$, and that it is an extension of $T$, namely $T(\alpha\varphi) = T(\varphi)$ if $\varphi \in C_c^\infty(U)$. This extension is useful in a context like the following. Consider the

function $f$ such that $f(x) = x$ (say in one variable $x$). If $T$ has compact support, then we can speak of the value $T(x) = Tf(x)$ using the definition we just made.

Using partitions of unity over a whole open set, one can prove the following result, left as an exercise.

*Let $\{U_i\}$ be an open covering of an open set $U$ in $\mathbf{R}^n$. For each $i$, let $T_i$ be a distribution on $U_i$, and assume that for each pair $i$, $j$ the restrictions of $T_i$ and $T_j$ to $U_i \cap U_j$ are equal. Then there exists a unique distribution $T$ on $U$ which is equal to $T_i$ on each $U_i$.*

We give one more example of the localization principle, using partitions of unity over a compact set. We say that a distribution $T$ is **locally of order** $\leqq m$ at a point $a$ if there exists a compact neighborhood $K$ of $a$ such that $T$ is of order $\leqq m$ on $K$.

**Theorem 2.5.** *If a distribution $T$ on $U$ is locally of order $\leqq m$ at every point of $U$, then $T$ is of order $\leqq m$ on every compact subset of $U$.*

*Proof.* Let $K$ be a compact subset of $U$. Let $\{\alpha_j\}$ $(j = 1,\ldots,k)$ be a $C^\infty$ partition of unity over $K$, such that each $\alpha_j$ has support in an open set $U_j$ whose closure $\overline{U}_j$ is compact, contained in $U$, and such that $T$ is of order $\leqq m$ on this closure. For any $\varphi \in C_c^\infty(K)$ we have

$$T(\varphi) = \sum_{j=1}^{k} T(\alpha_j \varphi).$$

We note that $\operatorname{supp}(\alpha_j \varphi) \subset U_j \subset \overline{U}_j$. Let $A_j$ be a number such that

$$|T(f)| \leqq A_j \pi_m(f), \qquad \text{all } f \in C_c^\infty(\overline{U}_j).$$

Then

$$|T(\varphi)| \leqq \sum_{j=1}^{k} A_j \pi_m(\alpha_j \varphi).$$

But

$$D^p(\alpha_j \varphi) = \sum_{|q| \leqq |p|} \psi_{jq} D^q \varphi$$

with suitable functions $\psi_{jq}$ determined by $\alpha_j$ and $p$. Thus

$$\|D^p(\alpha_j \varphi)\| \leqq \sum_{|q| \leqq |p|} \|\psi_{jq}\| \, \|D^q \varphi\|.$$

Hence there is a constant $B_j$ such that

$$\pi_m(\alpha_j \varphi) \leqq B_j \pi_m(\varphi), \qquad \text{all } \varphi \in C_c^\infty(K),$$

whence

$$|T(\varphi)| \leqq \sum_{j=1}^{k} A_j B_j \pi_m(\varphi), \qquad \text{all } \varphi \in C_c^\infty(K).$$

This proves our theorem.

## XI, §3. DERIVATION OF DISTRIBUTIONS

**Theorem 3.1.** *Let $D$ be a differential operator on the open set $U$ of $\mathbf{R}^n$. Then there exists a unique differential operator $D^*$ such that for any functions $f \in C^\infty(U)$ and $\varphi \in C_c^\infty(U)$ we have*

$$\int f D\varphi \, d\mu = \int (D^* f)\varphi \, d\mu.$$

*Proof.* For the existence, we may restrict ourselves to the case when $D = \alpha D^p$ for some $\alpha \in C^\infty(U)$. Then we have

$$D^* = (-1)^{|p|} D^p \circ M_\alpha,$$

because integration by parts shows that

$$\int f D\varphi = \int (f\alpha) D^p \varphi = (-1)^{|p|} \int D^p(\alpha f)\varphi.$$

This proves the existence. As for uniqueness, suppose that $D^*$ and $D'$ are differential operators such that

$$\int (D' f)\varphi = \int (D^* f)\varphi$$

for all $f \in C^\infty(U)$ and $\varphi \in C_c^\infty(U)$. Then

$$\int ((D^* - D')f)\varphi = 0$$

for all $\varphi$, so that $(D^* - D')f = 0$, whence $D^* = D'$.

We shall call $D^*$ the **adjoint** of $D$. The map $D \mapsto D^*$ is an anti-automorphism of the ring of differential operators; anti because

$$(D_1 D_2)^* = D_2^* D_1^*.$$

Let $T$ be a distribution on $U$ and $D$ a differential operator. We define

$$DT = T \circ D^* = TD^*$$

on $C_c^\infty(U)$. In particular, if $D = D_i$ is the $i$-th partial derivative, then $D_i^* = -D_i$ and

$$(D_i T)(\varphi) = -T\left(\frac{\partial \varphi}{\partial x_i}\right).$$

The reason for our definition is that if $f$ is a $C^1$ function, then recalling that $T_f(\varphi) = \int f\varphi \, d\mu$, we have the formula

$$DT_f = T_{Df},$$

as one sees from Theorem 3.1.

**Example.** Let $f$ be the locally integrable function on $\mathbf{R}$, such that $f(x) = 1$ if $x \geqq 0$ and $f(x) = 0$ if $x < 0$ (this is sometimes called the **Heaviside function**). A trivial integration in one variable shows that the derivative of $T_f$ is simply the Dirac distribution, i.e. we have

$$DT_f = \delta,$$

where $D = D_1$ is the derivative in one variable.

**Example.** Consider the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let

$$g(x, y) = \frac{1}{2\pi} \log r$$

where $r = (x^2 + y^2)^{1/2}$ as usual. Then $T_g$ is a distribution on the plane, and

$$\Delta T_g = \delta$$

is the Dirac distribution at the origin. See formula **L 3** of Chapter XIX, §3.

## XI, §4. DISTRIBUTIONS WITH DISCRETE SUPPORT

To investigate the structure of distributions with discrete support, it suffices to describe distributions whose support is one point, and then by translation, distributions whose support is at the origin. We can then give a complete description of such distributions.

**Theorem 4.1.** *Let $T$ be a distribution whose support is $\{0\}$. Then there exists an integer $m \geqq 0$ and constants $c_p$ such that*

$$T = \sum_{|p| \leqq m} c_p D^p \delta.$$

*In fact, $c_p = (-1)^{|p|} T(x^p)/p!$.*

*Proof.* First we recall from differential calculus that if $U$ is an open set containing 0, and if $f \in C^\infty(U)$ and $D^p f(0) = 0$ for $|p| \leqq k$, with $k \geqq 1$, then there exist $C^\infty$ functions $f_p$ such that

$$f(x) = \sum_{|p|=k+1} x^p f_p(x).$$

This is proved by starting with the formula

$$f(x) = f(x) - f(0) = \int_0^1 f'(tx)x\, dt = \int_0^1 f'(tx)\, dt \cdot x,$$

and continuing to integrate similarly the successive derivatives of $f$. We then write the Taylor expansion of $f$, namely

$$f(x) \sim \sum_p \frac{(D^p f)(0)}{p!} x^p.$$

Since $T$ has compact support, it has order $\leqq m$ for some $m$. We shall use the definition of $T$ on functions as in the discussion following Corollary 2.4.

We consider the Taylor expansion of $f$ up to the terms of order $m$, and consider the function

$$(1) \qquad g(x) = f(x) - \sum_{|p| \leqq m} \frac{(D^p f)(0)}{p!} x^p.$$

Then $(D^q g)(0) = 0$ if $|q| \leqq m$, and our preceding remark allows us to write $g$ as a sum of terms each of which is of type

$$x^k h_k(x)$$

with $|k| \geqq m + 1$. We shall prove below that $T(x^k h) = 0$ if $h$ is $C^\infty$ in some neighborhood of 0, $T$ has order $\leqq m$, and $|k| \geqq m + 1$. Once we have this, we conclude that $Tg = 0$, whence from (1), we obtain

$$Tf = \sum_{|p| \leqq m} \frac{(D^p f)(0)}{p!} T(x^p),$$

from which our theorem follows at once.

We now prove that $T(x^k h) = 0$ under the stated conditions. Let $\alpha$ be a $C^\infty$ function with support in the unit disc, and equal to 1 on some neighborhood of 0. Let $\alpha_r(x) = \alpha(rx)$, so that the support of $\alpha_r$ shrinks to the origin as $r \to \infty$, and in fact lies in the disc of radius $1/r$. Fix $q$ such that $|q| \geqq m + 1$. We have for $r > 0$:

$$T(x^q \alpha_r h) = T(x^q h),$$

and it will suffice to prove that this value tends to 0 as $r \to \infty$. Since $m$ is the order of $T$, there exists a constant $A$ such that

$$|T(x^q \alpha_r h)| \leqq A\pi_m(x^q \alpha_r h).$$

Thus we have to estimate

$$\|D^p(x^q \alpha_r h)\|, \qquad |p| \leqq m.$$

The support of $x^q \alpha_r h$ lies in the disc of radius $1/r$. The usual formula for the derivative of a product yields

$$D^p(x^q \alpha_r h) = \sum c_{jkl} x^{q-j} D^k \alpha_r D^l h$$

with $j + k + l = p$. (Addition is componentwise, and the coefficients $c_{jkl}$ are variations of binomial coefficients, determined universally by $p$ and $q$.) The derivatives $D^l h$ are uniformly bounded on a given neighborhood of the origin. We have

$$(D^k \alpha_r)(x) = r^{|k|}(D^k \alpha)(rx),$$

and hence $D^k \alpha_r$ is bounded by $r^{|k|}$ times a bound for the derivatives of $\alpha$ itself, up to order $m$. In the circle of radius $1/r$ we have

$$|x^{q-j}| \leqq 1/r^{|q-j|} = 1/r^{|q|-|j|}.$$

But $|k| \leqq m - |j| < |q| - |j|$. This proves that $D^p(x^q \alpha_r h)$ tends to 0 as $r \to \infty$, and concludes the proof of our theorem.

More generally, by a similar technique, one can prove that a distribution with compact support is equal to $DT_f$, where $f$ is a continuous function, $D$ is some differential operator, and $T_f$ is the distribution given by $T_f(\varphi) = \int f\varphi \, d\mu$. Our intent here is not to give an exhaustive theory, but merely to give the reader a brief acquaintance and feeling for functionals depending on a more involved topology than that of a norm, taking into account partial derivatives. For a concise and very useful summary of other facts, cf. the first two chapter of Hörmander [Hö], and also Palais [Pa 1].

We conclude with a remark which is technically useful. One can define distributions on a torus ($\mathbf{R}^n$ modulo $\mathbf{Z}^n$) using $C^\infty$ functions on $\mathbf{R}^n$ which are periodic. The advantage of doing this lies in the fact that in this case, every distribution has compact support, and estimates become easier to make. One can use certain open subsets of the torus, as local domains replacing open sets in euclidean space, and thus one avoids certain quasipathological types of distributions arising from open subsets in $\mathbf{R}^n$, due to exceptional growth along the boundary. In many cases, it is worth paying the price of periodicity to achieve this. For an exposition along these lines, cf. [Pa 1].

CHAPTER XII

# Integration on Locally Compact Groups

This chapter is independent of the others, but is interesting for its own sake. It gives examples of integration in a different setting from euclidean space, for instance integration on a group of matrices. For an application of integration and some functional analysis to compact groups, see Exercise 11.

## XII, §1. TOPOLOGICAL GROUPS

A **topological group** $G$ is a topological space $G$ together with a group law, i.e. maps

$$G \times G \to G \qquad \text{and} \qquad G \to G$$

which define a group law and the inverse mapping in the group, such that these maps are continuous. After this section, i.e. from §2 to the end of the chapter, we assume always in addition that $G$ is Hausdorff.

**Examples.** (1) Euclidean space $\mathbf{R}^p$ is a topological group under addition.

(2) The multiplicative group $\mathbf{R}^*$ of non-zero real numbers under multiplication. Similarly, the multiplicative group of non-zero complex numbers $\mathbf{C}^*$ under multiplication.

(3) The group of non-singular $n \times n$ matrices $\mathrm{Mat}_n(\mathbf{R})$ or $\mathrm{Mat}_n(\mathbf{C})$ under multiplication.

(4) The group $SL_n(\mathbf{R})$ or $SL_n(\mathbf{C})$ of matrices having determinant 1, under multiplication.

(5) The Galois group of the algebraic numbers over the rational numbers, with the Krull topology.

(6) The additive group of $p$-adic numbers.

(If you don't know these last two examples, don't panic; forget about them. They won't be used in this book.)

In the non-commutative case, we write the group multiplicatively as usual. In the commutative case, we write it either multiplicatively or additively, depending on situations.

If $G$ is a topological group, and $a \in G$, then we get a map

$$\tau_a: G \to G$$

called **translation** by $a$, and defined by $\tau_a x = ax$. More accurately we call this **left translation by** $a$. Multiplication being continuous, it is clear that $\tau_a$ is continuous, and is in fact a homeomorphism since it has an inverse, namely translation by $a^{-1}$.

The map $x \mapsto x^{-1}$ is also a homeomorphism of $G$ onto itself.

Let $e$ be the unit element of $G$. Let $U$ be an open set containing $e$. If $a \in G$, then $aU$ is an open set containing $a$. If $V$ is an open set containing $a$, then $a^{-1}V$ is an open set containing $e$. Thus neighborhoods of $e$ and neighborhoods of any point in $G$ differ only by translation.

The technique of $(\varepsilon, \delta)$ in metric spaces can be used in topological groups by using translations, to give a uniform way of describing closeness. For instance, if $a, b \in G$ and $U$ is an open neighborhood of $e$, we can say that $a, b$ are $U$-close if $a \in bU$. This relation is symmetric if we can select $U$ to be symmetric, i.e. $U = U^{-1}$ (where for any set $S$ in $G$, the set $S^{-1}$ is the set of all elements $x^{-1}$ with $x \in S$). This can always be done: if $V$ is an open neighborhood of $e$ in $G$, then $V \cap V^{-1}$ is a symmetric open neighborhood.

The $\varepsilon/2$ technique can also be used: given an open neighborhood $U$ of $e$, there exists an open neighborhood $V$ of $e$ such that $VV = V^2 \subset U$. Indeed, the map $G \times G \to G$ being continuous, the inverse image of $U$ is open in $G \times G$ and contains an open set $W \times W'$ containing $(e, e)$. We let $V = W \cap W'$. Similarly, we can find an open neighborhood $V$ of $e$ such that $VV^{-1} \subset U$.

We have the usual uniformity statements for continuous maps on compact sets. Let $S$ be a subset of $G$ and $f: S \to E$ be a map into a normed vector space. We say that $f$ is (left) **uniformly continuous** on $S$ if given $\varepsilon$, there exists a neighborhood $U$ of $e$ such that for all $x, y \in S$ with $y^{-1}x \in U$ we have

$$|f(x) - f(y)| < \varepsilon.$$

**Proposition 1.1.** *If $S$ is compact and $f: S \to E$ is continuous, then $f$ is uniformly continuous.*

*Proof.* Just the same as when $S$ is in a metric space. For each $x \in S$ we can find an open neighborhood $U_x$ of $e$ such that if $y \in xU_x$, then

$$|f(y) - f(x)| < \varepsilon.$$

Let $V_x$ be an open neighborhood of $e$ such that $V_x^2 \subset U_x$. There exists a finite covering $\{x_1 V_{x_1}, \ldots, x_n V_{x_n}\}$ of $S$ with $x_1, \ldots, x_n \in S$. Let

$$V = V_{x_1} \cap \cdots \cap V_{x_n}.$$

Let $x, y \in S$ and suppose that $x \in yV$. We have $y \in x_i V_{x_i}$ for some $i$, and hence

$$x \in x_i V_{x_i} V \subset x_i U_{x_i}$$

so that

$$\begin{aligned} |f(x) - f(y)| &\leqq |f(x) - f(x_i)| + |f(x_i) - f(y)| \\ &< 2\varepsilon, \end{aligned}$$

thus proving our assertion.

**Proposition 1.2.** *If $A$, $B$ are compact sets, then the product $AB$ is compact.*

*Proof. $AB$ is the image of $A \times B$* under the continuous composition law of $G$.

**Proposition 1.3.** *Let $A$ be a subset of $G$. Then the closure of $A$ satisfies*

$$\bar{A} = \bigcap_V AV,$$

*the intersection being taken over all open neighborhoods $V$ of $e$.*

*Proof.* Let $x \in \bar{A}$. For any open neighborhood $V$ of $e$ the open set $xV^{-1}$ contains $x$ and hence intersects $A$, i.e. there is some $y \in A$ such that $y = xv^{-1}$ with some $v \in V$. Then $x = yv$ and $x \in AV$, so we get one inclusion. Conversely, if $x$ is in all $AV$, then $xV^{-1}$ intersects $A$ for all $V$, whence $x$ lies in the closure of $A$. This proves our assertion.

**Proposition 1.4.** *Let $H$ be a subgroup of $G$. The closure $\bar{H}$ is a subgroup.*

*Proof.* This is proved purely formally using the fact that translations and the inverse map are homeomorphisms. Namely, if $h \in H$, then

$$hH = H \subset \bar{H}$$

and $H \subset h^{-1}\overline{H}$. Since $h^{-1}\overline{H}$ is closed it follows that $\overline{H} \subset h^{-1}\overline{H}$, whence $h\overline{H} \subset \overline{H}$. Hence $H\overline{H} \subset \overline{H}$. If $x \in \overline{H}$, then $Hx \subset \overline{H}$ and $H \subset \overline{H}x^{-1}$, which is closed. Hence

$$\overline{H} \subset \overline{H}x^{-1}$$

and therefore $\overline{H}x \subset \overline{H}$ so that $\overline{H}\,\overline{H} \subset \overline{H}$. Similarly, $H^{-1} = H \subset \overline{H}$, so that $H \subset \overline{H}^{-1}$, and since $\overline{H}^{-1}$ is closed, we get $\overline{H} \subset \overline{H}^{-1}$, whence $\overline{H}^{-1} \subset H$. Thus $\overline{H}$ is a subgroup.

**Example.** If we take $H = \{e\}$, then $\overline{H}$ is the smallest closed subgroup of $G$. By what we saw concerning the closure of a set, it is equal to the intersection of all open neighborhoods of $e$.

We now consider coset spaces and factor groups. Let $H$ be a subgroup of $G$. We have the set of left cosets $\{xH\}$, $x \in G$, which we denote by $G/H$, and a natural map

$$\pi\colon G \to G/H,$$

which to each $x \in G$ associates the left coset $xH$. We give $G/H$ the topology having the minimum amount of open sets making $\pi$ continuous. Thus a subset $W$ in $G/H$ is defined to be open if and only if $\pi^{-1}(W)$ is open in $G$. We have the following characterization of open sets in $G/H$:

**Proposition 1.5.** *A subset of $G/H$ is open if and only if it is of the form $\pi(V)$ for some open set $V$ in $G$.*

*Proof.* If $W$ is open in $G/H$, then $W = \pi(\pi^{-1}(W))$ and $\pi^{-1}(W)$ is open. Conversely, if $V$ is open in $G$, then

$$\pi^{-1}(\pi(V)) = VH$$

is open, so $\pi(V)$ is open.

In particular, we see that the map $\pi\colon G \to G/H$ is an open mapping, i.e. maps open sets onto open sets. All open subsets of $G/H$ may be written in the form $VH/H$ with $V$ open in $G$. In particular, $G/H$ is locally compact.

**Proposition 1.6.** *If $K'$ is a compact subset of $G/H$, then there exists a compact subset $K$ of $G$ such that $K' = \pi(K)$.*

*Proof.* Let $A$ be a compact neighborhood of $e$ in $G$. Then $\pi(A)$ is a compact neighborhood of the unit coset in $G/H$ (it is compact and we know that $\pi$ is an open mapping). Let $x_1, \ldots, x_n$ be elements of $G$ such

that the sets $\pi(x_i A)$ cover $K'$. Let

$$K = \pi^{-1}(K') \cap (x_1 A \cup \cdots \cup x_n A).$$

Then $K$ is compact and satisfies our requirements.

The preceding property will be useful when we consider continuous functions on $G/H$.

If we let $H = \bar{e}$ be the closure of the identity subgroup (interesting, if at all, only when the set consisting of $e$ alone is not closed), then $H$ is closed and it is easily seen that every point in $G/H$ is closed. It then follows that $G/H$ is Hausdorff, because of a general property of topological groups, namely:

*If each point of a topological group G is a closed set, then G is Hausdorff.*

*Proof.* Let $x, y \in G$ and $x \neq y$. Let $U$ be the complement of $xy^{-1}$ and let $V$ be a symmetric open neighborhood of $e$ such that $VV \subset U$. Then $V$ does not intersect $Vxy^{-1}$, and hence $Vx$, $Vy$ are disjoint open sets containing $x$ and $y$ respectively.

As an exercise, the reader can show that if $H$ is a subgroup of $G$, then $G/H$ is Hausdorff if and only if $H$ is closed. From now on, we deal only with Hausdorff groups, and take coset spaces or factor groups (with normal $H$) only when $H$ is closed.

**Example.** A subgroup $H$ of $G$ is said to be **discrete** if the induced topology on $H$ is the discrete topology, i.e. every point is open. Let $G = \mathbf{R}^p$ and let $v_1, \ldots, v_m$ $(m \leqq p)$ be vectors linearly independent over the reals. Let $\Gamma$ be the additive group of all linear combinations

$$k_1 v_1 + \cdots + k_m v_m$$

with integers $k_i$. Then $\Gamma$ is a subgroup, which is immediately verified to be discrete.

The group $\mathbf{Z}$ of integers is a discrete subgroup of $\mathbf{R}$, and the factor group $\mathbf{R}/\mathbf{Z}$ is isomorphic to the circle group (group of complex numbers of absolute value 1, under multiplication), under the map

$$t \to e^{2\pi i t}.$$

The group $GL(n, \mathbf{Z})$ of non-singular $n \times n$ matrices with integer components is a discrete subgroup of $GL(n, \mathbf{R})$.

## XII, §2. THE HAAR INTEGRAL, UNIQUENESS

By a (left) **Haar measure** on a locally compact group $G$ (assumed Hausdorff from now on) we mean a positive measure $\mu$ on the Borel sets which is $\sigma$-regular, non-zero on any non-empty open set (or equivalently on any Borel set containing a non-empty open set), and which is left invariant, meaning that

$$\mu(xA) = \mu(A)$$

for all measurable sets $A$, and all $x \in G$.

We shall get hold of a Haar measure by going through positive functionals. Thus by a **Haar functional** we shall mean a positive non-zero functional $\lambda$ on $C_c(G)$ which is left invariant, i.e.

$$\lambda(\tau_a f) = \lambda(f)$$

for all $f \in C_c(G)$. Here as usual, $\tau_a f = f_a$ is the $a$-translate of $f$, defined by

$$f_a(x) = f(a^{-1}x).$$

**Remark.** Observe the presence of the inverse $a^{-1}$. This is deliberate. We want the formula

$$\tau_{ab}(f) = \tau_a(\tau_b(f))$$

for $a$, $b \in G$, and this formula is true with the present definition. Functions form a contravariant system, i.e. if $T\colon X \to Y$ is a map of sets, then it induces a map in the reverse direction

$$T^*\colon \text{functions on } Y \to \text{functions on } X.$$

We can apply this when $T = \tau_a$ is the translation, thus forcing the inverse $a^{-1}$ when applying translation to functions.

The original proof for the existence of Haar measure due to Haar provides the standard model for all known proofs. We shall prove the existence of the functional in §2, following Weil's exposition [W]. Here, we discuss the relation between the measure and the functional; we prove uniqueness and give examples.

First we prove a lemma which shows that a locally compact group has a certain $\sigma$-finiteness built into it. We recall that a set is called $\sigma$-compact if it is a countable union of compact sets.

**Lemma 2.1.** *Let $G$ be a locally compact group. Then there exists an open and closed subgroup $H$ which is $\sigma$-compact.*

*Proof.* Let $K$ be a symmetric compact neighborhood of $e$. Then the sets $K^n = KK \cdots K$ ($n$ times) are compact neighborhoods of $e$, and form an increasing sequence since $e \in K^n$ for all $n$. Let $H = K^\infty$ be the union of all $K^n$ for all positive integers $n$. Then $H$ is $\sigma$-compact. Furthermore, $H$ is a subgroup (obvious). Next, $H$ is open, because if $x \in H$, then $x \in K^n$ for some $n$, whence

$$xK \subset K^{n+1} \subset H,$$

and $H$ is open. All cosets of $H$ are open, and we can write $G$ as a disjoint union of cosets of $H$. Then $H$ itself is the complement of an open set (union of all cosets $\neq H$), whence $H$ is closed. This proves our lemma.

In view of the lemma, we can write

$$G = \bigcup_{i \in I} x_i H$$

for $i$ in some indexing set $I$, and $H$ is open, closed, and $\sigma$-compact. Let $\mu$ be a Haar measure. By the remarks following Theorem 2.3, Chapter 14, it follows that the measure on each coset $x_i H$ is regular. If $A$ is an arbitrary measurable set, then we can write

$$A = \bigcup A_i \qquad \text{with} \quad A_i \subset x_i H$$

and the $A_i$ are disjoint. If we have proved the uniqueness of Haar measure on $H$ (i.e. the fact that two Haar measures differ by a constant $> 0$), then the reader will verify easily that the uniqueness follows for $G$ itself.

**Theorem 2.2.** *If $\mu$ is a Haar measure, then for any $f \in \mathscr{L}^1(\mu)$ and any $a \in G$ we have*

$$\int_G f(ax)\, d\mu(x) = \int_G f(x)\, d\mu(x).$$

*In particular, the functional $d\mu$ on $C_c(X)$ is left invariant, and therefore a Haar functional.*

*Proof.* If $\varphi$ is any step function then by linearity we see that

$$\int_G \tau_a \varphi \, d\mu = \int_G \varphi \, d\mu.$$

Let $\{\varphi_n\}$ be a sequence of step functions converging both $L^1$ and almost everywhere to some $f$ in $\mathscr{L}^1(\mu)$. Then $\{\tau_a \varphi_n\}$ converges almost every-

where to $\tau_a f$. On the other hand, $\{\tau_a \varphi_n\}$ is immediately seen to be $L^1$-Cauchy because we have remarked that the integral of $\tau_a \varphi$ is the same as the integral of $\varphi$ for any step function $\varphi$. The first assertion of our theorem follows at once. (*Note:* This is the same proof we gave for the invariance of Lebesgue measure on euclidean space.)

**Theorem 2.3.** *Let $\mu$ and $\nu$ be Haar measures on $G$. Let $d\mu$ and $d\nu$ be the functionals on $C_c(G)$ associated with $\mu$ and $\nu$. Then there exists a number $c > 0$ such that $d\nu = c \cdot d\mu$.*

*Proof.* By a previous remark, we may assume that $G$ is $\sigma$-compact, and hence $\sigma$-finite with respect to our Haar measures. We shall apply Fubini's theorem and refer the reader to Chapter IX, §6.

We shall first give a simpler proof when the Haar measure is also right invariant (which applies for instance when $G$ is commutative). Let $h \in C_c(G)$ be a positive function such that

$$\int_G h \, d\mu = 1.$$

We can find such $h$ by first selecting a non-empty open set $V$ with compact closure, a function $h_0$ such that $\bar{V} \prec h_0$, and then multiply $h_0$ by a suitable constant. (The symbol $\prec$ was defined in §2 of Chapter IX.) For an arbitrary $f \in C_c(G)$ we have:

$$\begin{aligned}
\int_G f \, d\nu = \int_G h \, d\mu \int_G f \, d\nu &= \int_G \int_G h(y) f(x) \, d\nu(x) \, d\mu(y) \\
&= \int_G \int_G h(y) f(xy) \, d\nu(x) \, d\mu(y) \\
&= \int_G \int_G h(y) f(xy) \, d\mu(y) \, d\nu(x) \\
&= \int_G \int_G h(x^{-1}y) f(y) \, d\mu(y) \, d\nu(x) \\
&= \int_G \int_G h(x^{-1}y) f(y) \, d\nu(x) \, d\mu(y) \\
&= c \int_G f \, d\mu
\end{aligned}$$

where

$$c = c(\nu) = \int_G h(x^{-1}) \, d\nu(x).$$

This proves our theorem in the present case. It also gives us an explicit determination of the constant $c$ involved in the statement of theorem.

The proof of uniqueness when the Haar measure is not also right invariant is slightly more involved, and runs as follows. For each nonzero positive function $f \in C_c(G)$, we consider the ratio of the integrals (taken over $G$):

$$r(f) = \frac{\int f \, d\mu}{\int f \, d\nu}.$$

It will suffice to show that this ratio is independent of $f$. We select a positive function $h = h_K$ with support equal to a compact neighborhood $K$ of the origin, such that $h$ is symmetric [i.e. $h(x) = h(x^{-1})$ for all $x$], and also

$$\int h \, d\nu = 1.$$

We can obviously satisfy these conditions with $K$ arbitrarily small, i.e. contained in a given open neighborhood of the origin. (To get the symmetry, use a product $h(x) = \psi(x)\psi(x^{-1})$ where $\psi$ has small support, and to normalize the integral, multiply by a suitable positive constant.) Now for any $f$ as above, we consider the difference

$$\int h \, d\mu \int f \, d\nu - \int h \, d\nu \int f \, d\mu = \int\int [h(x)f(y) - h(y)f(x)] \, d\mu(x) \, d\nu(y).$$

We change $x$ to $yx$ in the second term. We change $x$ to $y^{-1}x$ in the first term, and then replace $y^{-1}x$ by $x^{-1}y$ using the symmetry of $h$ to get $h(x^{-1}y)f(y)$ in the first term. We reverse the order of integration for the first term, change $y$ to $xy$ and change the order of integration once more. We then see that our difference is equal to

$$\int\int h(y)[f(xy) - f(yx)] \, d\mu(x) \, d\nu(y).$$

We now estimate this integral. If $K$ is small enough, then

$$|f(xy) - f(yx)| < \varepsilon$$

for all $x \in G$ and all $y \in K$. Furthermore if $y \in K$, the function

$$x \mapsto f(xy) - f(yx)$$

has support in the set $(\operatorname{supp} f)K^{-1} \cup K^{-1}(\operatorname{supp} f)$, which is compact, and whose $\mu$-measure is bounded by a fixed number $C_f$ depending only on $f$, as $K$ shrinks to the origin. Since $h$ is positive, we get the estimate (using the fact that $\int h\, d\nu = 1$):

$$\left|\int h\, d\mu \int f\, d\nu - \int h\, d\nu \int f\, d\mu\right| \leqq \varepsilon C_f \int h\, d\nu = \varepsilon C_f.$$

Dividing by $\int f\, d\nu$, we obtain

$$\lim_{K \to \{e\}} \int h_K\, d\mu = r(f),$$

with an obvious notation concerning the use of the limit symbol. The left-hand side is independent of $f$. This proves what we wanted, and concludes the proof of the uniqueness of Haar measure in general.

**Corollary 2.4.** *The map $\mu \mapsto d\mu$ is a bijection between the set of Haar measures on $G$ and the set of Haar functionals. If $\mu$, $\nu$ are Haar measures, then there exists $c > 0$ such that $\nu = c\mu$.*

*Proof.* Let $\lambda$ be a Haar functional. Let $\mu$ be the measure associated with $\lambda$ by Theorem 2.3 of Chapter IX. For any open set $V$ we have

$$\mu(V) = \sup \lambda f \qquad \text{for} \quad f \prec V,$$

and $f \prec V$ if and only if $f_a \prec aV$. Thus $\mu(aV) = \mu(V)$. For any Borel set $A$ we have

$$\mu(A) = \inf \mu(V) \qquad \text{for} \quad V \text{ open} \supset A,$$

and we conclude similarly that $\mu(aA) = \mu(A)$, so that $\mu$ is left invariant.

Let $V$ be a non-empty open set. Suppose that $\mu(V) = 0$. Any compact set $K$ can be covered by a finite number of translates of $V$, and consequently $\mu(K) = 0$ for all compact $K$. If $f \in C_c(G)$, $f \neq 0$, then $f/\|f\| \prec W$ for some open $W$ with compact closure. It follows that $0 \leqq \lambda f \leqq \mu(W)\|f\| = 0$, contradicting the non-triviality of $\lambda$. This proves that $\mu(V) > 0$, and hence that $\mu$ is a Haar measure. Thus the map $\mu \mapsto d\mu$ from Haar measures to Haar functionals is surjective. The map is injective by the Corollary of Theorem 2.7, Chapter IX. The last statement is now clear.

We are in a position to give examples of Haar measure and integrals. As is often the case, for any concretely given group, one can exhibit a specific Haar functional. The uniqueness theorem can then be used to guarantee that it is the only one possible.

**Examples.** (1) Let $G = \mathbf{R}^p$ be the additive group of euclidean space. Then Lebesgue measure is the Haar measure.

(2) Let $G = \mathbf{R}^*$ be the multiplicative group of non-zero real numbers. On $C_c(\mathbf{R}^*)$ we define a functional

$$f \mapsto \int_{-\infty}^{\infty} f(x) \frac{dx}{|x|}.$$

Thus we let $\mu^*$ be the measure such that $d\mu^*(x) = dx/|x|$. This is easily seen to be invariant under **multiplicative translations**. Namely, suppose that $a < 0$. We compute

$$\int_{-\infty}^{\infty} f(ax) \frac{dx}{|x|}.$$

We change variables, with $u = ax$, $du = a\,dx$. Then $|x| = |u|/|a|$, and

$$\frac{dx}{|x|} = -\frac{du}{|u|}.$$

But the limits of integration $\infty$ and $-\infty$ get reversed, and we conclude at once that our integral is equal to

$$\int_{-\infty}^{\infty} f(u) \frac{du}{|u|},$$

as desired. The case when $a > 0$ is even easier.

(3) Let $T$ be the circle group, i.e. the group of complex numbers of absolute value 1. The map $\lambda$ given by

$$\lambda f = \int_0^1 f(e^{2\pi i t})\, dt$$

is immediately seen to be a positive functional on $C_c(T)$, and also left invariant, so that it is a Haar functional.

(4) Let $G$ be a discrete group. The measure giving value 1 to a subset of $G$ consisting of one element is a Haar measure. What is its corresponding functional?

As a matter of notation, one usually writes $dx$ instead of $d\mu(x)$ for a Haar measure or its corresponding functional. For instance, in Example 1, we would say that $dx/|x|$ is a Haar measure on $\mathbf{R}^*$ if $dx$ is a Haar measure on $\mathbf{R}$.

Other examples will be given in the exercises.

As Weil pointed out in [W], most of classical Fourier analysis can be done on locally compact commutative groups. For this and other applications, besides [W], the reader can consult Loomis [Lo], Rudin [Ru 2] for commutative groups, and the collected works of Harish-Chandra for non-commutative groups.

For purposes of this book, the proof of existence given in the next section can be omitted.

## XII, §3. EXISTENCE OF THE HAAR INTEGRAL

*In this section we prove that a Haar functional exists.*

We let $L^+$ denote the set of all functions in $C_c(X, \mathbf{R}_{\geqq 0})$. Then $L^+$ is closed under addition and multiplication by real numbers $\geqq 0$. If $f$, $g \in L^+$ and if we assume that $g$ is not identically 0, then there exist numbers $c_i > 0$ and elements $s_i \in G$ $(i = 1,\dots,n)$ such that for all $x$ we have

$$f(x) \leqq \sum_{i=1}^{n} c_i g(s_i x).$$

For instance, we let $V$ be an open set and $m > 0$ such that $g \geqq m > 0$ on $V$. We can take all $c_i = \sup f/m$ and cover the support $K$ of $f$ by translates $s_1 V, \dots, s_n V$. We define

$$(f : g)$$

to be the inf of all sums $\sum c_i$ for all choices of $\{c_i\}$, $\{s_i\}$ satisfying the above inequality. If $g = 0$, we define $(f : g)$ to be $\infty$. The symbol $(f : g)$ satisfies the following properties. The first expresses an invariance under translation, where $f_a(x) = f(a^{-1}x)$.

(1) $$(f_a : g) = (f : g),$$

(2) $$(f_1 + f_2 : g) \leqq (f_1 : g) + (f_2 : g),$$

(3) $$(cf : g) = c(f : g) \text{ if } c \geqq 0.$$

(4) $$\text{If } f_1 \leqq f_2, \text{ then } (f_1 : g) \leqq (f_2 : g),$$

(5) $$(f : g) \leqq (f : h)(h : g),$$

(6) $$(f : g) \geqq \sup f/\sup g.$$

The first four properties are obvious. For (5), we note that if

$$f(x) \leqq \sum c_i h(s_i x) \qquad \text{and} \qquad h(x) \leqq \sum d_j g(t_j x),$$

then

$$f(x) \leqq \sum_{i,j} c_i d_j g(t_j s_i x).$$

For property (6), let $x$ be such that $f(x) = \sup f$. Then

$$\sup f \leqq \sum c_i g(s_i x) \leqq (\sum c_i) \sup g$$

whence (6) follows.

Let $h_0$ be a fixed non-zero function in $L^+$. We define

$$\lambda_g(f) = \frac{(f : g)}{(h_0 : g)}.$$

Then we have

$$\text{(7)} \qquad \frac{1}{(h_0 : f)} \leqq \lambda_g(f) \leqq (f : h_0).$$

For each fixed $g$, the map $\lambda_g$ will give an approximation of the Haar functional, which will be obtained below as a limit in a suitable sense.

We note that $\lambda_g$ is left invariant, and satisfies

$$\lambda_g(f_1 + f_2) \leqq \lambda_g(f_1) + \lambda_g(f_2), \qquad \lambda_g(cf) = c\lambda_g(f) \qquad c \geqq 0.$$

Furthermore, we shall now prove that if the support of $g$ is small, then $\lambda_g$ is almost additive.

**Lemma 3.1.** *Given $f_1, f_2 \in L^+$ and $\varepsilon$, there exists a neighborhood $V$ of $e$ such that*

$$\lambda_g(f_1) + \lambda_g(f_2) \leqq \lambda_g(f_1 + f_2) + \varepsilon$$

*for all $g \in L^+$ having support in $V$.*

*Proof.* Let $h$ be in $L^+$ and have the value 1 on the support of $f_1 + f_2$. Let

$$f = f_1 + f_2 + \delta h$$

with a number $\delta > 0$. Let

$$h_1 = f_1/f \qquad \text{and} \qquad h_2 = f_2/f.$$

We use the usual convention that $h_1$ and $h_2$ are 0 wherever $f$ is equal to 0. Then $h_1, h_2$ are in $L^+$, and are therefore uniformly continuous. Let $V$ be such that

$$|h_1(x) - h_1(y)| < \delta \qquad \text{and} \qquad |h_2(x) - h_2(y)| < \delta$$

whenever $y \in xV$. Let $g$ have support in $V$, and assume that $g \neq 0$. Let $c_1, \dots, c_n$ be positive numbers and $s_1, \dots, s_n \in G$ be such that

$$f(x) \leqq \sum c_i g(s_i x)$$

for all $x$. If $g(s_i x) \neq 0$, then $s_i x \in V$. We obtain

$$\begin{aligned} f_1(x) = f(x)h_1(x) &\leqq \sum c_i g(s_i x) h_1(x) \\ &\leqq \sum c_i g(s_i x)[h_1(s_i^{-1}) + \delta], \end{aligned}$$

and consequently

$$(f_1 : g) \leqq \sum c_i [h_1(s_i^{-1}) + \delta].$$

We have a similar inequality for $(f_2 : g)$. Since $h_1 + h_2 \leqq 1$, we obtain

$$(f_1 : g) + (f_2 : g) \leqq (\sum c_i)(1 + 2\delta).$$

Taking the inf over the families $\{c_i\}$, we find

$$\begin{aligned} (f_1 : g) + (f_2 : g) &\leqq (f : g)(1 + 2\delta) \\ &\leqq [(f_1 + f_2 : g) + \delta(h : g)](1 + 2\delta). \end{aligned}$$

Divide by $(h_0 : g)$. We obtain

$$\lambda_g(f_1) + \lambda_g(f_2) \leqq [\lambda_g(f_1 + f_2) + \delta\lambda_g(h)](1 + 2\delta)$$

But by (7), we know that $\lambda_g(f_1 + f_2)$ and $\lambda_g(h)$ are bounded from above by numbers depending only on $f_1$, $f_2$ (and $h$, which itself depends only on $f_1$, $f_2$). Hence for small $\delta$ we conclude the proof of the lemma.

For each non-zero $f \in L^+$ we let $I_f$ be the closed interval

$$\left[\frac{1}{(h_0 : f)}, (f : h_0)\right].$$

Let $I$ be the Cartesian product of all intervals $I_f$. Then $I$ is compact by Tychonoff's theorem. Each $g \in L^+$, $g \neq 0$ gives rise to a map $\lambda_g$, which is determined by its values $\lambda_g(f)$ for $f \in L^+$, $f \neq 0$. We may therefore view each $\lambda_g$ as a point of $I$. For each open neighborhood $V$ of $e$ in $G$ let $S_V$ be the closure of the set of all $\lambda_g$ with $g$ having support in $V$. Then $S_V$ is compact, and the collection of compact sets $S_V$ has the finite intersection property because

$$S_{V_1} \cap \cdots \cap S_{V_n} \supset S_{V_1 \cap \cdots \cap V_n}.$$

By compactness, there exists an element $\lambda$ in the intersection of all sets $S_V$. We contend that $\lambda$ is additive. This is immediate, because given $\varepsilon$ and any neighborhood $V$ of $e$, and $f_1$, $f_2$, $f_3 = f_1 + f_2$ we can find $g \in L^+$ having support in $V$ such that

$$|\lambda f_k - \lambda_g f_k| < \varepsilon.$$

Using Lemma 3.1, we conclude that $\lambda$ is additive.

If $f \in L$, we can write $f = f_1 - f_2$ with $f_1, f_2 \in L^+$. We define

$$\lambda f = \lambda f_1 - \lambda f_2.$$

The additivity of $\lambda$ shows that this is well defined, i.e. independent of the choice of $f_1$, $f_2$, and it is immediately verified that $\lambda$ is then linear on $L$. Furthermore, from the properties of $\lambda_g$, we also conclude that $\lambda$ is left invariant, and that for any $f \in L^+$ we have

$$\frac{1}{(h_0 : f)} \leqq \lambda f \leqq (f : h_0).$$

In particular, $\lambda$ is non-trivial. This concludes the proof of the existence of the Haar functional.

## XII, §4. MEASURES ON FACTOR GROUPS AND HOMOGENEOUS SPACES

Let $H$ be a closed subgroup of $G$ and let $dy$ be a Haar measure on $H$. Let $f \in C_c(G)$. Then the function

$$u \mapsto \int_H f(uy)\, dy$$

is continuous on $G$, as one verifies at once from the uniform continuity of $f$. Furthermore, it is constant on left cosets of $H$, because of the left invariance of the Haar measure on $H$, and has compact support on $G/H$. Thus if we write $u$ for elements of $G/H$, there exists a unique function $f^H \in C_c(G/H)$ such that

$$f^H(u) = \int_H f(uy)\, dy.$$

**Theorem 4.1.** *Let $H$ be a closed subgroup of $G$. The map*

$$f \mapsto f^H$$

*is a linear map of $C_c(G)$ onto $C_c(G/H)$. If $H$ is normal, and $du$ is a Haar measure on $G/H$, then the functional*

$$f \mapsto \int_{G/H} \int_H f(uy)\, dy\, du$$

*is a Haar functional on $G$.*

*Proof.* That the repeated integral is a positive functional and is left invariant is obvious. We must show that it is non-trivial. This will come from the first statement, valid even when $H$ is not normal, and easily proved as follows. Let $\pi: G \to G/H$ be the natural map. Let $f' \in C_c(G/H)$ and let $K'$ be the support of $f'$. We know from §1 that there exists a compact $K$ in $G$ such that $\pi(K) = K'$. Let $g \in C_c(X)$ be a positive function which is $> 0$ on $K$. Then $g^H$ will be $> 0$ on $K'$. Let

$$h(x) = \frac{f'(\pi(x))}{g^H(\pi(x))} \quad \text{if} \quad g^H(x) > 0,$$

$$h(x) = 0 \quad \text{if} \quad g^H(x) = 0.$$

Then $h$ is continuous on $G$, and is constant on cosets of $H$. Let $f = gh$. Then it is clear that $f^H = f'$, thus proving our first assertion, and the theorem.

When $H$ is not normal, we can still say something, and we cast it in a slightly more general context. Let $S$ be a locally compact Hausdorff space, and $G$ our group. We say that $G$ **operates** on $S$ if we are given a continuous map

$$G \times S \to S$$

satisfying the conditions

$$x(yu) = (xy)u \quad \text{and} \quad eu = u$$

for all $x, y \in G$, $u \in S$ and $e$ the unit element of $G$. Then for each $x \in G$ we have a homeomorphism $\tau_x: S \to S$ given by $\tau_x u = xu$.

The coset space of a closed subgroup $H$ is an example of the above, because for any coset $yH$ we can define $\tau_x(yH) = xyH$. The operation is obviously continuous. A **homogeneous space** is a space on which $G$ operates transitively. Such a space is the isomorphic to $G/H$ with some closed subgroup $H$.

Let $\lambda$ be a positive functional on $C_c(S)$. We shall say that $\lambda$ is **relatively invariant** (for the given operation of $G$) if for each $a \in G$ there exists

a number $\psi(a)$ such that for all $f \in C_c(S)$ we have

$$\lambda(\tau_a f) = \psi(a)\lambda(f).$$

As before, we define $\tau_a f(s) = f(a^{-1}s)$ for $a \in G$, $s \in S$. It follows immediately that $\psi: G \to \mathbf{R}^*$ is a continuous homomorphism into the multiplicative group of *positive* reals, called the **character** of the functional. We define a relatively invariant measure similarly.

As with Haar measures, we have

**Theorem 4.2.** *The map $\mu \mapsto d\mu$ is a bijection between the set of $\sigma$-regular positive relatively invariant measures on $S$ and the set of positive relatively invariant functionals on $C_c(S)$.*

In the case of the coset space, we then have the analog of Theorem 4.1.

**Theorem 4.3.** *Let $G$ be a locally compact group and $H$ a closed subgroup. Let $du$ be a relatively invariant non-zero positive functional on $C_c(G/H)$ with character $\psi$, and let $dy$ be a Haar measure on $H$. Then the map*

$$f \mapsto \int_{G/H} \int_H f(uy)\psi^{-1}(uy)\, dy\, du$$

*is a Haar functional on $G$.*

*Proof.* Our map is obviously linear, positive, and non-trivial because of the first statement of Theorem 4.1. There remains to show that it is left invariant. But

$$\int_{G/H} \int_H f(auy)\psi^{-1}(uy)\, dy\, du = \psi(a) \int_{G/H} \int_H f(auy)\psi^{-1}(auy)\, dy\, du.$$

The outer integral on the right is the integral of $(f\psi^{-1})^H$ translated by $a^{-1}$ and hence by the relative invariance of $du$ and the definition of the character, is the same as the integral of $(f\psi^{-1})^H$ times $\psi(a)^{-1}$. This yields what we wanted.

**Example 1.** In Exercise 4 you will show that the multiplicative group $\mathbf{C}^*$ of complex numbers is isomorphic to $\mathbf{R}^+ \times T$, where $T$ is the circle, and that Haar measure on $\mathbf{C}^*$ is given by $r^{-1}\, dr\, d\theta$ (in terms of the usual polar coordinates.

**Example 2.** Let $G = GL(n, \mathbf{R})$ be the group of invertible real $n \times n$ matrices. In Exercise 5 you will show that Haar measure on $G$ is given by $dx/|\det x|$ for $x \in G$.

**Example 3.** Let $G$ be the subgroup of $GL(2, \mathbf{R})$ consisting of all matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

In Exercise 9 you will show that right Haar measure on $G$ is the product measure on $\mathbf{R}^* \times \mathbf{R}$, and is not equal to left Haar measure.

**Example 4. SU(2).** This is the **special unitary group**, defined here to be the group of $2 \times 2$ complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{with} \quad \alpha\bar{\alpha} + \beta\bar{\beta} = 1.$$

Write $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$ with $a_i$, $b_i$ real. Then there is a bijection of $\mathbf{R}^4$ with the real vector space of matrices

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

and under this bijection, SU(2) corresponds to the sphere $\mathbf{S}^3$ defined by the equation

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1.$$

The reader can prove at once that euclidean measure on $\mathbf{R}^4$ given by

$$da_1 da_2 db_1 db_2$$

is invariant by the action of SU(2) on $\mathbf{R}^4$.

Define

$$r = (a_1^2 + a_2^2 + b_1^2 + b_2^2)^{1/2}.$$

and

$$u_1 = a_1/r, \qquad u_2 = a_2/r, \qquad \theta = \text{angle of } \beta.$$

Then we leave as an exercise to show that

$$da_1\, da_2\, db_1\, db_2 = r^3\, dr\, du_1\, du_2\, d\theta,$$

and $(u_1, u_2, \theta)$ define a $C^\infty$ isomorphism of $\mathbf{S}^3$ on SU(2). Finally, let

$f \in C_c(\mathrm{SU}(2))$, and let $F$ be the corresponding function in $C_c(\mathbf{S}^3)$, where $F = F(u_1, u_2, \theta)$ is a function of the above coordinates on $\mathbf{S}^3$. Then the Haar measure $\mu$ on SU(2) can be expressed in terms of the coordinates by the formula

$$\int_{\mathrm{SU}(2)} f\,d\mu = \frac{1}{2\pi^2}\int_{-1}^{1}\int_{-\sqrt{1-u_1^2}}^{\sqrt{1-u_1^2}}\int_0^{2\pi} F(u_1, u_2, \theta)\,d\theta\,du_2\,du_1.$$

We leave the verification as Exercise 12. This normalization gives the compact group SU(2) measure 1.

## XII, §5. EXERCISES

*We let $G$ be a locally compact Hausdorff group and $\mu$ a Haar measure.*

1. Identify $\mathbf{C}$ with $\mathbf{R}^2$. Let $\mu$ be Lebesgue (Haar) measure on $\mathbf{C}$. Let $\alpha \in \mathbf{C}$, and let $z$ denote the general element of $\mathbf{C}$. Show that $d\mu(\alpha z) = |\alpha|^2\,d\mu(z)$, as functionals on $C_c(\mathbf{C}) = C_c(\mathbf{R}^2)$.

2. Let $T$ be the circle group and let $H$ be a discrete abelian group which is not countable. Let $t$ be a fixed element of $T$ and let $\{x_i\}$ $(i \in I)$ be a non-countable subset of $H$. Let $S$ be the set of all pairs $(t, x_i)$, $i \in I$. Show that $S$ is discrete in $T \times H$, and that if $\mu$ is Haar measure on $T \times H$, then $S$ has infinite measure. Show that all compact subsets of $S$ have measure 0. Thus Haar measure is not regular.

3. Show that $G$ is compact if and only if $\mu(G)$ is finite. Show that $G$ is discrete if and only if the set consisting of $e$ alone has measure $> 0$.

4. Let $\mathbf{C}^*$ be the multiplicative group of complex numbers $\neq 0$. Let $\mathbf{R}^+$ denote the multiplicative group of real numbers $> 0$. Show that $\mathbf{C}^*$ is isomorphic to $\mathbf{R}^+ \times T$ (where $T$ is the circle group) under the map $(r, u) \mapsto ru$. We write $u = e^{2\pi i\theta}$. Show that the Haar integral on $\mathbf{C}^*$ is given by

$$f \mapsto \int_0^1 \int_0^\infty f(re^{2\pi i\theta})\frac{dr}{r}\,d\theta.$$

5. Let $G = GL(n, \mathbf{R})$ be the group of real $n \times n$ matrices. Show that Haar measure on $G$ is given by $dx/|\det x|$, if $dx$ is a Haar measure on the $n^2$-dimensional space of all $n \times n$ matrices. (Use the change of variables formula.)

6. Let $f \in C_c(G)$ and let $a \in G$. We denote by $r_a f$ or $f^a$ the **right translate** of $f$, defined by

$$f^a(x) = f(xa^{-1}).$$

For each fixed $a$, show that there exists a number $\Delta(a)$ such that for all

$f \in C_c(G)$ we have

$$\int_G f(xa^{-1})\,dx = \Delta(a)\int_G f(x)\,dx.$$

Show that $\Delta(ab) = \Delta(a)\Delta(b)$, and that $\Delta$ is continuous, as a map of $G$ into $\mathbf{R}^*$. We call $\Delta$ the **modular function** on $G$.

7. If $\Delta$ is the modular function on $G$, show that

$$\int_G f(x^{-1})\Delta(x^{-1})\,dx = \int_G f(x)\,dx,$$

where $dx$ is Haar measure, and that $\Delta(x^{-1})\,dx$ is right Haar measure.

8. If $G$ is compact and $\psi\colon G \to \mathbf{R}^+$ is a continuous homomorphism into $\mathbf{R}^+$, show that $\psi$ is trivial, i.e. $\psi(G) = 1$. In this case, Haar measure is also right invariant.

9. Compute the modular function for the group $G$ of all affine maps $x \mapsto ax + b$ with $a \in \mathbf{R}^*$ and $b \in \mathbf{R}$. In fact, show that $\Delta(a, b) = a$. In this case the right Haar measure is not equal to the left Haar measure. Show that the right Haar measure is the Cartesian product measure on $\mathbf{R}^* \times \mathbf{R}$.

10. Let $G$ be a locally compact group with Haar measure, let $\Delta$ be the modular function as in Exercise 7. Let $M^1$ be the set of regular complex measures on $G$.
 (a) Just as in Exercise 8 of Chapter IX, prove that $M^1$ is a Banach space, if we define the convolution $m * m'$ to be the measure associated with the functional

$$f \mapsto \iint f(xy)\,dm(x)\,dm'(y),$$

 for $f \in C_c(G)$.
 (b) For $m \in M^1$, define $m^\vee$ to be the direct image of $m$ under the mapping $x \mapsto x^{-1}$ of $G$. Show that $\|m^\vee\| = \|m\|$.
 (c) If $\mu$ is a right measure, prove that $\mu^\vee = \Delta\mu$.

11. Let $G$ be a compact abelian group. By a **character** of $G$ we mean a continuous homomorphism $\psi\colon G \to \mathbf{C}^*$ into the multiplicative group of non-zero complex numbers.
 (a) Show that the values of $\psi$ lie on the unit circle.
 (b) If $G = \mathbf{R}^n/\mathbf{Z}^n$ is an $n$-torus, show that the characters separate points. Assume this for the general case.
 (c) Let $\sigma\colon G \to G$ be a topological and algebraic automorphism of $G$, or an automorphism for short. Show that $\sigma$ preserves Haar measure, and induces a norm-preserving linear map $T\colon L^2(\mu) \to L^2(\mu)$ by $f \mapsto f \circ \sigma$.
 (d) If $\psi$ is a non-trivial character on $G$, show that $\int \psi\,d\mu = 0$. If $\psi$ is trivial, that is $\psi(G) = 1$, then $\int \psi\,d\mu = 1$, assuming that $\mu(G) = 1$, which we do. Prove that the characters generate an algebra which is dense for the sup norm in the algebra of continuous functions. Prove that the characters form a Hilbert basis for $L^2$.

(e) Let $f$ be an eigenvector for $T$ in $L^2$, with eigenvalue $\alpha$, that is $Tf = \alpha f$. Show that $\alpha$ is a root of unity. [*Hint:* First, $|\alpha| = 1$. Then write the Fourier expansion $F = \sum c_\psi \psi$ in $L^2$, and observe that $c_{T\psi} = \alpha^{-1} c_\psi$. Use the fact that $\sum |c_\psi|^2 < \infty$ whence if $c_\psi \neq 0$, then for some $n$, $T^n\psi = \psi$ and $\alpha^n = 1$.]

12. Prove the statements made in Example 4, concerning Haar measure on SU(2). You may use the following trick. Let $h \in C_c(\mathbf{R}^+)$ be such that

$$\int_0^\infty h(r) r^3 \, dr = 1.$$

Show that the functional

$$\begin{aligned} \lambda(F) &= \int_{\mathbf{R}^4} F(a_1/r, a_2/r, \theta) h(r) \, da_1 \, da_2 \, db_1 \, db_2 \\ &= \int_{-1}^{1} \int_{-\sqrt{1-u_1^2}}^{\sqrt{1-u_1^2}} \int_0^{2\pi} F(u_1, u_2, \theta) \, d\theta \, du_2 \, du_1 \end{aligned}$$

defines a functional on $C(\mathbf{S}^3)$ which is invariant under the action of SU(2).

# PART FOUR

# Calculus

The differential calculus is an essential tool, and some of it will be used in some of the later applications. Readers can bypass it until they come to a place where it is used. The differential calculus in Banach spaces was developed quite a while ago, in the 1920s, by Fréchet, Graves, Hildebrandt, and Michael. Its recent return to fashion, after a period during which it was somewhat forgotten, is due to increasingly fruitful applications to function spaces in various contexts of analysis and geometry.

For instance, at the end of Chapter XIV, in the exercises, we show how the calculus in Banach spaces can be used by describing the start of the calculus of variations.

In functional analysis, one considers at the very least functions from scalars into Banach or Hilbert spaces, giving rise to curves (real or complex) in such spaces.

We shall also use the elementary integral of Banach-valued continuous functions in the spectral theory of Chapter XVI, §1, in connection with Banach-valued analytic functions.

The Morse–Palais lemma of Chapter XVIII gives a nice illustration of the second derivative of a map in Hilbert space.

In the part on global analysis, readers will see the calculus used in dealing with finite dimensional manifolds, especially for integration theory, as in Stokes' theorem. For applications to infinite dimensional manifolds, see [La 2].

As a matter of exposition, we prefer to develop *ab ovo* the integral of step maps from an interval into a Banach space, and its extension to the uniform closure of the space of step maps, which can be done much more easily than the construction of the $L^1$ general integral.

CHAPTER XIII

# Differential Calculus

*Throughout this chapter and the next, we let E, F, G denote Banach spaces.*

## XIII, §1. INTEGRATION IN ONE VARIABLE

Let $[a, b]$ be a closed interval, and $E$ a Banach space. By a **step map** $f: [a, b] \to E$ we mean a map for which there exists a partition

$$P: a = a_0 \leqq a_1 \leqq \cdots \leqq a_n = b$$

and elements $v_1, \ldots, v_n \in E$ such that if $a_{i-1} < t < a_i$, then $f(t) = v_i$. We then say that $f$ is step with respect to $P$. The notion of a refinement of a partition is the usual one, and if $f$, $g$ are two step maps of $[a, b]$ into $E$, then there exists a partition $P$ such that both $f$, $g$ are step with respect to $P$. From this we see that the step maps form a subspace of the space of all bounded maps, and we deal with the sup norm on this space.

We define the **integral of a step map** $f$ **with respect to a partition** $P$ by

$$I_P(f) = \sum_{i=1}^{n} (a_i - a_{i-1}) v_i,$$

the notation being as above. This is in fact independent of $P$, and we write simply $I(f)$ or $I_a^b(f)$ to specify the interval $[a, b]$. It is then easily seen that $I$ is linear, and that $|I(f)| \leqq (b - a)\|f\|$, so $I$ is continuous, with bound $b - a$. We can therefore extend $I$ to the closure of the space of step maps by the linear extension theorem. If $f$ lies in this closure, we

denote $I(f)$ from now on by

$$\int_a^b f$$

and call it the **integral**. If $a \leqq c \leqq b$, then one verifies without difficulty that

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{1}$$

If $a \leqq c < d \leqq b$, we define

$$\int_d^c f = -\int_c^d f.$$

Then formula (1) actually holds for any three points $a$, $b$, $c$ in any order, lying in an interval on which $f$ is in the closure of the space of step maps.

Since a continuous map is uniformly continuous on a compact set, one concludes that the continuous maps of $[a, b]$ into $E$ lie in the closure of the space of step maps, so that the integral is defined over continuous maps.

If $E = E_1 \times \cdots \times E_n$ is a product of Banach spaces, and

$$f = (f_1, \ldots, f_n)$$

is represented by coordinate maps $f_i\colon [a, b] \to E_i$, then it is trivially verified that

$$\int_a^b f = \left(\int_a^b f_1, \ldots, \int_a^b f_n\right).$$

If $E = \mathbf{R}$, and $f \geqq 0$, then

$$\int_a^b f \geqq 0$$

as one sees first for step maps, and then by continuity for uniform limits of step maps.

For convenience, the closure of the space of step maps will be called the space of **regulated** maps. Thus a map is called regulated if it is a uniform limit of step maps.

**Proposition 1.1.** *Let $\lambda: E \to F$ be a continuous linear map. If $f: [a, b] \to E$ is regulated, then $\lambda \circ f$ is regulated and*

$$\int_a^b \lambda \circ f = \lambda\left(\int_a^b f\right).$$

This follows immediately from the definitions. Indeed, if $f$ is the uniform limit of a sequence of step maps $\{f_n\}$, then each $\lambda \circ f_n$ is a step map of $[a, b]$ into $F$, which clearly converges to $\lambda \circ f$. For a step map $f_n$ we have directly from the definition that

$$\int_a^b \lambda \circ f_n = \lambda\left(\int_a^b f_n\right).$$

Taking the limit proves our formula.

## XIII, §2. THE DERIVATIVE AS A LINEAR MAP

Let $U$ be open in $E$, and let $x \in U$. Let $f: U \to F$ be a map. We shall say that $f$ is **differentiable** at $x$ if there exists a continuous linear map $\lambda: E \to F$ and a map $\psi$ defined for all sufficiently small $h$ in $E$, with values in $F$, such that

$$\lim_{h \to 0} \psi(h) = 0,$$

and such that

$$(*) \qquad f(x + h) = f(x) + \lambda(h) + |h|\psi(h).$$

Setting $h = 0$ shows that we may assume that $\psi$ is defined at 0 and that $\psi(0) = 0$. The preceding formula still holds.

Equivalently, we could replace the term $|h|\psi(h)$ by a term $\varphi(h)$ where $\varphi$ is a map such that

$$\lim_{h \to 0} \frac{\varphi(h)}{|h|} = 0.$$

The limit is taken of course for $h \neq 0$, otherwise the quotient does not make sense.

A mapping $\varphi$ having the preceding limiting property is said to be $o(h)$ for $h \to 0$.

We view the definition of the derivative as stating that near $x$, the values of $f$ can be approximated by a linear map $\lambda$, except for the

additive term $f(x)$, of course, with an error term described by the limiting properties of $\psi$ or $\varphi$ described above.

It is clear that if $f$ is differentiable at $x$, then it is continuous at $x$.

We contend that if the continuous linear map $\lambda$ exists satisfying $(*)$, then it is uniquely determined by $f$ and $x$. To prove this, let $\lambda_1, \lambda_2$ be continuous linear maps having property $(*)$. Let $v \in E$. Let $t$ have real values $> 0$ and so small that $x + tv$ lies in $U$. Let $h = tv$. We have

$$\begin{aligned} f(x+h) - f(x) &= \lambda_1(h) + |h|\psi_1(h) \\ &= \lambda_2(h) + |h|\psi_2(h) \end{aligned}$$

with

$$\lim_{h \to 0} \psi_j(h) = 0$$

for $j = 1, 2$. Let $\lambda = \lambda_1 - \lambda_2$. Subtracting the two expressions for

$$f(x + tv) - f(x),$$

we find

$$\lambda_1(h) - \lambda_2(h) = |h|\big(\psi_2(h) - \psi_1(h)\big),$$

and setting $h = tv$, using the linearity of $\lambda$,

$$t\big(\lambda_1(v) - \lambda_2(v)\big) = t|v|\big(\psi_2(tv) - \psi_1(tv)\big).$$

We divide by $t$ and find

$$\lambda_1(v) - \lambda_2(v) = |v|\big(\psi_2(tv) - \psi_1(tv)\big).$$

Take the limit as $t \to 0$. The limit of the right side is equal to 0. Hence $\lambda_1(v) - \lambda_2(v) = 0$ and $\lambda_1(v) = \lambda_2(v)$. This is true for every $v \in E$, whence $\lambda_1 = \lambda_2$, as was to be shown.

In view of the uniqueness of the continuous linear map $\lambda$, we call it the **derivative of** $f$ **at** $x$ and denote it by $f'(x)$ or $Df(x)$. Thus $f'(x)$ is a continuous linear map, and we can write

$$f(x+h) - f(x) = f'(x)h + |h|\psi(h)$$

with

$$\lim_{h \to 0} \psi(h) = 0.$$

We have written $f'(x)h$ instead of $f'(x)(h)$ for simplicity, omitting a set of parentheses. In general we shall often write

$$\lambda h$$

instead of $\lambda(h)$ when $\lambda$ is a linear map.

If $f$ is differentiable at every point $x$ of $U$, then we say that $f$ is **differentiable on** $U$. In that case, the derivative $f'$ is a map

$$Df = f' : U \to L(E, F)$$

from $U$ into the space of continuous linear maps $L(E, F)$, and thus to each $x \in U$, we have associated the linear map $f'(x) \in L(E, F)$. If $f'$ is continuous, we say that $f$ is of class $C^1$. Since $f'$ maps $U$ into the Banach space $L(E, F)$, we can define inductively $f$ to be of class $C^p$ if all derivatives $D^k f$ exist and are continuous for $1 \leqq k \leqq p$.

If $f : [a, b] \to F$ is a map of a real variable, then its derivative

$$f'(t) : \mathbf{R} \to F$$

is a linear map into the vector space $F$. However, if $\lambda : \mathbf{R} \to F$ is any linear map, then for all $t \in \mathbf{R}$ we have

$$\lambda(t) = \lambda(t \cdot 1) = t\lambda(1).$$

Hence $\lambda$ is multiplication (on the right) by the vector $\lambda(1)$ in $F$, and we usually may identify $\lambda$ with this vector.

## XIII, §3. PROPERTIES OF THE DERIVATIVE

**Sum.** *Let $E$, $F$ be complete normed vector spaces, and let $U$ be open in $E$. Let $f, g : U \to F$ be maps which are differentiable at $x \in U$. Then $f + g$ is differentiable at $x$ and*

$$(f + g)'(x) = f'(x) + g'(x).$$

*If $c$ is a number, then*

$$(cf)'(x) = cf'(x).$$

*Proof.* Let $\lambda_1 = f'(x)$ and $\lambda_2 = g'(x)$ so that

$$f(x + h) - f(x) = \lambda_1 h + |h|\psi_1(h),$$
$$g(x + h) - g(x) = \lambda_2 h + |h|\psi_2(h),$$

where $\lim_{h \to 0} \psi_i(h) = 0$. Then

$$\begin{aligned}(f + g)(x + h) - (f + g)(x) &= f(x + h) + g(x + h) - f(x) - g(x) \\ &= \lambda_1 h + \lambda_2 h + |h|\big(\psi_1(h) + \psi_2(h)\big) \\ &= (\lambda_1 + \lambda_2)(h) + |h|\big(\psi_1(h) + \psi_2(h)\big).\end{aligned}$$

Since $\lim_{h \to 0} (\psi_1(h) + \psi_2(h)) = 0$, it follows by definition that

$$\lambda_1 + \lambda_2 = (f + g)'(x),$$

as was to be shown. The statement with the constant is equally clear.

**Product.** *Let $F_1$, $F_2$, $G$ be complete normed vector spaces, and let $F_1 \times F_2 \to G$ be a continuous bilinear map. Let $U$ be open in $E$ and let $f\colon U \to F_1$ and $g\colon U \to F_2$ be maps differentiable at $x \in U$. Then the product map $fg$ is differentiable at $x$ and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Before giving the proof, we make some comments on the meaning of the product formula. The linear map represented by the right-hand side is supposed to mean the map

$$v \mapsto (f'(x)v)g(x) + f(x)(g'(x)v).$$

Note that $f'(x)\colon E \to F_1$ is a linear map of $E$ into $F_1$, and when applied to $v \in E$ yields an element of $F_1$. Furthermore, $g(x)$ lies in $F_2$, and so we can take the product

$$(f'(x)v)g(x) \in G.$$

Similarly for $f(x)(g'(x)v)$. In practice we omit the extra set of parentheses, and write simply

$$f'(x)vg(x).$$

*Proof.* Changing the norm on $G$ if necessary, we may assume that

$$|vw| \leqq |v|\,|w|$$

*for* $v \in F_1$, $w \in F_2$.

We have:

$$\begin{aligned}
&f(x + h)g(x + h) - f(x)g(x) \\
&\quad = f(x + h)g(x + h) - f(x + h)g(x) + f(x + h)g(x) - f(x)g(x) \\
&\quad = f(x + h)(g(x + h) - g(x)) + (f(x + h) - f(x))g(x) \\
&\quad = f(x + h)(g'(x)h + |h|\psi_2(h)) + (f'(x)h + |h|\psi_1(h))g(x) \\
&\quad = f(x + h)g'(x)h + |h|f(x + h)\psi_2(h) + f'(x)hg(x) + |h|\psi_1(h)g(x) \\
&\quad = f(x)g'(x)h + f'(x)hg(x) + (f(x + h) - f(x))g'(x)h \\
&\qquad\qquad + |h|f(x + h)\psi_2(h) + |h|\psi_1(h)g(x).
\end{aligned}$$

The map

$$h \mapsto f(x)g'(x)h + f'(x)hg(x)$$

is the linear map of $E$ into $G$, which is supposed to be the desired derivative. It remains to be shown that each of the other three terms appearing on the right are of the desired type, namely $o(h)$. This is immediate. For instance,

$$|(f(x+h) - f(x))g'(x)h| \leqq |f(x+h) - f(x)|\,|g'(x)|\,|h|$$

and

$$\lim_{h \to 0} |f(x+h) - f(x)|\,|g'(x)| = 0$$

because $f$ is continuous, being differentiable. The others are equally obvious, and our property is proved.

**Quotient.** *Assume that $A$ is a Banach algebra with unit $e$, and let $U$ be the open set of invertible elements. Then the map $u \mapsto u^{-1}$ is differentiable on $U$, and its derivative at a point $u_0$ is given by*

$$v \mapsto -u_0^{-1}vu_0^{-1}.$$

*Proof.* We have

$$\begin{aligned}(u_0 + h)^{-1} - u_0^{-1} &= (u_0(e + u_0^{-1}h))^{-1} - u_0^{-1}\\ &= (e + u_0^{-1}h)^{-1}u_0^{-1} - u_0^{-1}\\ &= (e - u_0^{-1}h + o(h))u_0^{-1} - u_0^{-1}\\ &= -u_0^{-1}hu_0^{-1} + o(h).\end{aligned}$$

This proves that the derivative is what we said it is.

**Chain Rule.** *Let $U$ be open in $E$ and let $V$ be open in $F$. Let $f\colon U \to V$ and $g\colon V \to G$ be maps. Let $x \in U$. Assume that $f$ is differentiable at $x$ and $g$ is differentiable at $f(x)$. Then $g \circ f$ is differentiable at $x$ and*

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x).$$

Before giving the proof, we make explicit the meaning of the usual formula. Note that $f'(x)\colon E \to F$ is a liner map, and $g'(f(x))\colon F \to G$ is a linear map, and so these linear maps can be composed, and the composite is a linear map, which is continuous because both $g'(f(x))$ and $f'(x)$ are continuous. The composed linear map goes from $E$ into $G$, as it should.

*Proof.* Let $k(h) = f(x + h) - f(x)$. Then

$$g(f(x + h)) - g(f(x)) = g'(f(x))k(h) + |k(h)|\psi_1(k(h))$$

with $\lim_{k \to 0} \psi_1(k) = 0$. But

$$k(h) = f(x + h) - f(x) = f'(x)h + |h|\psi_2(h),$$

with $\lim_{h \to 0} \psi_2(h) = 0$. Hence

$$\begin{aligned} &g(f(x + h)) - g(f(x)) \\ &\qquad = g'(f(x))f'(x)h + |h|g'(f(x))\psi_2(h) + |k(h)|\psi_1(k(h)). \end{aligned}$$

The first term has the desired shape, and all we need to show is that each of the next two terms on the right is $o(h)$. This is obvious. For instance, we have the estimate

$$|k(h)| \leqq |f'(x)||h| + |h||\psi_2(h)|$$

and

$$\lim_{h \to 0} \psi_1(k(h)) = 0$$

from which we see that $|k(h)|\psi_1(k(h)) = o(h)$. We argue similarly for the other term.

**Map with Coordinates.** *Let $U$ be open in $E$, let*

$$f\colon U \to F_1 \times \cdots \times F_m,$$

*and let $f = (f_1, \ldots, f_m)$ be its expression in terms of coordinate maps. Then $f$ is differentiable at $x$ if and only if each $f_i$ is differentiable at $x$, and if this is the case, then*

$$f'(x) = (f_1'(x), \ldots, f_m'(x)).$$

*Proof.* This follows as usual by considering the coordinate expression

$$f(x + h) - f(x) = (f_1(x + h) - f(x), \ldots, f_m(x + h) - f_m(x)).$$

Assume that $f_i'(x)$ exists, so that

$$f_i(x + h) - f_i(x) = f_i'(x)h + \varphi_i(h)$$

where $\varphi_i(h) = o(h)$. Then

$$f(x + h) - f(x) = \big(f_1'(x)h, \ldots, f_m'(x)h\big) + \big(\varphi_1(h), \ldots, \varphi_m(h)\big)$$

and it is clear that this last term in $F_1 \times \cdots \times F_m$ is $o(h)$. (As always, we use the sup norm in $F_1 \times \cdots \times F_m$.) This proves that $f'(x)$ is what we said it is. The converse is equally easy and is left to the reader.

**Theorem 3.1.** *Let $\lambda\colon E \to F$ be a continuous linear map. Then $\lambda$ is differentiable at every point of $E$ and $\lambda'(x) = \lambda$ for every $x \in E$.*

*Proof.* This is obvious, because

$$\lambda(x + h) - \lambda(x) = \lambda(h) + 0.$$

Note therefore that the derivative of $\lambda$ is *constant* on $E$.

**Corollary 3.2.** *Let $f\colon U \to F$ be a differentiable map, and let $\lambda\colon F \to G$ be a continuous linear map. Then for each $x \in U$,*

$$(\lambda \circ f)'(x) = \lambda\big(f'(x)\big),$$

*so that for every $v \in E$ we have*

$$(\lambda \circ f)'(x)v = \lambda\big(f'(x)v\big).$$

*Proof.* This follows from Theorem 3.1 and the chain rule. Of course, one can also give a direct proof, considering

$$\begin{aligned}\lambda\big(f(x + h)\big) - \lambda\big(f(x)\big) &= \lambda\big(f(x + h) - f(x)\big)\\ &= \lambda\big(f'(x)h + |h|\psi(h)\big)\\ &= \lambda\big(f'(x)h\big) + |h|\lambda\big(\psi(h)\big),\end{aligned}$$

and noting that $\lim_{n \to 0} \big(\psi(h)\big) = 0$.

**Lemma 3.3.** *If $f$ is a differentiable map on an interval $[a, b]$ whose derivative is 0, then $f$ is constant.*

*Proof.* Suppose that $f(t) \neq f(a)$ for some $t \in [a, b]$. By the Hahn–Banach theorem, let $\lambda$ be a functional such that $\lambda\big(f(t)\big) \neq \lambda\big(f(a)\big)$. The map $\lambda \circ f$ is differentiable, and its derivative is equal to 0. Hence $\lambda \circ f$ is constant on $[a, b]$, contradiction.

**Fundamental Theorem of Calculus.** *Let $f$ be regulated on $[a, b]$, and assume that $f$ is continuous at a point $c$ of $[a, b]$. Then the map*

$$t \mapsto \int_a^t f = \varphi(t)$$

*is differentiable at $c$ and its derivative is $f(c)$.*

*Proof.* The standard proof works, namely

$$\varphi(c + h) - \varphi(c) = \int_c^{c+h} f$$

and

$$\varphi(c + h) - \varphi(c) - hf(c) = \int_c^{c+h} (f - f(c)).$$

The right-hand side is estimated by

$$|h| \sup |f(t) - f(c)|$$

for $t$ between $c$ and $c + h$, thus proving that the derivative is $f(c)$.

In particular,

$$f(b) - f(a) = \int_a^b f'(t)\, dt.$$

## XIII, §4. MEAN VALUE THEOREM

The mean value theorem essentially relates the values of a map at two different points by means of the intermediate values of the map on the line segment between these two points. In vector spaces, we give an integral form for it.

We shall be integrating curves in the space of continuous linear maps $L(E, F)$.

We shall also deal with the association

$$L(E, F) \times E \to F$$

given by

$$(\lambda, y) \mapsto \lambda(y)$$

for $\lambda \in L(E, F)$ and $y \in E$. It is a continuous bilinear map.

Let $\alpha: J \to L(E, F)$ be a continuous map from a closed interval $J = [a, b]$ into $L(E, F)$. For each $t \in J$, we see that $\alpha(t) \in L(E, F)$ is a linear

map. We can apply it to an element $y \in E$ and $\alpha(t)y \in F$. On the other hand, we can integrate the curve $\alpha$, and

$$\int_a^b \alpha(t)\,dt$$

is an element of $L(E, F)$. If $\alpha$ is differentiable, then $d\alpha(t)/dt$ is also an element of $L(E, F)$.

**Lemma 4.1.** *Let $\alpha\colon J \to L(E, F)$ be a continuous map from a closed interval $J = [a, b]$ into $L(E, F)$. Let $y \in E$. Then*

$$\int_a^b \alpha(t)y\,dt = \int_a^b \alpha(t)\,dt \cdot y$$

*where the dot on the right means the application ofthe linear map*

$$\int_a^b \alpha(t)\,dt$$

*to the vector $y$.*

*Proof.* Here $y$ is fixed, and the map

$$\lambda \mapsto \lambda(y) = \lambda y$$

is a continuous linear map of $L(E, F)$ into $F$. Hence our lemma follows from the last property of the integral proved in §1.

**Theorem 4.2.** *Let $U$ be open in $E$ and let $x \in U$. Let $y \in E$. Let $f\colon U \to F$ be a $C^1$ map. Assume that the line segment $x + ty$ with $0 \leqq t \leqq 1$ is contained in $U$. Then*

$$f(x + y) - f(x) = \int_0^1 f'(x + ty)y\,dt = \int_0^1 f'(x + ty)\,dt \cdot y.$$

*Proof.* Let $g(t) = f(x + ty)$. Then $g'(t) = f'(x + ty)y$. By the fundamental theorem of calculus we find that

$$g(1) - g(0) = \int_0^1 g'(t)\,dt.$$

But $g(1) = f(x + y)$ and $g(0) = f(x)$. Our theorem is proved, taking into account the lemma which allows us to pull the $y$ out of the integral.

**Corollary 4.3.** *Let $U$ be open in $E$ and let $x$, $z \in U$ be such that the line segment between $x$ and $z$ is contained in $U$ (that is the segment $x + t(z - x)$ with $0 \leqq t \leqq 1$). Let $f\colon U \to F$ be of class $C^1$. Then*

$$|f(z) - f(x)| \leqq |z - x| \sup |f'(v)|,$$

*the* sup *being taken for all $v$ in the segment.*

*Proof.* We estimate the integral, letting $x + y = z$. We find

$$\left| \int_0^1 f'(x + ty)y \, dt \right| \leqq (1 - 0) \sup |f'(x + ty)| \, |y|,$$

the sup being taken for $0 \leqq t \leqq 1$. Our corollary follows.

(*Note.* The sup of the norms of the derivative exists because the segment is compact and the map $t \mapsto |f'(x + ty)|$ is continuous.)

**Corollary 4.4.** *Let $U$ be open in $E$ and let $x$, $z$, $x_0 \in U$. Assume that the segment between $x$ and $z$ lies in $U$. Then*

$$|f(z) - f(x) - f'(x_0)(z - x)| \leqq |z - x| \sup |f'(v) - f'(x_0)|,$$

*the* sup *being taken for all $v$ on the segment between $x$ and $z$.*

*Proof.* We can either apply Corollary 4.3 to the map $g$ such that $g(x) = f(x) - f'(x_0)x$, or argue directly with the integral:

$$f(z) - f(x) = \int_0^1 f'\big(x + t(z - x)\big)(z - x) \, dt.$$

We write

$$f'\big(x + t(z - x)\big) = f'\big(x + t(z - x)\big) - f'(x_0) + f'(x_0),$$

and find

$$f(z) - f(x) = f'(x_0)(z - x) + \int_0^1 \big[f'\big(x + t(z - x)\big) - f'(x_0)\big](z - x) \, dt.$$

We then estimate the integral on the right as usual.

We shall call Theorem 4.2 or either one of its two corollaries the **mean value theorem** in vector spaces. In practice, the integral form of the remainder is always preferable and should be used as a conditioned

reflex. One big advantage it has over the others is that the integral, as a function of $y$, is just as smooth as $f'$, and this is important in some applications. In others, one needs only an intermediate value estimate, and then Corollary 4.3, or especially Corollary 4.4, may suffice.

## XIII, §5. THE SECOND DERIVATIVE

Let $U$ be open in $E$ and let $f\colon U \to F$ be differentiable. Then

$$Df = f'\colon U \to L(E, F)$$

and we know that $L(E, F)$ is again a complete normed vector space. Thus we are in a position to define the second derivative

$$D^2 f = f^{(2)}\colon U \to L\big(E, L(E, F)\big).$$

We have seen in Chapter IV, §1 that we can identify $L\big(E, L(E, F)\big)$ with $L(E, E; F)$, which we denote by $L^2(E, F)$, i.e. the space of continuous bilinear maps of $E$ into $F$.

**Theorem 5.1.** *Let $\omega\colon E_1 \times E_2 \to F$ be a continuous bilinear map. Then $\omega$ is differentiable, and for each $(x_1, x_2) \in E_1 \times E_2$ and every*

$$(v_1, v_2) \in E_1 \times E_2$$

*we have*

$$D\omega(x_1, x_2)(v_1, v_2) = \omega(x_1, v_2) + \omega(v_1, x_2),$$

*so that $D\omega\colon E_1 \times E_2 \to L(E_1 \times E_2, F)$ is linear. Hence $D^2\omega$ is constant, and $D^3\omega = 0$.*

*Proof.* We have by definition

$$\omega(x_1 + h_1, x_2 + h_2) - \omega(x_1, x_2) = \omega(x_1, h_2) + \omega(h_1, x_2) + \omega(h_1, h_2).$$

This proves the first assertion, and also the second, since each term on the right is linear in both $(x_1, x_2) = x$ and $h = (h_1, h_2)$. We know that the derivative of a linear map is constant, and the derivative of a constant map is 0, so the rest is obvious.

We consider especially a bilinear map

$$\lambda\colon E \times E \to F$$

and say that $\lambda$ is symmetric if we have

$$\lambda(v, w) = \lambda(w, v)$$

for all $v, w \in E$. In general, a multilinear map

$$\lambda\colon E \times \cdots \times E \to F$$

is said to be **symmetric** if

$$\lambda(v_1, \ldots, v_n) = \lambda(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

for any permutation $\sigma$ of the indices $1, \ldots, n$. In this section we look at the symmetric bilinear case in connection with the second derivative.

We see that we may view a second derivative $D^2f(x)$ as a continuous bilinear map. Our next theorem will be that this map is symmetric. We need a lemma.

**Lemma 5.2.** *Let $\lambda\colon E \times E \to F$ be a bilinear map, and assume that there exists a map $\psi$ defined for all sufficiently small pairs $(v, w) \in E \times E$ with values in $F$ such that*

$$\lim_{(v,w)\to(0,0)} \psi(v, w) = 0,$$

*and that*

$$|\lambda(v, w)| \leqq |\psi(v, w)|\,|v|\,|w|.$$

*Then $\lambda = 0$.*

*Proof.* This is like the argument which gave us the uniqueness of the derivative. Take $v, w \in E$ arbitrary, and let $s$ be a positive real number sufficiently small so that $\psi(sv, sw)$ is defined. Then

$$|\lambda(sv, sw)| \leqq |(sv, sw)|\,|sv|\,|sw|,$$

whence

$$s^2|\lambda(v, w)| \leqq s^2|\psi(sv, sw)|\,|v|\,|w|.$$

Divide by $s^2$ and let $s \to 0$. We conclude that $\lambda(v, w) = 0$, as desired.

**Theorem 5.3.** *Let $U$ be open in $E$ and let $f\colon U \to F$ be twice differentiable, and such that $D^2f$ is continuous. Then for each $x \in U$, the bilinear map $D^2f(x)$ is symmetric, that is*

$$D^2f(x)(v, w) = D^2f(x)(w, v)$$

*for all $v, w \in E$.*

*Proof.* Let $x \in U$ and suppose that the open ball of radius $r$ in $E$ centered at $x$ is contained in $U$. Let $v, w \in E$ have lengths $< r/2$. Let

$$g(x) = f(x + v) - f(x).$$

Then

$$\begin{aligned} f(x + v + w) &- f(x + w) - f(x + v) + f(x) \\ &= g(x + w) - g(x) = \int_0^1 g'(x + tw)w\,dt \\ &= \int_0^1 [Df(x + v + tw) - Df(x + tw)]w\,dt \\ &= \int_0^1 \int_0^1 D^2f(x + sv + tw)v\,ds \cdot w\,dt. \end{aligned}$$

Let

$$\psi(sv, tw) = D^2f(x + sv + tw) - D^2f(x).$$

Then

$$\begin{aligned} g(x + w) - g(x) &= \int_0^1 \int_0^1 D^2f(x)(v, w)\,ds\,dt \\ &\quad + \int_0^1 \int_0^1 \psi(sv, tw)v \cdot w\,ds\,dt \\ &= D^2f(x)(v, w) + \varphi(v, w) \end{aligned}$$

where $\varphi(v, w)$ is the second integral on the right, and satisfies the estimate

$$|\varphi(v, w)| \leqq \sup_{s,t} |\psi(sv, tw)|\,|v|\,|w|.$$

The sup is taken for $0 \leqq s \leqq 1$ and $0 \leqq t \leqq 1$. If we had started with

$$g_1(x) = f(x + w) - f(x)$$

and considered $g_1(x + v) - g_1(x)$, we would have found another expression for the expression

$$f(x + v + w) - f(x + w) - f(x + v) + f(x),$$

namely

$$D^2f(x)(w, v) + \varphi_1(v, w)$$

where

$$|\varphi_1(v, w)| \leqq \sup_{s,t} |\psi_1(sv, tw)|\,|v|\,|w|.$$

But then

$$D^2f(x)(w, v) - D^2f(x)(v, w) = \varphi(v, w) - \varphi_1(v, w).$$

By the lemma, and the continuity of $D^2f$ which guarantees that

$$\sup_{s,t} |\psi(sv, tw)| \quad \text{and} \quad \sup_{s,t} |\psi_1(sv, tw)|$$

satisfy the limit condition of the lemma, we now conclude that

$$D^2f(x)(w, v) = D^2f(x)(v, w),$$

as was to be shown.

For an application of the second derivative, cf. the Morse–Palais lemma in Chapter XIII. It describes the behavior of a function in a neighborhood of a critical point in a manner used for instance in the calculus of variations.

## XIII, §6. HIGHER DERIVATIVES AND TAYLOR'S FORMULA

We may now consider higher derivatives. We define

$$D^pf(x) = D(D^{p-1}f)(x).$$

Thus $D^pf(x)$ is an element of $L(E, L(E, \ldots, L(E, F)\ldots))$ which we denote by $L^p(E, F)$. We say that $f$ is of class $C^p$ on $U$ or is a $C^p$ map if $D^kf(x)$ exists for each $x \in U$, and if

$$D^kf\colon U \to L^k(E, F)$$

is continuous for each $k = 0, \ldots, p$.

We have trivially $D^qD^rf(x) = D^pf(x)$ if $q + r = p$ and if $D^pf(x)$ exists. Also the $p$-th derivative $D^p$ is linear in the sense that

$$D^p(f + g) = D^pf + D^pg \quad \text{and} \quad D^p(cf) = cD^pf.$$

If $\lambda \in L^p(E, F)$ we write

$$\lambda(v_1)(v_2)\cdots(v_p) = \lambda(v_1, \ldots, v_p).$$

If $q + r = p$, we can evaluate $\lambda(v_1, \ldots, v_p)$ in two steps, namely

$$\lambda(v_1, \ldots, v_q)\cdot(v_{q+1}, \ldots, v_p).$$

We regard $\lambda(v_1, \ldots, v_q)$ as the element of $L^{p-q}(E, F)$ given by

$$\lambda(v_1, \ldots, v_q) \cdot (v_{q+1}, \ldots, v_p) = \lambda(v_1, \ldots, v_p).$$

**Lemma 6.1.** *Let $v_2, \ldots, v_p$ be fixed elements of $E$. Assume that $f$ is $p$ times differentiable on $U$. Let*

$$g(x) = D^{p-1}f(x)(v_2, \ldots, v_p).$$

*Then $g$ is differentiable on $U$ and*

$$Dg(x)(v) = D^p f(x)(v, v_2, \ldots, v_p).$$

*Proof.* The map $g: U \to F$ is a composite of the maps

$$D^{p-1}f: U \to L^{p-1}(E, F) \quad \text{and} \quad \lambda: L^{p-1}(E, F) \to F,$$

where $\lambda$ is given by the evaluation at $(v_2, \ldots, v_p)$. Thus $\lambda$ is continuous and linear. It is an old theorem that

$$D(\lambda \circ D^{p-1}f) = \lambda \circ DD^{p-1}f = \lambda \circ D^p f,$$

namely the corollary of Theorem 3.1. Thus

$$Dg(x)v = \big(D^p f(x)v\big)(v_2, \ldots, v_p),$$

which is precisely what we wanted to prove.

**Theorem 6.2.** *Let $f$ be of class $C^p$ on $U$. Then for each $x \in U$ the map $D^p f(x)$ is multilinear symmetric.*

*Proof.* By induction on $p \geqq 2$. For $p = 2$ this is Theorem 5.3. In particular, if we let $g = D^{p-2}f$ we know that for $v_1, v_2 \in E$,

$$D^2 g(x)(v_1, v_2) = D^2 g(x)(v_2, v_1),$$

and since $D^p f = D^2 D^{p-2} f$ we conclude that

$$(*) \qquad \begin{aligned} D^p f(x)(v_1, \ldots, v_p) &= \big(D^2 D^{p-2} f(x)\big)(v_1, v_2) \cdot (v_3, \ldots, v_p) \\ &= \big(D^2 D^{p-2} f(x)\big)(v_2, v_1) \cdot (v_3, \ldots, v_p) \\ &= D^p f(x)(v_2, v_1, v_3, \ldots, v_p). \end{aligned}$$

Let $\sigma$ be a permutation of $(2, \ldots, p)$. By induction,

$$D^{p-1}f(x)(v_{\sigma(2)}, \ldots, v_{\sigma(p)}) = D^{p-1}f(x)(v_2, \ldots, v_p).$$

By the lemma, we conclude that

$$(**) \qquad D^p f(x)(v_1, v_{\sigma(2)}, \ldots, v_{\sigma(p)}) = D^p f(x)(v_1, \ldots, v_p).$$

From (*) and (**) we conclude that $D^p f(x)$ is symmetric because any permutation of $(1, \ldots, p)$ can be expressed as a composition of the permutations considered in (*) or (**). This proves the theorem.

For the higher derivatives, we have similar statements to those obtained with the first derivative in relation to linear maps. Observe that if $\omega \in L^p(E, F)$ is a multilinear map, and $\lambda \in L(F, G)$ is linear, we may compose these

$$E \times \cdots \times E \xrightarrow{\omega} F \xrightarrow{\lambda} G$$

to get $\lambda \circ \omega$, which is a multilinear map of $E \times \cdots \times E \to G$. Furthermore, $\omega$ and $\lambda$ being continuous, it is clear that $\lambda \circ \omega$ is also continuous. Finally, the map

$$\lambda_*: L^p(E, F) \to L^p(E, G)$$

given by "composition with $\lambda$", namely

$$\omega \mapsto \lambda \circ \omega,$$

is immediately verified to be a continuous *linear* map, that is for $\omega_1$, $\omega_2 \in L^p(E, F)$ and $c \in \mathbf{R}$ we have

$$\lambda \circ (\omega_1 + \omega_2) = \lambda \circ \omega_1 + \lambda \circ \omega_2 \qquad \text{and} \qquad \lambda \circ (c\omega_1) = c\lambda \circ \omega_1,$$

and for the continuity,

$$|\lambda \circ \omega(v_1, \ldots, v_n)| \leqq |\lambda| |\omega| |v_1| \cdots |v_n|$$

so

$$|\lambda \circ \omega| \leqq |\lambda| |\omega|.$$

**Theorem 6.3.** *Let $f: U \to F$ be $p$ times differentiable and let $\lambda: F \to G$ be a continuous linear map. Then for every $x \in U$ we have*

$$D^p(\lambda \circ f)(x) = \lambda \circ D^p f(x).$$

*Proof.* Consider the map $x \mapsto D^{p-1}(\lambda \circ f)(x)$. By induction,

$$D^{p-1}(\lambda \circ f)(x) = \lambda \circ D^{p-1} f(x).$$

By the Corollary 3.2 concerning the derivative

$$D(\lambda_* \circ D^{p-1} f),$$

namely the derivative of the composite map

$$U \xrightarrow{D^{p-1}f} L^{p-1}(E, F) \xrightarrow{\lambda_*} L^{p-1}(E, G),$$

we get the assertion of our theorem.

If one wishes to omit the $x$ from the notation in Theorem 6.3, then one must write

$$D^p(\lambda \circ f) = \lambda_* \circ D^p f.$$

Occasionally, one omits the lower $*$ and writes simply $D^p(\lambda \circ f) = \lambda \circ D^p f$.

**Taylor's Formula.** *Let $U$ be open in $E$ and let $f\colon U \to F$ be of class $C^p$. Let $x \in U$ and let $y \in E$ be such that the segment $x + ty$, $0 \leqq t \leqq 1$, is contained in $U$. Denote by $y^{(k)}$ the $k$-tuple $(y, y, \ldots, y)$. Then*

$$f(x + y) = f(x) + \frac{Df(x)y}{1!} + \cdots + \frac{D^{p-1}f(x)y^{(p-1)}}{(p-1)!} + R_p$$

*where*

$$R_p = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x + ty)y^{(p)}\, dt.$$

*Proof.* We can give two proofs, the first by integration by parts as usual, starting with the mean value theorem,

$$f(x + y) = f(x) + \int_0^1 Df(x + ty)y\, dt.$$

We consider the map $t \mapsto Df(x + ty)y$ of the interval into $F$, and the usual product

$$\mathbf{R} \times F \to F,$$

which consists in multiplying vectors of $F$ by numbers. We let

$$u = Df(x + ty)y, \qquad v = -(1 - t), \qquad \text{and} \qquad dv = dt.$$

This gives the next term, and then we proceed by induction, letting

$$u = D^p f(x + ty)y^{(p)} \qquad \text{and} \qquad dv = \frac{(1-t)^{p-1}}{(p-1)!} dt$$

at the $p$-th state. Integration by parts yields the next term of Taylor's formula, plus the next remainder term.

The other proof can be given by using the Hahn–Banach theorem and applying a continuous linear function to the formula. This reduces the proof to the ordinary case of functions of one variable, that is with values in **R**. Of course, in that case, we also proceed by induction, so there is really not much to choose from between the two proofs.

The remainder term $R_p$ can also be written in the form

$$R_p = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x+ty)\, dt \cdot y^{(p)}.$$

The mapping

$$y \mapsto \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x+ty)\, dt$$

is continuous. If $f$ is infinitely differentiable, then this mapping is infinitely differentiable since we shall see later that one can differentiate under the integral sign.

**Estimate of the Remainder.** *With notation as in Taylor's formula, we can also write*

$$f(x+y) = f(x) + \frac{Df(x)y}{1!} + \cdots + \frac{D^p f(x) y^{(p)}}{p!} + \theta(y)$$

*where*

$$|\theta(y)| \leqq \sup_{0 \leqq t \leqq 1} \frac{|D^p f(x+ty) - D^p f(x)|}{p!} |y|^p$$

*and*

$$\lim_{y \to 0} \frac{\theta(y)}{|y|^p} = 0.$$

*Proof.* We write

$$D^p f(x+ty) - D^p f(x) = \psi(ty).$$

Since $D^p f$ is continuous, it is bounded in some ball containing $x$, and

$$\lim_{y \to 0} \psi(ty) = 0$$

uniformly in $t$. On the other hand, the remainder $R_p$ given above can be written as

$$\int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x) y^{(p)}\, dt + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \psi(ty) y^{(p)}\, dt.$$

We integrate the first integral to obtain the desired $p$-th term, and esti-

mate the second integral by

$$\sup_{0 \leqq t \leqq 1} |\psi(ty)| \, |y|^p \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \, dt,$$

where we can again perform the integration to get the estimate for the error term $\theta(y)$.

**Theorem 6.4.** *Let $U$ be open in $E$ and let $f\colon U \to F_1 \times \cdots \times F_m$ be a map with coordinate maps $(f_1, \ldots, f_n)$. Then $f$ is of class $C^p$ if and only if each $f_i$ is of class $C^p$, and if that is the case, then*

$$D^p f = (D^p f_1, \ldots, D^p f_m).$$

*Proof.* We proved this for $p = 1$ in §3, and the general case follows by induction.

**Theorem 6.5.** *Let $U$ be open in $E$ and $V$ open in $F$. Let $f\colon U \to V$ and $g\colon V \to G$ be $C^p$ maps. The $g \circ f$ is of class $C^p$.*

*Proof.* We have

$$D(g \circ f)(x) = Dg\big(f(x)\big) \circ Df(x).$$

Thus $D(g \circ f)$ is obtained by composing a lot of maps, namely as represented in the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{Dg} & L(F, G) \\ {\scriptstyle f}\uparrow & & \times \\ U & \xrightarrow[Df]{} & L(E, F) \end{array} \Bigg\} \to L(E, G)$$

If $p = 1$, then all mappings occurring on the right are continuous and so $D(g \circ f)$ is continuous. By induction, $Dg$ and $Df$ are of class $C^{p-1}$, and all the maps used to obtain $D(g \circ f)$ are of class $C^{p-1}$ (the last one on the right is a composition of linear maps, and is continuous bilinear, so infinitely differentiable by Theorem 5.1). Hence $D(g \circ f)$ is of class $C^{p-1}$, whence $g \circ f$ is of class $C^p$, as was to be shown.

## XIII, §7. PARTIAL DERIVATIVES

Consider a product $E = E_1 \times \cdots \times E_n$ of complete normed vector spaces. Let $U_i$ be open in $E_i$ and let

$$f\colon U_1 \times \cdots \times U_n \to F$$

be a map. We write an element $x \in U_1 \times \cdots \times U_n$ in terms of its "coordinates", namely $x = (x_1, \ldots, x_n)$ with $x_i \in U_i$.

We can form partial derivatives just as in the simple case when $E = \mathbf{R}^n$. Indeed, for $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ fixed, we consider the partial map

$$x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_n)$$

of $U_i$ into $F$. If this map is differentiable, we call its derivative the **partial derivative of** $f$ and denote it by $D_i f(x)$ at the point $x$. Thus, if it exists,

$$D_i f(x) = \lambda\colon E_i \to F$$

is the unique continuous linear map $\lambda \in L(E_i, F)$ such that

$$f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n) = \lambda(h) + o(h),$$

for $h \in E_i$ and small enough that the left-hand side is defined.

**Theorem 7.1.** *Let $U_i$ be open in $E_i$ $(i = 1, \ldots, n)$ and let*

$$f\colon U_1 \times \cdots \times U_n \to F$$

*be a map. This map is of class $C^p$ if and only if each partial derivative*

$$D_i f\colon U_1 \times \cdots \times U_n \to L(E_i, F)$$

*is of class $C^{p-1}$. If this is the case, and*

$$v = (v_1, \ldots, v_n) \in E_1 \times \cdots \times E_n,$$

*then*

$$Df(x)v = \sum_{i=1}^{n} D_i f(x) v_i.$$

*Proof.* We shall give the proof just for $n = 2$, to save space. We assume that the partial derivatives are continuous, and want to prove that the derivative of $f$ exists and is given by the formula of the theorem. We let $(x, y)$ be the point at which we compute the derivative, and let $h = (h_1, h_2)$. We have

$$\begin{aligned} &f(x + h_1, y + h_2) - f(x, y) \\ &\qquad = f(x + h_1, y + h_2) - f(x + h_1, y) + f(x + h_1, y) - f(x, y) \\ &\qquad = \int_0^1 D_2 f(x + h_1, y + th_2) h_2 \, dt + \int_0^1 D_1 f(x + th_1, y) h_1 \, dt. \end{aligned}$$

Since $D_2 f$ is continuous, the map $\psi$ given by

$$\psi(h_1, th_2) = D_2 f(x + h_1, y + th_2) - D_2 f(x, y)$$

satisfies

$$\lim_{h \to 0} \psi(h_1, th_2) = 0.$$

Thus we can write the first integral as

$$\int_0^1 D_2 f(x + h_1, y + th_2) h_2 \, dt = \int_0^1 D_2 f(x, y) h_2 \, dt + \int_0^1 \psi(h_1, th_2) h_2 \, dt$$
$$= D_2 f(x, y) h_2 + \int_0^1 \psi(h_1, th_2) h_2 \, dt.$$

Estimating the error term given by this last integral, we find

$$\left| \int_0^1 \psi(h_1, th_2) h_2 \, dt \right| \leqq \sup_{0 \leqq t \leqq 1} |\psi(h_1 th_2)| \, |h_2|$$
$$\leqq |h| \, sup \, |\psi(h_1, th_2)|$$
$$= o(h).$$

Similarly, the second integral yields

$$D_1 f(x, y) h_1 + o(h).$$

Adding these terms, we find that $Df(x, y)$ exists and is given by the formula, which also shows that the map $Df = f'$ is continuous, so $f$ is of class $C^1$. If each partial is of class $C^p$, then it is clear that $f$ is $C^p$. We leave the converse to the reader.

It will be useful to have a notation for linear maps of products into products. We treat the special case of two factors. We wish to describe linear maps

$$\lambda: E_1 \times E_2 \to F_1 \times F_2.$$

We contend that such a linear map can be represented by a matrix

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

where each $\lambda_{ij}: E_j \to F_i$ is itself a linear map. We thus take matrices whose components are not numbers any more but are themselves linear maps. This is done as follows.

Suppose we are **given** four linear maps $\lambda_{ij}$ as above. An element of $E_1 \times E_2$ may be viewed as a pair of elements $(v_1, v_2)$ with $v_1 \in E_1$ and $v_2 \in E_2$. We now write such a pair as a column vector

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and define $\lambda(v_1, v_2)$ to be

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_{11}v_1 + \lambda_{12}v_2 \\ \lambda_{21}v_1 + \lambda_{22}v_2 \end{pmatrix}$$

so that we multiply just as we would with numbers. Then it is clear that $\lambda$ is a linear map of $E_1 \times E_2$ into $F_1 \times F_2$.

Conversely, let $\lambda: E_1 \times E_2 \to F_1 \times F_2$ be a linear map. We write an element $(v_1, v_2) \in E_1 \times E_2$ in the form

$$(v_1, v_2) = (v_1, 0) + (0, v_2).$$

We also write $\lambda$ in terms of its coordinate maps $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_1: E_1 \times E_2 \to F_1$ and $\lambda_2: E_1 \times E_2 \to F_2$ are linear. Then

$$\begin{aligned} \lambda(v_1, v_2) &= (\lambda_1(v_1, v_2), \lambda_2(v_1, v_2)) \\ &= (\lambda_1(v_1, 0) + \lambda_1(0, v_1), \lambda_2(v_1, 0) + \lambda_2(0, v_2)). \end{aligned}$$

The map

$$v_1 \mapsto \lambda_1(v_1, 0)$$

is a linear map of $E_1$ into $F_1$ which we call $\lambda_{11}$. Similarly, we let

$$\begin{aligned} \lambda_{11}(v_1) &= \lambda_1(v_1, 0), &\quad \lambda_{12}(v_2) &= \lambda_1(0, v_2), \\ \lambda_{21}(v_1) &= \lambda_2(v_1, 0), &\quad \lambda_{22}(v_2) &= \lambda_2(0, v_2). \end{aligned}$$

Then we can represent $\lambda$ as the matrix

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

as explained in the preceding discussion, and we see that $\lambda(v_1, v_2)$ is given by the multiplication of the above matrix with the vertical vector formed with $v_1$ and $v_2$.

Finally, we observe that if all $\lambda_{ij}$ are continuous, then the map $\lambda$ is also continuous, and conversely.

We can apply this to the case of partial derivatives, and we formulate the result as a corollary.

**Corollary 7.2.** *Let $U$ be open in $E_1 \times E_2$ and let $f\colon U \to F_1 \times F_2$ be a $C^p$ map. Let $f = (f_1, f_2)$ be represented by its coordinate maps*

$$f_1\colon U \to F_1 \quad \text{and} \quad f_2\colon U \to F_2.$$

*Then for any $x \in U$, the linear map $Df(x)$ is represented by the matrix*

$$\begin{pmatrix} D_1 f_1(x) & D_2 f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) \end{pmatrix}.$$

*Proof.* This follows by applying Theorem 7.1 to each one of the maps $f_1$ and $f_2$, and using the definitions of the preceding discussion.

Observe that except for the fact that we deal with linear maps, all that precedes is treated just like the standard way for functions on open sets of $n$-space, where the derivatives follow exactly the same formalism with respect to the partial derivatives.

**Theorem 7.3.** *Let $U$ be open in $E_1 \times E_2$ and let $f\colon U \to F$ be a map such that $D_1 f$, $D_2 f$, $D_1 D_2 f$, and $D_2 D_1 f$ exist and are continuous. Then $D_1 D_2 f = D_2 D_1 f$.*

*Proof.* The proof is entirely analogous to the standard proof of the similar result for functions of two variables, and will be left to the reader. Actually, if we assume that $f$ is of class $C^2$, then the second derivative $D^2 f(x)$ is represented by the matrix $(D_i D_j f(x))$, with $i, j = 1, 2$. By Theorem 5.3, we know that $D^2 f(x)$ is symmetric, whence we conclude that $D_1 D_2 f(x) = D_2 D_1 f(x)$.

## XIII, §8. DIFFERENTIATING UNDER THE INTEGRAL SIGN

**Theorem 8.1.** *Let $U$ be open in $E$ and let $J = [a, b]$ be an interval. Let $f\colon J \times U \to F$ be a continuous map such that $D_2 f$ exists and is continuous. Let*

$$g(x) = \int_a^b f(t, x)\, dt.$$

*Then $g$ is differentiable on $U$ and*

$$Dg(x) = \int_a^b D_2 f(t, x)\, dt.$$

*Proof.* Differentiability is a property relating to a point, so let $x \in U$. Selecting a sufficiently small open neighborhood $V$ of $x$, we can assume

that $D_2 f$ is bounded on $J \times V$. Let $\lambda$ be the linear map

$$\lambda = \int_a^b D_2 f(t, x)\, dt.$$

We investigate

$$\begin{aligned} g(x + h) - g(x) - \lambda h &= \int_a^b [f(t, x + h) - f(t, x) - D_2 f(t, x)h]\, dt \\ &= \int_a^b \left[\int_a^b D_2 f(t, x + uh)h\, du - D_2 f(t, x)h\right] dt \\ &= \int_a^b \left\{\int_a^b [D_2 f(t, x + uh) - D_2 f(t, x)]h\, du\right\} dt. \end{aligned}$$

We estimate:

$$|g(x + h) - g(x) - \lambda h| \leqq \max |D_2 f(t, x + uh) - D_2 f(t, x)|\, |h|,$$

the maximum being taken for $0 \leqq u \leqq 1$ and $0 \leqq t \leqq 1$. By the relative uniform continuity of $D_2 f$ with respect to the compact set $J \times \{x\}$, we conclude that given $\varepsilon$ there exists $\delta$ such that whenever $|h| < \delta$ then this maximum is $< \varepsilon$. This proves that $\lambda$ is the derivative $g'(x)$, as desired.

## XIII, §9. DIFFERENTIATION OF SEQUENCES

**Theorem 9.1.** *Let $U$ be an open subset of a Banach space $E$, and let $\{f_n\}$ be a sequence of $C^1$ maps of $U$ into a Banach space $F$. Assume that $\{f_n\}$ converges pointwise to a map $f$, and also that the sequence of derivatives $\{f_n'\}$ converges uniformly, to a mapping*

$$g\colon U \to L(E, F).$$

*Then $f$ is differentiable, and $f' = g$.*

*Proof.* Let $x_0 \in U$. Differentiability being a local property, we can assume without loss of generality that $U$ is an open ball cetnered at $x_0$. For $x \in U$, we have by the mean value theorem applied to $f_n - f_m$:

$$|f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| \leqq |x - x_0| \sup_{y \in U} |f_n'(y) - f_m'(y)|.$$

Given $\varepsilon$ there exists $N$ such that if $m, n > N$, then

$$\|f_n' - f_m'\| < \varepsilon \qquad \text{and} \qquad \|f_n' - g\| < \varepsilon.$$

Letting $m$ tend to infinity, we conclude that for $n > N$ we have

$$(1) \qquad |f_n(x) - f(x) - (f_n(x_0) - f(x_0))| \leqq |x - x_0|\varepsilon.$$

Fix $n > N$. Again by the mean value theorem, there exists $\delta$ such that if $|x - x_0| < \delta$ we have

$$(2) \qquad |f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0)| \leqq |x - x_0|\varepsilon.$$

Finally, use the fact that $\|f_n' - g\| < \varepsilon$. We conclude from (1) and (2) that

$$|f(x) - f(x_0) - g(x_0)(x - x_0)| \leqq 3|x - x_0|\varepsilon.$$

This proves our theorem.

## XIII, §10. EXERCISES

1. Let $U$ be open in $E$ and $V$ pen in $F$. Let

$$f\colon U \to V \qquad \text{and} \qquad g\colon V \to G$$

be of class $C^p$. Let $x_0 \in U$. Assume that $D^k f(x_0) = 0$ for all $k = 0, \dots, p$. Show that $D^k(g \circ f)(x_0) = 0$ for $0 \leqq k \leqq p$. [*Hint:* Induction.] Also prove that if $D^k g(f(x_0)) = 0$ for $0 \leqq k \leqq p$, then $(D^k(g \circ f))(x_0) = 0$ for $0 \leqq k \leqq p$.

2. Let $f(t) = \sum c_n t^n$ be a power series with real coefficients, converging in a circle of radius $r$. Let $A$ be a Banach algebra. Show that the map

$$u \mapsto \sum c_n u^n$$

is a $C^1$ (or even $C^\infty$) map on the disc of radius $r$ centered at the origin in $A$.

3. Let $E$, $F$ be Banach spaces, and Lis$(E, F)$ the set of toplinear isomorphisms between $E$ and $F$. Show that the map $u \mapsto u^{-1}$ from Lis$(E, F)$ to Lis$(F, E)$ is differentiable, and find its derivative (as in the case of Banach algebras).

4. Let $A$ be a Banach algebra with unit $e$. Show that one can define a square root function in a neighborhood of $e$, in such a manner that it is of class $C^1$ (or even $C^\infty$).

5. Let $Z$ be a compact topological space, $E$ a Banach space, and $F = C^0(Z, E)$ the Banach space of continuous maps of $Z$ into $E$, with the sup norm. Let $U$ be open in $E$, and let $V$ be the subset of $F$ consisting of all maps $f\colon Z \to U$ which map $Z$ into $U$, so $V = C^0(Z, U)$. Let $g\colon U \to G$ be a map of $U$ into a Banach space $G$.
(a) If $g$ is continuous, show that the map

$$f \mapsto g \circ f$$

of $V$ into $C^0(Z, G)$ is continuous.

(b) If $g$ is of class $C^1$, show that the above map is of class $C^1$, and find a formula for its derivative.
(c) If $g$ is of class $C^p$, show that the above map is of class $C^p$.

6. Let $J = [a, b]$ be a closed interval, and let $U$ be open in a Banach space $E$. Let $g\colon U \to G$ be a $C^1$ map. Let $C^0(J, U)$ be the set of continuous maps of $J$ into $U$. Show that $C^0(J, U)$ is open in $C^0(J, E)$, and that the map

$$\alpha \mapsto \int_a g \circ \alpha = S_g(\alpha)$$

is of class $C^1$. The notation means that

$$S_g(\alpha)(t) = \int_a^t g(\alpha(u))\, du.$$

Find an expression for the derivative of $S_g$.

7. Let $f$ be a map of class $C^1$ on a Banach space $E$ such that $f(tx) = tf(x)$ for all real $t$ and all $x \in E$. Show that $f$ is linear, and in fact that $f(x) = f'(0)x$.

8. Let $f$ be a map of class $C^2$ on a Banach space $E$ such that $f(tx) = t^2 f(x)$ for all real $t$ and all $x \in E$. Show that $f$ is quadratic, and that in fact

$$f(x) = D^2 f(0)(x, x).$$

Generalize Exercises 7 and 8.

9. Let $E$ be a Banach space, and $J = [a, b]$ a closed interval. For each $C^1$ curve $\alpha\colon J \to E$ let the $C^1$ norm of $\alpha$ be defined by

$$\|\alpha\|_1 = \|\alpha\| + \|\alpha'\|$$

where $\alpha'$ is the derivative of $\alpha$. Show that this is a norm, and that the space $C^1(J, E)$ of $C^1$ curves is complete under this norm.

10. Let $U$ be open in a Banach space and let $BC^p(U, F)$ be the space of maps $f\colon U \to F$ into a Banach space $F$ which are of class $C^p$, and such that all derivatives $D^k f$ are bounded, for $k = 0, \ldots, p$. Show that $BC^p(U, F)$ is a Banach space, under the norm

$$\|f\|_{C^p} = \sup \|D^k f\|,$$

the sup being taken for $0 \leqq k \leqq p$.

11. This exercise is a starting point for the calculus of variations. Let $E$ be a Banach space and $U$ an open subset of $\mathbf{R} \times E \times E$. Let

$$H\colon U \to \mathbf{R}$$

be a $C^p$ function. Let $J$ be a closed interval $[a, b]$. Let $V$ be the subset of

$C^1(J, E)$ consisting of all curves $\alpha \in C^1(J, E)$ such that the curve

$$t \mapsto (t, \alpha(t), \alpha'(t)), \qquad t \in J,$$

lies in $U$.

(a) Show that $V$ is open in $C^1(J, E)$.

(b) Show that the map

$$\alpha \mapsto \int_a^b H(t, \alpha(t), \alpha'(t))\, dt = f(\alpha)$$

is a $C^p$ function on $V$, and determine its derivative.

(c) Let $g: J \to L(E, \mathbf{R})$ be a continuous function such that

$$\int_a^b g(t)\sigma(t)\, dt = 0$$

for every $C^1$ curve $\sigma: J \to E$ having the property that

$$\sigma(a) = \sigma(b) = 0.$$

Show that $g = 0$.

(d) Let $C^1(J, E, 0)$ be the subset of curves $\alpha$ in $C^1(J, E)$ such that

$$\alpha(a) = \alpha(b) = 0.$$

Show that $C^1(J, E, 0)$ is a closed subspace. Restrict the function $f$ to the open set

$$V_0 = V \cap C^1(J, E, 0)$$

of this closed subspace. Show that if an element $\alpha \in V_0$ is a local minimum of $f$ in $V_0$, then

$$D_1 D_3 H(t, \alpha(t), \alpha'(t)) = D_2 H(t, \alpha(t), \alpha'(t))$$

for all $t \in J$. [*Hint:* Show that if a function has a local minimum at a point, then its derivative is 0 at that point, and use (c).]

CHAPTER XIV

# Inverse Mappings and Differential Equations

## XIV, §1. THE INVERSE MAPPING THEOREM

Both the inverse mapping theorem and the existence theorem for differential equations will be based on a basic and simple lemma in complete metric spaces.

**Lemma 1.1 (Shrinking Lemma).** *Let $M$ be a complete metric space, and let $T: M \to M$ be a mapping. Assume that there exists a number $K$ with $0 < K < 1$ such that for all $x, y \in M$ we have*

$$d(Tx, Ty) \leqq Kd(x, y),$$

*where $d$ is the distance function in $M$. Then $T$ has a unique fixed point $z$, that is a point such that $Tz = z$. If $x \in M$, then*

$$z = \lim_{n\to\infty} T^n x.$$

*Proof.* For simplicity of notation, we assume that $M$ is a closed subset of a Banach space. We first observe that a fixed point $z$, if it exists, is unique because if $z_1$ is also fixed, then

$$|z - z_1| = |Tz - Tz_1| \leqq K|z - z_1|,$$

so $z - z_1 = 0$. Now for existence, let $m$, $n$ be positive integers and say $n \geqq m$, $n = m + r$. Then for any $x$ we have

$$|T^n x - T^m x| \leqq K^m|x - T^r x|$$

and

$$|x - T^r x| \leqq |x - Tx| + |Tx - T^2 x| + \cdots + |T^{r-1}x - T^r x|$$
$$\leqq (1 + K + \cdots + K^{r-1})|x - Tx|.$$

This shows that the sequence $\{T^n x\}$ is Cauchy, converging to some element $z \in M$. This element $z$ is a fixed point because

$$|Tz - TT^n x| \leqq K|z - T^n x|$$

and for $n$ sufficiently large, $TT^n x$ approaches $z$ and also $Tz$. This proves the shrinking lemma.

We shall call $K$ in the lemma a **shrinking constant** for $T$.

Let $U$ be open in a Banach space $E$, and let $f: U \to F$ be a $C^p$ map ($p \geqq 1$). We shall say that $f$ is a **$C^p$-isomorphism** or is **$C^p$-invertible on** $U$ if the image $f(U)$ is an open set $V$ in $F$, and if there exists a $C^p$ map

$$g: V \to U$$

such that $g \circ f$ and $f \circ g$ are the identity maps on $U$ and $V$ respectively. We say that $f$ is a **local** $C^p$-isomorphism at a point $x$ in $U$, or is **locally $C^p$-invertible** at $x$, if there exists an open set $U_1$ contained in $U$ and containing $x$ such that the restriction of $f$ to $U_1$ is $C^p$-invertible on $U_1$.

It is clear that the composite of two $C^p$-isomorphisms is again a $C^p$-isomorphism, and that the composite of two locally $C^p$-invertible maps is also locally $C^p$-invertible. In other words, if $f$ is locally $C^p$-invertible at $x$, if $f(x)$ is contained in some open set $V$, and if $g: V \to G$ is locally $C^p$-invertible at $f(x)$, then $g \circ f$ is locally $C^p$-invertible at $x$.

The inverse mapping theorem provides a criterion for a map to be locally $C^p$-invertible, in terms of its derivative.

**Theorem 1.2 (Inverse Mapping Theorem).** *Let $U$ be open in a Banach space $E$, and let $f: U \to F$ be a $C^p$ map. Let $x_0 \in U$ and assume that $f'(x_0): E \to F$ is a toplinear isomorphism (i.e. invertible as a continuous linear map). Then $f$ is a local $C^p$-isomorphism at $x_0$.*

*Proof.* Let $\lambda = f'(x_0)$. It suffices to prove that $\lambda^{-1} \circ f$ is locally invertible at $x_0$ because we may consider $\lambda^{-1} \circ f$ instead of $f$ itself. Thus we have reduced our theorem to the case where $E = F$ and $f'(x_0)$ is the identity mapping. Next, making translations, it suffices to prove our theorem when $x_0 = 0$ and $f(x_0) = 0$ also. From now on, we make these additional assumptions.

Let $g(x) = x - f(x)$. Then $g'(0) = 0$ and by continuity there exists

$r > 0$ such that if $|x| \leqq 2r$, then

$$|g'(x)| \leqq \tfrac{1}{2}.$$

From the mean value theorem we see that $|g(x)| \leqq \frac{1}{2}|x|$, and hence that $g$ maps the closed ball $\bar{B}_r(0)$ into $\bar{B}_{r/2}(0)$. We contend that given $y \in \bar{B}_{r/2}(0)$, there exists a unique element $x \in \bar{B}_r(0)$ such that $f(x) = y$. We prove this by considering the map

$$g_y(x) = y + x - f(x).$$

If $|y| \leqq r/2$ and $|x| \leqq r$, then $|g_y(x)| \leqq r$ and hence $g_y$ may be viewed as a mapping of the complete metric space $\bar{B}_r(0)$ into itself. The bound of $\frac{1}{2}$ on the derivative together with the mean value theorem shows that $g_y$ is a shrinking map, i.e. that

$$|g_y(x_1) - g_y(x_2)| = |g(x_1) - g(x_2)| \leqq \tfrac{1}{2}|x_1 - x_2|$$

for $x_1$, $x_2 \in \bar{B}_r(0)$. By the shrinking lemma, it follows that $g_y$ has a unique fixed point, which is precisely the solution of the equation $f(x) = y$. This proves our contention.

We obtain a local inverse for $f$, which we denote by $f^{-1}$. This inverse is continuous, because writing $x = x - f(x) + f(x)$ we see that

$$\begin{aligned} |x_1 - x_2| &\leqq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| \\ &\leqq |f(x_1) - f(x_2)| + \tfrac{1}{2}|x_1 - x_2|, \end{aligned}$$

whence

$$|x_1 - x_2| \leqq 2|f(x_1) - f(x_2)|.$$

We shall now see that this inverse is differentiable on the open ball $B_{r/2}(0)$. Indeed, fix $y_1 \in B_{r/2}(0)$ and let $y_1 = f(x_1)$ with $x_1 \in \bar{B}_r(0)$. Let $y \in B_{r/2}(0)$, and let $y = f(x)$ with $x \in \bar{B}_r(0)$. Then:

$$(*) \qquad \begin{aligned} &|f^{-1}(y) - f^{-1}(y_1) - f'(x_1)^{-1}(y - y_1)| \\ &\qquad = |x - x_1 - f'(x_1)^{-1}(f(x) - f(x_1))|. \end{aligned}$$

From the differentiability of $f$, we can write

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + o(x - x_1).$$

If we substitute this in $(*)$, we find the expression

$$|f'(x_1)^{-1}o(x - x_1)|.$$

Of course, $f'(x_1)^{-1}$ is bounded by a fixed constant and by what we have already seen, we have

$$|x - x_1| \leqq 2|y - y_1|.$$

From the definition of differentiability, we conclude that $f^{-1}$ is differentiable at $y_1$ and that its derivative is given by

$$f'(x_1)^{-1} = f'(f^{-1}(y_1))^{-1}.$$

Since the mappings $f^{-1}$, $f'$, "inverse" are continuous, it follows that $D(f^{-1})$ is continuous and thus that $f^{-1}$ is of class $C^1$. Since taking inverses is $C^\infty$, it follows inductively that $f^{-1}$ is $C^p$, as was to be shown.

We shall generalize part of the inverse mapping theorem in Chapter XV, §3.

In some applications it is necessary to know that if the derivative of a map is close to the identity, then the image of a ball contains a ball of only slightly smaller radius. The precise statement follows. In this book, it will be used only in the proof of the change of variables formula, and therefore may be omitted until the reader needs it.

**Lemma 1.3.** *Let $U$ be open in $E$, and let $f\colon U \to E$ be of class $C^1$. Assume that $f(0) = 0$, $f'(0) = I$. Let $r > 0$ and assume that $\bar{B}_r(0) \subset U$. Let $0 < s < 1$, and assume that*

$$|f'(z) - f'(x)| \leqq s$$

*for all $x$, $z \in \bar{B}_r(0)$. If $y \in E$ and $|y| \leqq (1 - s)r$, then there exists a unique $x \in \bar{B}_r(0)$ such that $f(x) = y$.*

*Proof.* The map $g_y$ given by $g_y(x) = x - f(x) + y$ is defined for $|x| \leqq r$ and $|y| \leqq (1 - s)r$, and maps $\bar{B}_r(0)$ into itself because, from the estimate

$$\begin{aligned} |f(x) - x| = |f(x) - f(0) - f'(0)x| &\leqq |x| \sup |f'(z) - f'(0)| \\ &\leqq sr, \end{aligned}$$

we obtain

$$|g_y(x)| \leqq sr + (1 - s)r = r.$$

Furthermore, $g_y$ is a shrinking map because, from the mean value theorem, we get

$$\begin{aligned} |g_y(x_1) - g_y(x_2)| &= |x_1 - x_2 - (f(x_1) - f(x_2))| \\ &= |x_1 - x_2 - f'(0)(x_1 - x_2) + \delta(x_1, x_2)| \\ &= |\delta(x_1, x_2)| \end{aligned}$$

where

$$|\delta(x_1, x_2)| \leqq |x_1 - x_2| \sup |f'(z) - f'(0)|$$
$$\leqq s|x_1 - x_2|.$$

Hence $g_y$ has a unique fixed point $x \in \bar{B}_r(0)$ which is such that $f(x) = y$. This proves the lemma.

## XIV, §2. THE IMPLICIT MAPPING THEOREM

Its statement is as follows.

**Theorem 2.1.** *Let $U$, $V$ be open sets in Banach spaces $E$, $F$ respectively, and let*

$$f: U \times V \to G$$

*be a $C^p$ mapping. Let $(a, b) \in U \times V$, and assume that*

$$D_2 f(a, b): F \to G$$

*is a toplinear isomorphism. Let $f(a, b) = 0$. Then there exists a continuous map $g: U_0 \to V$ defined on an open neighborhood $U_0$ of $a$ such that $g(a) = b$ and such that*

$$f(x, g(x)) = 0$$

*for all $x \in U_0$. If $U_0$ is taken to be a sufficiently small ball, then $g$ is uniquely determined, and is also of class $C^p$.*

*Proof.* Let $\lambda = D_2 f(a, b)$. Replacing $f$ by $\lambda^{-1} \circ f$ we may assume without loss of generality that $D_2 f(a, b)$ is the identity. Consider the map

$$\varphi: U \times V \to E \times G$$

given by

$$\varphi(x, y) = (x, f(x, y)).$$

Then the derivative of $\varphi$ at $(a, b)$ is immediately computed to be represented by the matrix

$$D\varphi(a, b) = \begin{pmatrix} I_E & O \\ D_1 f(a, b) & D_2 f(a, b) \end{pmatrix} = \begin{pmatrix} I_E & O \\ D_1 f(a, b) & I_G \end{pmatrix}$$

whence $\varphi$ is locally invertible at $(a, b)$ since the inverse of $D\varphi(a, b)$ exists and is the matrix

$$\begin{pmatrix} I_E & O \\ -D_1 f(a, b) & I_G \end{pmatrix}.$$

We denote the local inverse of $\varphi$ by $\psi$. We can write

$$\psi(x, z) = (x, h(x, z))$$

where $h$ is some mapping of class $C^p$. We define

$$g(x) = h(x, 0).$$

Then certainly $g$ is of class $C^p$ and

$$\begin{aligned}(x, f(x, g(x))) = \varphi(x, g(x)) &= \varphi(x, h(x, 0)) \\ &= \varphi(\psi(x, 0)) = (x, 0).\end{aligned}$$

This proves the existence of a $C^p$ map $g$ satisfying our requirements.

Now for the uniqueness, suppose that $g_0$ is a continuous map defined near $a$ such that $g_0(a) = b$ and $f(x, g_0(x)) = 0$ for all $x$ near $a$. Then $g_0(x)$ is near $b$ for such $x$, and hence

$$\varphi(x, g_0(x)) = (x, 0).$$

Since $\varphi$ is invertible near $(a, b)$ it follows that there is a unique point $(x, y)$ near $(a, b)$ such that $\varphi(x, y) = (x, 0)$. Let $U_0$ be a small ball on which $g$ is defined. If $g_0$ is also defined on $U_0$, then the above argument shows that $g$ and $g_0$ coincide on some smaller neighborhood of $a$. Let $x \in U_0$ and let $v = x - a$. Consider the set of those numbers $t$ with $0 \leqq t \leqq 1$ such that $g(a + tv) = g_0(a + tv)$. This set is not empty. Let $s$ be its least upper bound. By continuity, we have $g(a + sv) = g_0(a + sv)$. If $s < 1$, we can apply the existence and that part of the uniqueness just proved to show that $g$ and $g_0$ are in fact equal in a neighborhood of $a + sv$. Hence $s = 1$, and our uniqueness statement is proved, as well as the theorem.

**Note.** The particular value $f(a, b) = 0$ in the preceding theorem is irrelevant. If $f(a, b) = c$ for some $c \neq 0$, then the above proof goes through replacing 0 by $c$ everywhere.

## XIV, §3. EXISTENCE THEOREM FOR DIFFERENTIAL EQUATIONS

Let $E$ be a Banach space and $U$ an open set in $E$. By a **vector field** on $U$ we simply mean a mapping $f: U \to E$, which we interpret as assigning a vector to each point of $U$. We shall assume our vector field is $C^p$ with $p \geqq 1$. Let $x_0$ be a point of $U$. An **integral curve** for $f$ with **initial**

**condition** $x_0$ is a mapping of class $C^r$ ($r \geqq 1$)

$$\alpha: J \to U$$

defined on an open interval $J$ containing 0, such that $\alpha(0) = x_0$ and such that

$$\alpha'(t) = f(\alpha(t)).$$

We visualize this as saying that the velocity (tangent) vector of the curve $\alpha$ at a point is equal to the vector associated to that point by the vector field. We observe that an integral curve can also be viewed as a solution of the integral equation

$$\alpha(t) = x_0 + \int_0^t f(\alpha(s))\, ds.$$

Namely, any solution of this integral equation is obviously an integral curve of $f$ with the specified initial condition, and conversely, such an integral curve satisfies the integral equation. Furthermore, we observe that an integral curve for $f$ is then necessarily of class $C^{p+1}$, by induction.

We shall say that $f$ satisfies a **Lipschitz condition** on $U$ if there exists a number $K > 0$ such that

$$|f(x) - f(y)| \leqq K|x - y|$$

for all $x$, $y \in U$. We then call $K$ a **Lipschitz constant for** $f$. If $f$ is of class $C^1$, it follows at once from the mean value theorem that $f$ is locally Lipschitz, that is Lipschitz in the neighborhood of every point, and that it is bounded on such a neighborhood.

Let $f: U \to E$ be a vector field and $x_0 \in U$. By a **local flow** at $x_0$ we mean a mapping

$$\alpha: J \times U_0 \to U$$

where $J$ is an open interval containing 0, and $U_0$ is an open subset of $U$ containing $x_0$, such that for each $x$ in $U_0$ the map

$$t \mapsto \alpha_x(t) = \alpha(t, x)$$

is an integral curve for $f$ with initial condition $x$ (namely $\alpha(0, x) = x$). We define a local flow with the eventual intent to analyze its dependence on $x$. However, for this section, the occurrence of $x$ is still incidental, and is introduced only to get some uniformity results. We shall prove that a local flow always exists if $f$ satisfies a Lipschitz condition.

**Theorem 3.1.** *Let $f: U \to E$ be a vector field satisfying a Lipschitz condition with constant $K > 0$. Let $x_0 \in U$. Let $0 < a < 1$, assume that the closed ball $\bar{B}_{2a}(x_0)$ is contained in $U$, and that $f$ is bounded by a constant $L > 0$ on this ball. If $b$ is a number $> 0$ such that $b < a/L$ and $b < 1/K$, then there exists a unique local flow*

$$\alpha: J_b \times B_a(x_0) \to U$$

*where $J_b$ is the open interval $-b < t < b$, and $B_a(x_0)$ is the open ball of radius $a$ centered at $x_0$.*

*Proof.* Let $I_b$ be the closed interval $-b \leqq t \leqq b$ and let $x$ be a point in $\bar{B}_a(x_0)$. Let $M$ be the set of continuous maps

$$\alpha: I_b \to \bar{B}_{2a}(x_0)$$

of the closed interval into the closed ball of center $x_0$ and radius $2a$, such that $\alpha(0) = x$. We view $M$ as a subset of the space of continuous maps of $I_b$ into $E$, with the sup norm. Then $M$ is complete. For each $\alpha$ in $M$ we define the curve $S\alpha$ by

$$S\alpha(t) = x + \int_0^t f(\alpha(u))\, du.$$

Then $S\alpha$ is certainly continuous, and $S\alpha(0) = x$. The distance of any point of $S\alpha$ from $x$ is bounded by the norm of the integral, and we have the estimate

$$|S\alpha(t) - x| \leqq bL < a.$$

Hence $S\alpha$ lies in $M$, so $S$ maps $M$ into itself. Furthermore, $S$ is a shrinking map, because for $\alpha$, $\beta$ in $M$ we have

$$\begin{aligned}\|S\alpha - S\beta\| &\leqq b \sup |f(\alpha(u)) - f(\beta(u))| \\ &\leqq bK\|\alpha - \beta\|.\end{aligned}$$

We can now apply the shrinking lemma to conclude the proof of our theorem.

If we fix the initial condition $x$, then each integral curve $\alpha_x$ is of course differentiable. However, we shall be interested in the dependence on $x$, and it is already easy to show continuity.

**Corollary 3.2.** *The local flow $\alpha$ in Theorem 3.1 is continuous. Furthermore, the map $x \mapsto \alpha_x$ of $\bar{B}_a(x_0)$ into the space of curves is continuous, and in fact satisfies a Lipschitz condition.*

*Proof.* The second statement obviously implies the first. So fix $x$ in $\bar{B}_a(x_0)$ and take $y$ close to $x$ in $\bar{B}_a(x_0)$. We let $S_x$ be the shrinking map of the theorem, corresponding to the initial condition $x$. Then

$$\|\alpha_x - S_y\alpha_x\| = \|S_x\alpha_x - S_y\alpha_x\| \leqq |x - y|.$$

Let $C = bK$ so $0 < C < 1$. Then

$$\begin{aligned}\|\alpha_x - S_y^n\alpha_x\| &\leqq \|\alpha_x - S_y\alpha_x\| + \|S_y\alpha_x - S_y^2\alpha_x\| + \cdots + \|S_y^{n-1}\alpha_x - S_y^n\alpha_x\| \\ &\leqq (1 + C + \cdots + C^{n-1})|x - y|.\end{aligned}$$

Since the limit of $S_y^n\alpha_x$ is equal to $\alpha_y$ as $n$ goes to infinity, the continuity of the map $x \mapsto \alpha_x$ follows at once. In fact, the map satisfies a Lipschitz condition as stated.

It is easy to formulate a uniqueness theorem for integral curves over their whole domain of definition.

**Theorem 3.3.** *Let $U$ be open in $E$ and let $f: U \to E$ be a vector field of class $C^p$, $p \geqq 1$. Let*

$$\alpha_1: J_1 \to U \qquad \textit{and} \qquad \alpha_2: J_2 \to U$$

*be two integral curves for $f$ with the same initial condition $x_0$. Then $\alpha_1$ and $\alpha_2$ are equal on $J_1 \cap J_2$.*

*Proof.* Let $Q$ be the set of numbers $b$ such that $\alpha_1(t) = \alpha_2(t)$ for $0 \leqq t < b$. Then $Q$ contains some number $b > 0$ by the local uniqueness theorem. If $Q$ is not bounded from above, the equality of $\alpha_1(t)$ and $\alpha_2(t)$ for all $t > 0$ follows at once. If $Q$ is bounded from above, let $b$ be its least upper bound. We must show that $b$ is the right end point of $J_1 \cap J_2$. Suppose that this is not the case. Define curves $\beta_1$ and $\beta_2$ near 0 by

$$\beta_1(t) = \alpha_1(b + t) \qquad \text{and} \qquad \beta_2(t) = \alpha_2(b + t).$$

Then $\beta_1$ and $\beta_2$ are integral curves of $f$ with the initial conditions $\alpha_1(b)$ and $\alpha_2(b)$ respectively. The values $\beta_1(t)$ and $\beta_2(t)$ are equal for small negative $t$ because $b$ is the least upper bound of $Q$. By continuity it follows that $\alpha_1(b) = \alpha_2(b)$, and finally we see from the local uniqueness theorem that

$$\beta_1(t) = \beta_2(t)$$

for all $t$ in some neighborhood of 0, whence $\alpha_1$ and $\alpha_2$ are equal in a neighborhood of $b$, contradicting the fact that $b$ is a least upper bound

of $Q$. We can argue the same way towards the left end points, and thus prove our theorem.

For each $x \in U$, let $J(x)$ be the union of all open intervals containing 0 on which integral curves for $f$ are defined, with initial condition equal to $x$. Then Theorem 3.3 allows us to define the integral curve uniquely on all of $J(x)$.

**Remark.** The choice of 0 as the initial time value is made for convenience. From Theorem 3.3 one obtains at once (making a time translation) the analogous statement for an integral curve defined on any open interval; in other words, if $J_1$, $J_2$ do not necessarily contain 0, and $t_0$ is a point in $J_1 \cap J_2$ such that $\alpha_1(t_0) = \alpha_2(t_0)$, and also we have the differential equations

$$\alpha_1'(t) = f(\alpha_1(t)) \qquad \text{and} \qquad \alpha_2'(t) = f(\alpha_2(t)),$$

then $\alpha_1$ and $\alpha_2$ are equal on $J_1 \cap J_2$. One can also repeat the proof of Theorem 3.3 in this case.

In practice, one meets vector fields which may be time dependent, and also depend on parameters. We discuss these to show that their study reduces to the study of the standard case.

### Time-Dependent Vector Fields

Let $J$ be an open interval, $U$ open in a Banach space $E$, and

$$f\colon J \times U \to E$$

a $C^p$ map, which we view as depending on time $t \in J$. Thus for each $t$, the map $x \mapsto f(t, x)$ is a vector field on $U$. Define

$$\bar{f}\colon J \times U \to \mathbf{R} \times E$$

by

$$\bar{f}(t, x) = (1, f(t, x))$$

and view $\bar{f}$ as a time-independent vector field on $J \times U$. Let $\bar{\alpha}$ be its flow, so that

$$D_1 \bar{\alpha}(t, s, x) = \bar{f}(\bar{\alpha}(t, s, x)), \qquad \bar{\alpha}(0, s, x) = (s, x).$$

We note that $\bar{\alpha}$ has its values in $J \times U$ and thus can be expressed in terms of two components. In fact, it follows at once that we can write $\bar{\alpha}$ in the form

$$\bar{\alpha}(t, s, x) = (t + s, \bar{\alpha}_2(t, s, x)).$$

Then $\bar{\alpha}_2$ satisfies the differential equation

$$D_1\bar{\alpha}_2(t, s, x) = f(t + s, \bar{\alpha}_2(t, s, x))$$

as we see from the definition of $\bar{f}$. Let

$$\beta(t, x) = \bar{\alpha}_2(t, 0, x).$$

Then $\beta$ is a flow for $f$, i.e. satisfies the differential equation

$$D_1\beta(t, x) = f(t, \beta(t, x)), \qquad \beta(0, x) = x.$$

Given $x \in U$, any value of $t$ such that $\alpha$ is defined at $(t, x)$ is also such that $\bar{\alpha}$ is defined at $(t, 0, x)$ because $\alpha_x$ and $\beta_x$ are integral curves of the same vector field, with the same initial condition, hence are equal. Thus the study of time-dependent vector fields is reduced to the study of time-independent ones.

**Dependence on Parameters**

Let $V$ be open in some space $F$ and let

$$g: J \times V \times U \to E$$

be a map which we view as a time-dependent vector field on $U$, also depending on parameters in $V$. We define

$$G: J \times V \times U \to F \times E$$

by

$$G(t, z, y) = (0, g(t, z, y))$$

for $t \in J$, $z \in V$, and $y \in U$. This is now a time-dependent vector field on $V \times U$. A local flow for $G$ depends on three variables, say $\beta(t, z, y)$, with initial condition $\beta(0, z, y) = (z, y)$. The map $\beta$ has two components, and it is immediately clear that we can write

$$\beta(t, z, y) = (z, \alpha(t, z, y))$$

for some map $\alpha$ depending on three variables. Consequently $\alpha$ satisfies the differential equation

$$D_1\alpha(t, z, y) = g(t, z, \alpha(t, z, y)), \qquad \alpha(0, z, y) = y,$$

which gives the flow of our original vector field $g$ depending on the

parameters $z \in V$. This procedure reduces the study of differential equations depending on parameters to those which are independent of parameters.

## XIV, §4. LOCAL DEPENDENCE ON INITIAL CONDITIONS

We shall now see that the map $x \mapsto \alpha_x$ in fact depends differentiably on $x$. The proof, which depends on a very simple application of the implicit mapping theorem in Banach spaces, was found independently by Pugh and Robbin.

Let $U$ be open in $E$ and let $f: U \to E$ be a $C^p$ map (which we call a vector field). Let $b > 0$ and let $I_b$ be the closed interval of radius $b$ centered at 0. Let

$$F = C^0(I_b, E)$$

be the Banach space of continuous maps of $I_b$ into $E$. We let $V$ be the subset of $F$ consisting of all continuous curves

$$\sigma: I_b \to U$$

mapping $I_b$ into our open set $U$. Then it is clear that $V$ is open in $F$ because for each curve $\sigma$ the image $\sigma(I_b)$ is compact, hence at a finite distance from the complement of $U$, so that any curve close to it is also contained in $U$.

We define a map

$$T: U \times V \to F$$

by

$$T(x, \sigma) = x + \int_0 f \circ \sigma - \sigma.$$

Here we omit the dummy variable of integration, and $x$ stands for the constant curve with value $x$. If we evaluate the curve $T(x, \sigma)$ at $t$, then by definition we have

$$T(x, \sigma)(t) = x + \int_0^t f(\sigma(u))\, du - \sigma(t).$$

**Lemma 4.1.** *The map $T$ is of class $C^p$, and its second partial derivative is given by the formula*

$$D_2 T(x, \sigma) = \int_0 Df \circ \sigma - I$$

*where I is the identity. In terms of t, this reads:*

$$D_2 T(x, \sigma)h(t) = \int_0^t Df(\sigma(u))h(u)\,du - h(t).$$

*Proof.* It is clear that the first partial derivative $D_1 T$ exists and is continuous, in fact $C^\infty$, being linear in $x$ up to a translation. To determine the second partial, we apply the definition of the derivative. The derivative of the map $\sigma \mapsto \sigma$ is of course the identity. We have to get the derivative with respect to $\sigma$ of the integral expression. We have for small $h$:

$$\left\| \int_0 f \circ (\sigma + h) - \int_0 f \circ \sigma - \int_0 (Df \circ \sigma)h \right\|$$
$$\leqq \int_0 |f \circ (\sigma + h) - f \circ \sigma - (Df \circ \sigma)h|.$$

We estimate the expression inside the integral at each point $u$, with $u$ between 0 and the upper variable of integration. From the mean value theorem, we get

$$|f(\sigma(u) + h(u)) - f(\sigma(u)) - Df(\sigma(u))h(u)|$$
$$\leqq \|h\| \sup |Df(z_u) - Df(\sigma(u))|$$

where the sup is taken over all points $z_u$ on the segment between $\sigma(u)$ and $\sigma(u) + h(u)$. Since $Df$ is continuous, and using the fact that the image of the curve $\sigma(I_b)$ is compact, we conclude (as in the case of uniform continuity) that as $\|h\| \to 0$, the expression

$$\sup |Df(z_u) - Df(\sigma(u))|$$

also goes to 0. (Put the $\varepsilon$ and $\delta$ in yourself.) By definition, this gives us the derivative of the integral expression in $\sigma$. The derivative of the final term is obviously the identity, so this proves that $D_2 T$ is given by the formula which we wrote down.

This derivative does not depend on $x$. It is continuous in $\sigma$. Namely, we have

$$D_2 T(x, \tau) - D_2 T(x, \sigma) = \int_0 [Df \circ \tau - Df \circ \sigma].$$

If $\sigma$ is fixed and $\tau$ is close to $\sigma$, then $Df \circ \tau - Df \circ \sigma$ is small, as one proves easily from the compactness of $\sigma(I_b)$, as in the proof of uniform continuity. Thus $D_2 T$ is continuous. By Theorem 7.1 of Chapter XIII we now conclude that $T$ is of class $C^1$.

The derivative of $D_2 T$ with respect to $\sigma$ can again be computed as before if $Df$ is itself of class $C^1$, and thus by induction, if $f$ is of class $C^p$ we conclude that $D_2 T$ is of class $C^{p-1}$ so that by Theorem 7.1 of Chapter XIII, we conclude that $T$ itself is of class $C^p$. This proves our lemma.

We observe that a solution of the equation

$$T(x, \sigma) = 0$$

is precisely an integral curve for the vector field, with initial condition equal to $x$. Thus we are in a situation where we want to apply the implicit mapping theorem.

**Lemma 4.2.** *Let $x_0 \in U$. Let $a > 0$ be such that $Df$ is bounded, say by a number $C_1 > 0$, on the ball $B_a(x_0)$ (we can always find such a since $Df$ is continuous at $x_0$). Let $b < 1/C_1$. Then $D_2 T(x, \sigma)$ is invertible for all $(x, \sigma)$ in $B_a(x_0) \times V$.*

*Proof.* We have an estimate

$$\left| \int_0^t Df(\sigma(u))h(u)\, du \right| \leqq bC_1 \|h\|.$$

This means that

$$|D_2 T(x, \sigma) + I| < 1,$$

and hence that $D_2 T(x, \sigma)$ is invertible, as a continuous linear map, thus proving Lemma 4.2.

**Theorem 4.3.** *Let $p$ be a positive integer, and let $f: U \to E$ be a $C^p$ vector field. Let $x_0 \in U$. Then there exist numbers $a, b > 0$ such that the local flow*

$$\alpha: J_a \times B_a(x_0) \to U$$

*is of class $C^p$.*

*Proof.* We take $a$ so small and then $b$ so small that the local flow exists and is uniquely determined by Theorem 3.1. We then take $b$ smaller and $a$ smaller so as to satisfy the hypotheses of Lemma 4.2. We can then apply the implicit mapping theorem to conclude that the map $x \mapsto \alpha_x$ is of class $C^p$. Of course, we have to consider the flow $\alpha$ and still must show that $\alpha$ itself is of class $C^p$. It will suffice to prove that $D_1\alpha$ and $D_2\alpha$ are of class $C^{p-1}$, by Theorem 7.1 of Chapter XIII. We first consider the case $p = 1$.

We could derive the continuity of $\alpha$ from Corollary 3.2 but we can also get it as an immediate consequence of the continuity of the map

$x \mapsto \alpha_x$. Indeed, fixing $(s, y)$ we have

$$\begin{aligned}|\alpha(t, x) - \alpha(s, y)| &\leqq |\alpha(t, x) - \alpha(t, y)| + |\alpha(t, y) - \alpha(s, y)| \\ &\leqq \|\alpha_x - \alpha_y\| + |\alpha_y(t) - \alpha_y(s)|.\end{aligned}$$

Since $\alpha_y$ is continuous (being differentiable), we get the continuity of $\alpha$. Since

$$D_1\alpha(t, x) = f(\alpha(t, x)),$$

we conclude that $D_1\alpha$ is a composite of continuous maps, whence continuous.

Let $\varphi$ be the derivative of the map $x \mapsto \alpha_x$, so that

$$\varphi\colon B_a(x_0) \to L(E, C^0(I_b, E)) = L(E, F)$$

is of class $C^{p-1}$. Then

$$\alpha_{x+w} - \alpha_x = \varphi(x)w + |w|\psi(w)$$

where $\psi(w) \to 0$ as $w \to 0$. Evaluating at $t$, we find

$$\alpha(t, x + w) - \alpha(t, x) = (\varphi(x)w)(t) + |w|\psi(w)(t),$$

and from this we see that

$$D_2\alpha(t, x)w = (\varphi(x)w)(t)$$

Then

$$\begin{aligned}|D_2\alpha(t, x)w - D_2\alpha(s, y)w| \leqq\ &|(\varphi(x)w)(t) - (\varphi(y)w)(t)| \\ &+ |(\varphi(y)w)(t) - (\varphi(y)w)(s)|.\end{aligned}$$

The first term on the right is bounded by

$$|\varphi(x) - \varphi(y)|\,|w|$$

so that

$$|D_2\alpha(t, x) - D_2\alpha(t, y)| \leqq |\varphi(x) - \varphi(y)|.$$

We shall prove below that

$$|(\varphi(y)w)(t) - (\varphi(y)w)(s)|$$

is uniformly small with respect to $w$ when $s$ is close to $t$. This proves the continuity of $D_2\alpha$, and concludes the proof that $\alpha$ is of class $C^1$.

The following proof that $|(\varphi(y)w)(t) - (\varphi(y)w)(s)|$ is uniformly small was shown to be by Professor Yamanaka. We have

$$\alpha(t, x) = x + \int_0^t f(\alpha(u, x))\, du. \tag{1}$$

Replacing $x$ with $x + \lambda w$ ($w \in E$, $\lambda \neq 0$), we obtain

$$\alpha(t, x + \lambda w) = x + \lambda w + \int_0^t f(\alpha(u, x + \lambda w))\, du. \tag{2}$$

Therefore

$$\frac{\alpha(t, x + \lambda w) - \alpha(t, x)}{\lambda} = w + \int_0^t \frac{1}{\lambda}[f(\alpha(u, x + \lambda w)) - f(\alpha(u, x))]\, du. \tag{3}$$

On the other hand, we have already seen in the proof of Theorem 4.3 that

$$\alpha(t, x + \lambda w) - \alpha(t, x) = \lambda(\varphi(x)w)(t) + |\lambda|\,|w|\,\psi(\lambda w)(t). \tag{4}$$

Substituting (4) in (3), we obtain:

$$\begin{aligned}(\varphi(x)w)(t) + \frac{|\lambda|}{\lambda}|w|\,\psi(\lambda w)(t) &= w + \int_0^t \frac{1}{\lambda}[f(\alpha(u, x + \lambda w)) - f(\alpha(u, x))]\, du \\ &= w + \int_0^t \int_0^1 G(u, \lambda, v)\, dv\, du,\end{aligned}$$

where

$$G(u, \lambda, v) = Df(\alpha(u, x) + v\varepsilon_1(\lambda))((\varphi(x)w)(u) + \varepsilon_2(\lambda))$$

with

$$\varepsilon_1(\lambda) = \lambda(\varphi(x)w)(u) + |\lambda|\,|w|\,\psi(\lambda w)(u), \qquad \varepsilon_2(\lambda) = \frac{|\lambda|}{\lambda}\psi(\lambda w)(u).$$

Letting $\lambda \to 0$, we have

$$(\varphi(x)w)(t) = w + \int_0^t Df(\alpha(u, x))(\varphi(x)w)(u)\, du. \tag{5}$$

By (5) we have

$$|(\varphi(x)w)(t) - (\varphi(x)w)(s)| \leqq \left| \int_s^t Df(\alpha(u, x))(\varphi(x)w)(u))\, du \right|$$
$$\leqq bC_1 |\varphi(x)| \cdot |w| \cdot |t - s|,$$

from which we immediately obtain the desired uniformity.

We have

$$\alpha(t, x) = x + \int_0^t f(\alpha(u, x))\, du.$$

We can differentiate under the integral sign with respect to the parameter $x$ and thus obtain

$$D_2\alpha(t, x) = I + \int_0^t Df(\alpha(u, x))D_2\alpha(u, x)\, du,$$

where $I$ is a constant linear map (the identity). Differentiating with respect to $t$ yields the linear differential equation satisfied by $D_2\alpha$, namely

$$\boxed{D_1D_2\alpha(t, x) = Df(\alpha(t, x))D_2\alpha(t, x)}$$

and this differential equation depends on time and parameters. We have seen in §3 how such equations can be reduced to the ordinary case. We now conclude that locally, by induction, $D_2\alpha$ is of class $C^{p-1}$ since $Df$ is of class $C^{p-1}$. Since

$$D_1\alpha(t, x) = f(\alpha(t, x)),$$

we conclude by induction that $D_1\alpha$ is $C^{p-1}$. Hence $\alpha$ is of class $C^p$ by Theorem 7.1 of Chapter 5. Note that each time we use induction, the domain of the flow may shrink. In the next section, we shall prove a more global result. In any case, we have proved Theorem 4.3.

## XIV, §5. GLOBAL SMOOTHNESS OF THE FLOW

Let $U$ be open in a Banach space $E$, and let $f: U \to E$ be a $C^p$ vector field. We let $J(x)$ be the domain of the integral curve with initial condition equal to $x$.

Let $\mathfrak{D}(f)$ be the set of all points $(t, x)$ in $\mathbf{R} \times U$ such that $t$ lies in

$J(x)$. Then we have a map

$$\alpha\colon \mathfrak{D}(f) \to U$$

defined on all of $\mathfrak{D}(f)$, letting $\alpha(t, x) = \alpha_x(t)$ be the integral curve on $J(x)$ having $x$ as initial condition. We call this the **flow** determined by $f$, and we call $\mathfrak{D}(f)$ its **domain of definition.**

**Theorem 5.1.** *Let $f\colon U \to E$ be a $C^p$ vector field on the open set $U$ of $E$, and let $\alpha$ be its flow. Abbreviate $\alpha(t, x)$ by $tx$ if $(t, x)$ is in the domain of definition of the flow. Let $x \in U$. If $t_0$ lies in $J(x)$, then*

$$J(t_0 x) = J(x) - t_0$$

*(translation of $J(x)$ by $-t_0$), and we have for all $t$ in $J(x) - t_0$:*

$$t(t_0 x) = (t + t_0)x.$$

*Proof.* The two curves defined by

$$t \mapsto \alpha(t, \alpha(t_0, x)) \qquad \text{and} \qquad t \mapsto \alpha(t + t_0, x)$$

are integral curves of the same vector field, with the same initial condition $t_0 x$ at $t = 0$. Hence they have the same domain of definition $J(t_0 x)$. Hence $t_1$ lies in $J(t_0 x)$ if and only if $t_1 + t_0$ lies in $J(x)$. This proves the first assertion. The second assertion comes from the uniqueness of the integral curve having given initial condition, whence the theorem follows.

**Theorem 5.2.** *If $f$ is of class $C^p$ (with $p \leqq \infty$), then its flow is of class $C^p$ on its domain of definition.*

*Proof.* First let $p$ be an integer $\geqq 1$. We know that the flow is locally of class $C^p$ at each point $(0, x)$, by Theorem 4.3. Let $x_0 \in U$ and let $J(x_0)$ be the maximal interval of definition of the integral curve having $x_0$ as initial condition. Let $\mathfrak{D}(f)$ be the domain of definition of the flow, and let $\alpha$ be the flow. Let $Q$ be the set of numbers $b > 0$ such that for each $t$ with $0 \leqq t < b$ there exists an open interval $J$ containing $t$ and an open set $V$ containing $x_0$ such that $J \times V$ is contained in $\mathfrak{D}(f)$ and such that $\alpha$ is of class $C^p$ on $J \times V$. Then $Q$ is not empty by Theorem 4.3. If $Q$ is not bounded from above, then we are done looking toward the right end point of $J(x_0)$. If $Q$ is bounded from above, we let $b$ be its least upper bound. We must prove that $b$ is the right end point of $J(x_0)$. Suppose that this is not the case. Then $\alpha(b, x_0)$ is defined. Let $x_1 = \alpha(b, x_0)$. By the local Theorem 4.3, we have a unique local flow at $x_1$, which we

denote by $\beta$:

$$\beta\colon J_a \times B_a(x_1) \to U, \qquad \beta(0, x) = x,$$

defined for some open interval $J_a = (-a, a)$ and open ball $B_a(x_1)$ of radius $a$ centered at $x_1$. Let $\delta$ be so small that whenever $b - \delta < t < b$ we have

$$\alpha(t, x_0) \in B_{a/4}(x_1).$$

We can find such $\delta$ because

$$\lim_{t \to b} \alpha(t, x_0) = x_1$$

by continuity. Select a point $t_1$ such that $b - \delta < t_1 < b$. By the hypothesis on $b$, we can select an open interval $J_1$ containing $t_1$ and an open set $U_1$ containing $x_0$ so that

$$\alpha\colon J_1 \times U_1 \to B_{a/2}(x_1)$$

maps $J_1 \times U_1$ into $B_{a/2}(x_1)$. We can do this because $\alpha$ is continuous at $(t_1, x_0)$, being in fact $C^p$ at this point. If $|t - t_1| < a$ and $x \in U_1$, we define

$$\varphi(t, x) = \beta(t - t_1, \alpha(t_1, x)).$$

Then

$$\varphi(t_1, x) = \beta(0, \alpha(t_1, x)) = \alpha(t_1, x)$$

and

$$\begin{aligned} D_1\varphi(t, x) &= D_1\beta(t - t_1, \alpha(t_1, x)) \\ &= f(\beta(t - t_1, \alpha(t_1, x))) \\ &= f(\varphi(t, x)). \end{aligned}$$

Hence both $\varphi_x$ and $\alpha_x$ are integral curves for $f$ with the same value at $t_1$. They coincide on any interval on which they are defined by Theorem 3.3. If we take $\delta$ very small compared to $a$, say $\delta < a/4$, we see that $\varphi$ is an extension of $\alpha$ to an open set containing $(t_1, x_0)$, and also containing $(b, x_0)$. Furthermore, $\varphi$ is of class $C^p$, thus contradicting the fact that $b$ is strictly smaller than the end point of $J(x_0)$. Similarly, one proves the analogous statement on the other side, and we therefore see that $\mathfrak{D}(f)$ is open in $\mathbf{R} \times U$ and that $\alpha$ is of class $C^p$ on $\mathfrak{D}(f)$, as was to be shown.

The idea of the above proof is very simple geometrically. We go as far to the right as possible in such a way that the given flow $\alpha$ is of class $C^p$ locally at $(t, x_0)$. At the point $\alpha(b, x_0)$ we then use the flow $\beta$ to extend differentiably the flow $\alpha$ in case $b$ is not the right-hand point of

$J(x_0)$. The flow $\beta$ at $\alpha(b, x_0)$ has a fixed local domain of definition, and we simply take $t$ close enough to $b$ so that $\beta$ gives an extension of $\alpha$, as described in the above proof.

Of course, if $f$ is of class $C^\infty$, then we have shown that $\alpha$ is of class $C^p$ for each positive integer $p$, and therefore the flow is also of class $C^\infty$.

## XIV, §6. EXERCISES

1. (*Tate*) Let $E$, $F$ be complete normed vector spaces. Let $f: E \to F$ be a map having the following property. There exists a number $C > 0$ such that for all $x, y \in E$ we have

$$|f(x + y) - f(x) - f(y)| \leqq C.$$

Show that there exists a unique linear map $g: E \to F$ such that $g - f$ is bounded for the sup norm. [*Hint*: Show that the limit

$$g(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists.]

2. Generalize Exercise 1 to the bilinear case. In other words, let $f: E \times F \to G$ be a map and assume that there is a constant $C$ such that

$$|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)| \leqq C,$$
$$|f(x, y_1 + y_2) - f(x, y_1) - f(x, y_2)| \leqq C$$

for all $x, x_1, x_2 \in E$ and $y, y_1, y_2 \in F$. Show that there exists a unique bilinear map $g: E \times F \to G$ such that $f - g$ is bounded for the sup norm.

3. Prove the following statement. Let $\bar{B}_r$ be the closed ball of radius $r$ centered at 0 in $E$. Let $f: \bar{B}_r \to E$ be a map such that:
(a) $|f(x) - f(y)| \leqq b|x - y|$ with $0 < b < 1$.
(b) $|f(0)| \leqq r(1 - b)$.
Show that there exists a unique point $x \in \bar{B}_r$ such that $f(x) = x$.

4. With notation as in Exercise 3, let $g$ be another map of $\bar{B}_r$ into $E$ and let $c > 0$ be such that $|g(x) - f(x)| \leqq c$ for all $x$. Assume that $g$ has a fixed point $x_2$, and let $x_1$ be the fixed point of $f$. Show that $|x_2 - x_1| \leqq c/(1 - b)$.

5. Let $K$ be a continuous function of two variables, defined for $(x, y)$ in the square $a \leqq x \leqq b$ and $a \leqq y \leqq b$. Assume that $\|K\| \leqq C$ for some constant $C > 0$. Let $f$ be a continuous function on $[a, b]$ and let $r$ be a real number satisfying the inequality

$$|r| < \frac{1}{C(b - a)}.$$

Show that there is one and only one function $g$ continuous on $[a, b]$ such

that

$$f(x) = g(x) + r \int_a^b K(t, x)g(t)\,dt.$$

6. **Newton's method.** This method serves the same purpose as the shrinking lemma but sometimes is more efficient and converges more rapidly. It is used to find zeros of mappings.

   Let $B_r$ be a ball of radius $r$ centered at a point $x_0 \in E$. Let $f\colon B_r \to E$ be a $C^2$ mapping, and assume that $f''$ is bounded by some number $C \geqq 1$ on $B_r$. Assume that $f'(x)$ is invertible for all $x \in B_r$ and that $|f'(x)^{-1}| \leqq C$ for all $x \in B_r$. Show that there exists a number $\delta$ depending only on $C$ such that if $|f(x_0)| \leqq \delta$ then the sequence defined by

$$x_{n+1} = x_n - f'(x_n)^{-1}f(x_n)$$

   lies in $B_r$ and converges to an element $x$ such that $f(x) = 0$. *Hint:* Show inductively that

$$|x_{n+1} - x_n| \leqq C|f(x_n)|,$$
$$|f(x_{n+1})| \leqq |x_{n+1} - x_n|^2 C,$$

   and hence that

$$|f(x_n)| \leqq C^{3(1+2+\cdots+2^n)}\delta^{2n},$$
$$|x_{n+1} - x_n| \leqq CC^{3(1+2+4+\cdots+2^n)}\delta^{2n}.$$

7. Apply Newton's method to prove the following statement. Assume that $f\colon U \to E$ is of class $C^2$ and that for some point $x_0 \in U$ we have $f(x_0) = 0$ and $f'(x_0)$ is invertible. Show that given $y$ sufficiently close to 0, there exists $x$ close to $x_0$ such that $f(x) = y$. [*Hint*: Consider the map $g(x) = f(x) - y$.]

   *Note.* The point of the Newton method is that it often gives procedure which converges much faster than the procedure of the shrinking lemma. Indeed, the shrinking lemma converges more or less like a geometric series. The Newton method converges with an *exponent* of $2^n$. For an interesting application of the Newton method, see the Nash–Moser implicit mapping theorem [Nas], [Mo 2], [Ha]. See also the partial axiomatization which I gave in [La 4]. These show that the calculus in Banach spaces is insufficient and leads to calculus in Fréchet spaces, where the inverse mapping theorem and existence theorem for differential equations is much more subtle.

8. The following is a reformulation due to Tate of a theorem of Michael Shub.
   (a) Let $n$ be a positive integer, and let $f\colon \mathbf{R} \to \mathbf{R}$ be a differentiable function such that $f'(x) \geqq r > 0$ for all $x$. Assume that $f(x + 1) = f(x) + n$. Show that there exists a strictly increasing continuous map $\alpha\colon \mathbf{R} \to \mathbf{R}$ satisfying

$$\alpha(x + 1) = \alpha(x) + 1$$

   such that

$$f(\alpha(x)) = \alpha(nx).$$

   [*Hint*: Follow Tate's proof. Show that $f$ is continuous, strictly increasing,

and let $g$ be its inverse function. You want to solve $\alpha(x) = g(\alpha(nx))$. Let $M$ be the set of all continuous functions which are increasing (not necessarily strictly) and satisfying $\alpha(x + 1) = \alpha(x) + 1$. On $M$, define the norm

$$\|\alpha\| = \sup_{0 \leqq x \leqq 1} |\alpha(x)|.$$

Let $T: M \to M$ be the map such that

$$(T\alpha)(x) = g(\alpha(nx)).$$

Show that $T$ maps $M$ into $M$ and is a shrinking map. Show that $M$ is complete, and that a fixed point for $T$ solves the problem.] Since one can write

$$nx = \alpha^{-1}(f(\alpha(x))),$$

one says that the map $x \mapsto nx$ is conjugate to $f$. Interpreting this on the circle, one gets the statement originally due to Shub that a differentiable function on the circle, with positive derivative, is conjugate to the $n$-th power for some $n$.

(b) Show that the differentiability condition can be replaced by the weaker condition: There exist numbers $r_1$, $r_2$ with $1 < r_1 < r_2$ such that for all $x \geqq 0$ we have

$$r_1 s \leqq f(x + s) - f(x) \leqq r_2 s.$$

9. Let $M$ be a complete metric space (or a closed subset of a complete normed vector space if you wish), and let $S$ be a topological space. Let $T: S \times M \to M$ be a continuous map, such that for each $u \in S$ the map $T_u: M \to M$ given by $T_u(x) = T(u, x)$ is a shrinking map with constant $K_u$, $0 < K_u < 1$. Assume that there is some $K$ with $0 < K < 1$ such that $K_u \leqq K$ for all $u \in S$. Let $\varphi: S \to M$ be the map such that $\varphi(u)$ is the fixed point of $T_u$. Show that $\varphi$ is continuous.

10. Exercises 10 and 11 develop a special case of a theorem of Anosov, by a proof due to Moser.

First we make some definitions. Let $A: \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map. We say that $A$ is **hyperbolic** if there exist numbers $b > 1$, $c < 1$, and two linearly independent vectors $v$, $w$ in $\mathbf{R}^2$ such that $Av = bv$ and $Aw = cw$. As an example, show that the matrix (linear map)

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

has this property.

Next we introduce the $C^1$ norm. If $f$ is a $C^1$ map, such that both $f$ and $f'$ are bounded, we define the $C^1$ norm to be

$$\|f\|_1 = \max(\|f\|, \|f'\|),$$

where $\| \ \|$ is the usual sup norm. In this case, we also say that $f$ is $C^1$-bounded.

The theorem we are after runs as follows:

**Theorem.** *Let $A: \mathbf{R}^2 \to \mathbf{R}^2$ be a hyperbolic linear map. There exists $\delta$ having the following property. If $f: \mathbf{R}^2 \to \mathbf{R}^2$ is a $C^1$ map such that*

$$\|f - A\|_1 < \delta,$$

*then there exists a continuous bounded map $h: \mathbf{R}^2 \to \mathbf{R}^2$ satisfying the equation*

$$f \circ h = h \circ A.$$

First prove a lemma.

**Lemma.** *Let $M$ be the vector space of continuous bounded maps of $\mathbf{R}^2$ into $\mathbf{R}^2$. Let $T: M \to M$ be the map defined by $Tp = p - A^{-1} \circ p \circ A$. Then $T$ is a continuous linear map, and is invertible.*

To prove the lemma, write

$$p(x) = p^+(x)v + p^-(x)w$$

where $p^+$ and $p^-$ are functions, and note that symbolically,

$$Tp^+ = p^+ - b^{-1}p^+ \circ A,$$

that is $Tp^+ = (I - S)p^+$ where $\|S\| < 1$. So find an inverse for $T$ on $p^+$. Analogously, show that $Tp^- = (I - S_0^{-1})p^-$ where $\|S_0\| < 1$, so that $S_0 T = S_0 - I$ is invertible on $p^-$. Hence $T$ can be inverted componentwise, as it were.

To prove the theorem, write $f = A + g$ where $g$ is $C^1$-small. We want to solve for $h = I + p$ with $p \in M$, satisfying $f \circ h = h \circ A$. Show that this is equivalent to solving

$$Tp = -A^{-1} \circ g \circ h,$$

or equivalently,

$$p = -T^{-1}(A^{-1} \circ g \circ (I + p)).$$

This is then a fixed point condition for the map $R: M \to M$ given by

$$R(p) = -T^{-1}(A^{-1} \circ g \circ (I + p)).$$

Show that $R$ is a shrinking map to conclude the proof.

11. One can formulate a variant of the preceding exercise (actually the very case dealt with by Anosov–Moser). Assume that the matrix $A$ with respect to the standard basis of $\mathbf{R}^2$ has integer coefficients. A vector $z \in \mathbf{R}^2$ is called an **integral** vector if its coordinates are integers. A map $p: \mathbf{R}^2 \to \mathbf{R}^2$ is said to be **periodic** if $p(x + z) = p(x)$ for all $x \in \mathbf{R}^2$ and all integral vectors $z$. Prove:

**Theorem.** *Let $A$ be hyperbolic, with integer coefficients. There exists $\delta$ having the following property. If $g$ is a $C^1$, periodic map, and $\|g\|_1 < \delta$, and if $f = A + g$, then there exists a periodic continuous map $h$ satisfying the equation*

$$f \circ h = h \circ A.$$

*Note.* With only a bounded amount of extra work, one can show that the map $h$ itself is $C^0$-invertible, and so $f = h \circ A \circ h^{-1}$.

12. (a) Let $f$ be a $C^1$ vector field on an open set $U$ in $E$. If $f(x_0) = 0$ for some $x_0 \in U$, if $\alpha: J \to U$ is an integral curve for $f$, and there exists some $t_0 \in J$ such that $\alpha(t_0) = x_0$, show that $\alpha(t) = x_0$ for all $t \in J$. (A point $x_0$ such that $f(x_0) = 0$ is called a critical point of the vector field.)
(b) Let $f$ be a $C^1$ vector field on an open set $U$ of $E$. Let $\alpha: J \to U$ be an integral curve for $f$. Assume that all numbers $t > 0$ are contained in $J$, and that there is a point $P$ in $U$ such that

$$\lim_{t\to\infty} \alpha(t) = P.$$

Prove that $f(P) = 0$. (Exercises 12(a) and 12(b) have many applications, notably when $f = \text{grad } g$ for some function $g$. In this case we see that $P$ is a critical point of the function $g$.)

13. Let $U$ be open in the (real) Hilbert space $E$ and let $g: U \to \mathbf{R}$ be a $C^2$ function. Then $g': U \to L(E, \mathbf{R})$ is a $C^1$ map into the dual space, and we know that $E$ is self dual. Thus there is a $C^1$ map $f: U \to E$ such that

$$g'(x)v = \langle v, f(x)\rangle$$

for all $x \in U$ and $v \in E$. We call $f$ the **gradient** of $g$.

Let $g: U \to \mathbf{R}$ be a function of class $C^2$. Let $x_0 \in U$ and assume that $x_0$ is a critical point of $g$ (that is $g'(x_0) = 0$). Assume also that $D^2g(x_0)$ is negative definite. By definition, take this to mean that there exists a number $c > 0$ such that for all vectors $v$ we have

$$D^2g(x_0)(v, v) \leqq -c|v|^2.$$

Prove that if $x_1$ is a point in the ball $B_r(x_0)$ of radius $r$, centered at $x_0$, and if $r$ is sufficiently small, then the integral curve $\alpha$ of grad $g$ having $x_1$ as initial condition is defined for all $t \geqq 0$ and

$$\lim_{t\to\infty} \alpha(t) = x_0.$$

*Hint*: Let $\psi(t) = (\alpha(t) - x_0)\cdot(\alpha(t) - x_0)$ be the square of the distance from $\alpha(t)$ to $x_0$. Show that $\psi$ is strictly decreasing, and in fact satisfies

$$\psi'(t) \leqq -c\psi(t).$$

Divide by $\psi(t)$ and integrate to see that

$$\log \psi(t) - \log \psi(0) \leqq -ct.$$

14. Let $U$ be open in $E$ and let $f: U \to E$ be a $C^1$ vector field on $U$. Let $x_0 \in U$ and assume that $f(x_0) = v \neq 0$. Let $\alpha$ be a local flow for $f$ at $x_0$. Let $F$ be a subspace of $E$ which is complementary to the one-dimensional space gener-

ated by $v$, that is the map

$$\mathbf{R} \times F \to E$$

given by $(t, y) \mapsto tv + y$ is an invertible continuous linear map.

(a) If $E = \mathbf{R}^n$ show that such a subspace exists. (The general case can be proved by the Hahn–Banach theorem.)

(b) Show that the map $\beta: (t, y) \mapsto (t, x_0 + y)$ is a local $C^1$ isomorphism at $(0, 0)$. Compute $D\beta$ in terms of $D_1\alpha$ and $D_2\alpha$.

(c) The map $\sigma: (t, y) \mapsto x_0 + y + tv$ is obviously a $C^1$ isomorphism, because it is composed of a translation and an invertible linear map. Define locally at $x_0$ the map $\varphi$ by $\varphi = \beta \circ \sigma^{-1}$, so that by definition,

$$\varphi(x_0 + y + tv) = \alpha(t, x_0 + y).$$

Using the chain rule, show that for all $x$ near $x_0$ we have

$$D\varphi(x)v = f(\varphi(x)).$$

In the language of charts (Chapter XXI) this expresses the fact that if a vector field is not zero at a point, then after a change of charts, this vector field can be made to be constant in a neighborhood of that point.

15. Let $J$ be an open interval $(a, b)$ and let $U$ be open in $E$. Let $f: J \times U \to E$ be a continuous map which is Lipschitz on $U$ uniformly for every compact subinterval of $J$. Let $\alpha$ be an integral curve of $f$, defined on a maximal open subinterval $(a_0, b_0)$ of $J$. Assume:
(a) There exists $\varepsilon > 0$ such that the closure $\overline{\alpha((b_0 - \varepsilon, b_0))}$ is contained in $U$.
(b) There exists $C > 0$ such that $|f(t, \alpha(t))| \leqq C$ for all $t$ in $(b_0 - \varepsilon, b_0)$.
Then $b_0 = b$.

16. **Linear differential equations.** Let $J$ be an open interval containing 0, and let $V$ be open in a Banach space $E$. Let $L$ be a Banach space. Let $A: J \times V \to L$ be a continuous map, and let $L \times E \to E$ be a continuous bilinear map. Let $w_0 \in E$. Then there exists a unique map $\lambda: J \times V \to E$, which for each $x \in V$ is a solution of the differential equation

$$D_1\lambda(t, x) = A(t, x)\lambda(t, x), \qquad \lambda(0, x) = w_0.$$

This map $\lambda$ is continuous. [*Hint*: Use Exercise 15. We see that in the linear case, the integral curve is defined over the whole interval $J$.]

17. Let $U$ be open in a Banach space $E$ and let $f: U \to E$ be a $C^1$ vector field. Assume that $f$ is bounded. Let $\alpha$ be an integral curve for $f$, and let $J$ be its maximal interval of definition. Suppose that $J$ does not contain all positive real numbers, and let $b$ be its right end point. Show that

$$\lim_{t \to b} \alpha(t)$$

exists, and that it is a boundary point of $U$. Cf. [La 1] and [La 2] to see Exercises 12–17 worked out.

# PART FIVE

# Functional Analysis

In this part, we present some basic and substantial results of functional analysis, which are extremely widely used. The part splits into essentially independent considerations dealing with Banach spaces in general, and then with the special case of hermitian operators in Hilbert space.

First we have a chapter on some general theorems which extend to Banach spaces some basic algebraic results on finite dimensional spaces, taking into account the linear topology. The algebraic theorems concerning the existence of complementary subspaces, or criteria for a linear map to be surjective, need a more systematic study in light of the additional structure provided by the norms and the continuity of linear maps.

The rest of the part handles systematically the spectrum of an operator in various contexts. First we deal with the spectrum in a fairly general context of Banach algebras. Then we study spectral decompositions for specific types of operators, starting with compact operators which are closest to those in finite dimensional spaces. Then we go into the systematic study of hermitian operators in Hilbert space. Note that the proof of the spectral theorem in this context is self-contained, independent of the chapter on Banach spaces. Knowing just the self duality of Hilbert space and the existence of orthogonal complements for closed subspaces constitutes a sufficient tool for the applications to the rest of the book. The spectral theorems are included so that readers can push forward in these particular directions if they are so inclined by taste rather than by formal requirements for the present basic course.

The functional analysis is principally concerned with the study of a space with an operator, giving as simple a description as possible for the way in which this operator operates. The two spectral theorems give examples of the standard manner in which such a description can be

made, i.e. either by describing a basis for the space on which the effect of the operator is obvious, or by giving a structure theorem for the algebra generated by the operator. These two ways permeate functional analysis.

CHAPTER XV

# The Open Mapping Theorem, Factor Spaces, and Duality

## XV, §1. THE OPEN MAPPING THEOREM

We begin with a general theorem on metric spaces.

**Theorem 1.1 (Baire's Theorem).** *Let $X$ be a complete metric space, and assume that $X$ is the union of a sequence of closed subsets $S_n$. Then some $S_n$ contains a non-empty open ball.*

*Proof.* Suppose that this is not the case. We find $x_1$ in the complement of $S_1$ (which cannot be the whole space) and some closed ball $B_{r_1}(x_1)$ centered at $x_1$ of radius $r_1 > 0$, contained in this complement. By assumption, there is some $x_2$ in $B_{r_1}(x_1)$ contained in the complement of $S_2$ and some closed ball $B_{r_2}(x_2)$ contained in $B_{r_1}(x_1)$, and which lies in the complement of $S_2$. We continue inductively using a sequence $r_1$, $r_2$, ... such that $r_n > 0$ and $r_n \to 0$. We thus obtain a sequence of closed balls

$$B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \cdots \supset B_{r_n}(x_n)$$

such that $B_{r_n}(x_n)$ is disjoint from $S_1 \cup \cdots \cup S_n$. We then select $x_{n+1}$ and $B_{r_{n+1}}(x_{n+1}) \subset B_{r_n}(x_n)$ disjoint from $S_{n+1}$. Then the sequence $\{x_n\}$ is a Cauchy sequence, converging to a point $x$, and $x$ lies in every $B_{r_n}(x_n)$ for all $n$. Hence $x$ does not lie in $S_n$ for any $n$, contradicting the hypothesis that the union of all $S_n$ is equal to $X$. This proves Baire's theorem.

**Corollary 1.2.** *Let $X$ be a complete metric space, and $\{U_n\}$ a sequence of open dense sets. Then the intersection $\bigcap U_n$ is not empty.*

*Proof.* Take the complement of the sets in Baire's theorem.

**Theorem 1.3 (Open Mapping Theorem).** *Let $E$, $F$ be Banach spaces, and let $\varphi: E \to F$ be a continuous linear map, which is surjective. Then $\varphi$ is open.*

*Proof.* For $s > 0$ we denote by $B_s$ the open ball of radius $s$ in $E$, centered at the origin, and by $C_s$ the open ball in $F$ centered at the origin. Let $S_n = \overline{\varphi(B_n)}$. Then $S_n$ is closed, and the union of all sets $S_n$ is equal to $F$. By Baire's theorem, some $\varphi(B_n)$ contains a set which is dense in some non-empty open ball $V$ in $F$, centered at a point $y$. If $y = \varphi(x)$, for some $x \in E$, then translating by $y$, we conclude that there is some $k \geqq 1$ and $r > 0$ such that $\varphi(B_{kr})$ contains a set which is dense in $C_r$. By homogeneity (i.e. the fact that $B_{ts} = tB_s$ for $s$, $t > 0$), it follows that this last statement holds if we replace $r$ by any number $s > 0$. We shall prove that in fact, $\varphi(B_{kr})$ contains $C_r$. Select $0 < \delta < 1$, and let $y \in F$, $|y| < r$. There exists $x_1 \in E$ with $|x_1| < kr$ such that

$$|y - \varphi(x_1)| < \delta r.$$

Inductively, there exist $x_1, \ldots, x_n \in E$ such that $|x_n| < k\delta^{n-1}r$ and

$$|y - \varphi(x_1) - \cdots - \varphi(x_n)| < \delta^n r.$$

Then the sum $x_1 + \cdots + x_n$ converges to an element $x$ such that $y = \varphi(x)$, and furthermore,

$$|x| < kr/(1 - \delta).$$

Hence $\varphi(B_{kr})$ contains the ball $C_{r(1-\delta)}$ of radius $r(1 - \delta)$. This is true for every $\delta > 0$ whence our assertion follows that $\varphi(B_{kr})$ contains $C_r$.

Now to conclude the proof of Theorem 1.3, let $U$ be an open set in $E$, and let $x \in U$. Let $B$ be an open ball centered at the origin in $E$ such that $x + B \subset U$. Then $\varphi(x) + \varphi(B)$ is contained in $\varphi(U)$. But $\varphi(B)$ contains an open ball centered at the origin in $F$. This proves that $\varphi(U)$ is open, and concludes the proof of the open mapping theorem.

**Corollary 1.4.** *Let $\varphi: E \to F$ be a continuous linear map, which is bijective. Then $\varphi$ is a toplinear isomorphism.*

*Proof.* The inverse of $\varphi$ is also continuous, so we are done.

**Corollary 1.5.** *Let $F$, $G$ be closed subspaces of $E$ such that $F + G = E$ and $F \cap G = \{0\}$. Then the map*

$$F \times G \to E$$

*such that $(x, y) \mapsto x + y$ is a toplinear isomorphism.*

*Proof.* It is continuous and bijective, so that Corollary 1.4 applies.

In Corollary 1.5, we shall also say that $E$ is the **direct sum** of $F$, $G$ and we write

$$E = F \oplus G.$$

We say that $F$, $G$ are **complementary** subspaces.

Let $E$ be a Banach space and $F$ a closed subspace. Then we can define a norm on the factor space $E/F$ by

$$|x + F| = \inf_{y \in F} |x + y|.$$

*Then $E/F$ is complete under this norm, i.e. is also a Banach space.* To see this, let

$$\varphi: E \to E/F$$

be the canonical map which to each $x \in E$ associates the coset $\varphi(x) = x + F$. Then $\varphi$ is a continuous linear map, and

$$|\varphi(x)| \leqq |x|.$$

Let $\{\xi_n\}$ be a Cauchy sequence in $E/F$. Taking a subsequence if necessary, we may assume without loss of generality that

$$|\xi_n - \xi_{n-1}| < \frac{1}{2^n}.$$

We find inductively elements $x_n \in E$ such that $\varphi(x_n) = \xi_n$ and such that

$$|x_n - x_{n-1}| < \frac{1}{2^n}.$$

Indeed, suppose that we have found $x_1, \ldots, x_n$ satisfying these conditions. Since $|\xi_{n+1} - \xi_n| < \frac{1}{2^{n+1}}$, we can find $y$ such that

$$\varphi(y) = \xi_{n+1} - \xi_n \qquad \text{and} \qquad |y| < \frac{1}{2^{n+1}}$$

by the definition of the norm on $E/F$. We let $x_{n+1} = y + x_n$ to achieve what we want. Then the sequence $\{x_n\}$ is a Cauchy sequence in $E$, and converges to some element $x$. It follows that $\varphi(x_n) = \xi_n$ converges to $\varphi(x)$ since $\varphi$ is continuous, as was to be shown.

Let $\varphi: E \to G$ be a continuous linear map where $E$, $G$ are Banach

spaces. The image $\varphi(E)$ is a subspace, which is not necessarily closed. Let $F$ be the kernel of $\varphi$. Then we have the usual linear map

$$E/F \to G$$

induced by $\varphi$, namely the map such that $x + F \mapsto \varphi(x) = \varphi(x + F)$. This map is in fact continuous, because there exists $C > 0$ such that for all $x \in E$ we have

$$|\varphi(x)| \leqq C|x + F|$$

Since $\varphi(x) = \varphi(x + y)$ for all $y \in F$ it follows that

$$|\varphi(x)| \leqq C|x + F|$$

whence the continuity of $E/F \to G$. Consequently, by Corollary 1.4 of the open mapping theorem, if $\varphi$ is surjective, it follows that the map $E/F \to G$ is a toplinear isomorphism.

Let $E$ be a vector space and $F$ a subspace. If $E/F$ has finite dimension, then we say that $F$ has **finite codimension**, and we call dim $E/F$ its **codimension.**

**Corollary 1.6.** *Let $E$ be a Banach space and $F$ a closed subspace of finite dimension or finite codimension. Then $F$ has a complementary closed subspace.*

*Proof.* Assume that $F$ is finite dimensional. The proof is then independent of the open mapping theorem, namely we let $\{\varphi_1, \ldots, \varphi_n\}$ be a basis of the dual space of $F$. By Hahn–Banach, we extend each $\varphi_i$ to a functional on $E$, denoted by the same letter, and we map

$$x \mapsto (\varphi_1 x, \ldots, \varphi_n x)$$

for $x \in E$. Let $G$ be the kernel of this map. Then $G$ is closed, and it is immediately verified that $G$ is a complement of $F$.

Next, assume that $F$ has finite codimension, and let $\{y_1, \ldots, y_n\}$ be a basis of $E/F$. Let $x_1, \ldots, x_n$ be elements of $E$ mapping into $y_1, \ldots, y_n$ respectively in the natural map

$$E \to E/F.$$

Let $G$ be the space generated by $x_1, \ldots, x_n$. Then $G$ is finite dimensional, hence closed, and $F \cap G = \{0\}$ while $F + G = E$. We can apply Corollary 1.5 to conclude the proof of this case.

Later in discussing Fredholm operators, we shall also need the following completely elementary fact:

**Proposition 1.7.** *If $F$ is closed in $E$, and $E/F$ is finite dimensional, and if $G$ is a subspace of $E$ such that $F \subset G \subset E$, then $G$ is closed.*

*Proof.* The image of $G$ in the factor space $E/F$ is in a finite dimensional space, hence closed. Since $G$ is the inverse image in $E$ of its image in $E/F$, it follows that $G$ is closed.

**Corollary 1.8.** *Let $E$, $G$ be Banach spaces. Let $\varphi: E \to G$ be a continuous linear map such that the image $\varphi(E)$ is finite codimensional. Then $\varphi(E)$ is closed.*

*Proof.* We can find in the usual way (as in Corollary 1.6) a finite dimensional subspace $F$ of $G$ such that $G = \varphi(E) + F$. Of course, so far, this is an algebraic direct sum, not yet topological. Factoring out the kernel of $\varphi$, we may assume without loss of generality that $\varphi$ is injective. We compose $\varphi$ with the natural map $G \to G/F$. Then the composite

$$E \xrightarrow{\varphi} G \xrightarrow{\psi} G/F$$

is a bijective continuous linear map of $E$ on $G/F$, hence is a toplinear isomorphism by Corollary 1.4. Hence the inverse map $(\psi \circ \varphi)^{-1}$ is continuous, and therefore so is the map

$$\varphi \circ (\psi \circ \varphi)^{-1},$$

which maps $G/F$ on $\varphi(E)$. Hence $\varphi(E)$ is toplinearly isomorphic with $G/F$. Since $G/F$ is complete, it follows that $\varphi(E)$ is complete, and consequently $\varphi(E)$ is closed in $G$, as was to be shown.

## XV, §2. ORTHOGONALITY

We could now deal with either the real or complex case. We deal with the latter, since it is useful to get used to the complex conjugation which occurs, and introduces only a change of notation.

Let $E$ be a Banach space over the complex numbers. We let $E^*$ be the space of **anti-linear** maps $\varphi: E \to \mathbf{C}$, i.e. continuous maps which are $\mathbf{R}$-linear and satisfy

$$\varphi(\alpha x) = \bar{\alpha}\varphi(x).$$

Elements of this space will be called **anti-functionals** or **semi-functionals**. This space is obtained from the dual space $E'$ very simply, namely if $\varphi$ is a functional in $E'$, then the map $\bar{\varphi}$ defined by

$$\bar{\varphi}(x) = \overline{\varphi(x)}$$

is an anti-functional, i.e. an element of $E^*$, and conversely. We shall apply to elements of $E^*$ certain results proved for elements of $E'$, e.g. the Hahn–Banach theorem. Let $E$, $F$ be Banach spaces, and let

$$u: E \to F$$

be a continuous linear map. Then $u$ induces a map

$$u^*: F^* \to E^*$$

such that

$$\varphi \mapsto \varphi \circ u,$$

and it is clear that $u^*$ is linear and continuous. It is convenient here to use a notation as in Hilbert space. We define a map

$$E \times E^* \to \mathbf{C}$$

by

$$(x, \varphi) \mapsto \langle x, \varphi \rangle = \overline{\varphi(x)}.$$

This map is continuous sesquilinear, and we shall see that it behaves very much like the scalar product of Hilbert space for the basic formalism of duality.

First the remark that the map

$$u \mapsto u^*$$

is **anti-linear** from $L(E, F)$ to $L(F^*, E^*)$. By definition, we have

$$\langle ux, \varphi \rangle = \langle x, u^*\varphi \rangle$$

for all $x \in E$, $\varphi \in F^*$. Thus we call $u^*$ the **adjoint** of $u$. We note that $u^*$ is the unique element of $L(F^*, E^*)$ which satisfies this formula. We have

$$\boxed{|u| = |u^*|.} \tag{1}$$

To prove this, observe that for any $\varphi \in F^*$ we have

$$|(u^*\varphi)(x)| = |\varphi(ux)| \leqq |\varphi||u||x|$$

so that $|u^*| \leqq |u|$. Conversely, for each $x \in E$, by the Hahn–Banach theorem, there exists $\varphi \in F^*$ such that $|\varphi(ux)| = |ux|$, and $|\varphi| \leqq 1$. Then for

this $\varphi$, we get

$$|ux| = |\varphi(ux)| = |(u^*\varphi)(x)| \leqq |u^*||\varphi||x| \leqq |u^*||x|.$$

Hence $|u| \leqq |u^*|$, thus proving our assertion.

We have the following duality between spaces, subspaces, and factor spaces. Let $F$ be a closed subspace of $E$. We denote by $F^\perp$ the set of all elements $\varphi \in E^*$ such that $\varphi(F) = 0$. (This is similar to the situation in Hilbert space, but here we have the natural map

$$E \times E^* \to \mathbf{C}$$

instead of the hermitian product of Hilbert space.) Then $F^\perp$ is clearly a closed subspace. We have a natural continuous linear map

$$E^* \to F^*$$

by restriction, i.e. each $\varphi \in E^*$ induces by restriction an element of $F^*$. The kernel is precisely $F^\perp$. Furthermore, our map is surjective, because an anti-functional $\psi$ on $F$ can be extended to an anti-functional $\varphi$ on $E$ by the Hahn–Banach theorem. *Hence we have a natural toplinear isomorphism*

$$E^*/F^\perp \xrightarrow{\approx} F^*. \tag{2}$$

We observe that the notion of perpendicularity can be defined on the other side as well, i.e. given a subset $S$ of $E^*$, we let $S^\perp$ be the set of $x \in E$ such that $\langle x, \varphi \rangle = 0$ for all $\varphi \in S$. Then $S^\perp$ is a closed subspace of $E$.

*We have a natural toplinear isomorphism*

$$F^\perp \xrightarrow{\approx} (E/F)^*. \tag{3}$$

Indeed, each $\varphi \in F$ defines an element of $(E/F)^*$ since $\langle F, \varphi \rangle = 0$. It is clear that this map induces our stated isomorphism.

*Let $F$ be a subspace of $E$. Then*

$$F^{\perp\perp} = \overline{F}. \tag{4}$$

Indeed, it is clear that $F \subset F^{\perp\perp}$, and $F^{\perp\perp}$ is closed so that $\overline{F} \subset F^{\perp\perp}$. Conversely, suppose that $x \notin \overline{F}$. Then there is some anti-functional $\varphi$ on $E/\overline{F}$ such that $\varphi(x) \neq 0$ by the Hahn–Banach theorem. Hence $x \notin F^{\perp\perp}$, thus proving our assertion.

We also have the duality associated with a continuous linear map

$$u \colon E \to G$$

namely

$$\text{Ker } u^* = (\text{Im } u)^\perp, \tag{5}$$

*and if the image of u is closed, then so is the image of $u^*$ and*

$$\text{Im } u^* = (\text{Ker } u)^\perp. \tag{6}$$

We leave (5) as an exercise, and prove (6). We have for $x \in E$:

$$\langle x, u^*G^* \rangle = 0 \qquad \text{if and only if} \qquad \langle ux, G^* \rangle = 0.$$

Hence $\text{Im } u^* \subset (\text{Ker } u)^\perp$. Conversely, let $\varphi \in E^*$ and $\varphi \perp \text{Ker } u$. We have a toplinear isomorphism

$$\sigma\colon E/\text{Ker } u \to u(E)$$

by Corollary 1.4 ($\sigma$ is continuous bijective). We view $\varphi$ as an anti-functional on $E/\text{Ker } u$, and then $\varphi \circ \sigma^{-1}$ is an anti-functional on $u(E)$, which is a closed subspace of $G$. We can extend $\varphi \circ \sigma^{-1}$ to an anti-functional $\psi$ of $G$ by the Hahn–Banach theorem. Then it is clear that $u^*\psi = \varphi$, whence $\varphi \in \text{Im } u^*$. This proves that $\text{Im } u^* = (\text{Ker } u)^\perp$, and in particular proves that $\text{Im } u^*$ is closed, thus proving (6).

*In particular, if again the image of u is closed, then we have toplinear isomorphisms*

$$\text{Ker } u^* \approx (E/uE)^*, \tag{7}$$

$$(\text{Ker } u)^* \approx E^*/u^*G^*, \tag{8}$$

*in a natural way.*

The reader acquainted with the language of exact sequences will see that our results can be expressed as follows. If

$$0 \to F \to E \to G \to 0$$

*is an exact sequence of Banach spaces, then the adjoint sequence*

$$0 \leftarrow F^* \leftarrow E^* \leftarrow G^* \leftarrow 0$$

*is also exact.*

## XV, §3. APPLICATIONS OF THE OPEN MAPPING THEOREM

The results of this section will not be used at all throughout the rest of this book, and are included only for the sake of completeness. The first two give criteria for a linear map to be continuous.

As usual, if $\varphi: E \to F$ is a map, we define the **graph** of $\varphi$ to be the set of all points $(x, \varphi(x))$ in $E \times F$. If $\varphi$ is linear, then the graph of $\varphi$ is obviously a subspace of $E \times F$.

**Theorem 3.1 (Closed Graph Theorem).** *Let $\varphi: E \to F$ be a linear map from one Banach space into another, and assume that the graph is closed. Then $\varphi$ is continuous.*

*Proof.* Let $G$ be the graph of $\varphi$, so that by assumption $G$ is a closed subspace of $E \times F$. The projection

$$G \to E \qquad \text{given by} \qquad (x, \varphi(x)) \mapsto x$$

is obviously continuous and bijective. By Corollary 1.4 of the open mapping theorem, it follows that this projection is a toplinear isomorphism, and thus has a continuous linear inverse. If we compose this continuous linear inverse with the projection on $F$, then we obtain $\varphi$, thus proving that $\varphi$ is continuous, as desired.

**Theorem 3.2 (Principle of Uniform Boundedness).** *Let $E$, $F$ be Banach spaces, and let $\{T_i\}_{i \in I}$ be a family of continuous linear maps from $E$ into $F$. Assume that for each $x \in E$ the set $\{T_i x\}_{i \in I}$ is bounded. Let $B$ be a bounded subset of $E$. Then the set*

$$\bigcup_{i \in I} T_i(B)$$

*is bounded.*

*Proof.* For each positive integer $n$ let $C_n$ be the set of all $x \in E$ such that $|T_i x| \leqq n$ for all $i \in I$. Since each $T_i$ is continuous, it follows that $C_n$ is closed. By assumption, we have

$$E = \bigcup_{n=1}^{\infty} C_n.$$

By Baire's theorem (Theorem 1.1) it follows that some $C_m$ contains an open ball $B_r(x_0)$ with $r > 0$. This means that if $|x| < r$, then for all $i \in I$ we have

$$|T_i(x_0 + x)| \leqq m,$$

whence

$$|T_i(x)| \leqq |T_i(x + x_0)| + |T_i(x_0)|$$
$$\leqq 2m.$$

Our theorem follows by homogeneity.

**Corollary 3.3.** *Let $\{T_n\}$ be a sequence of continuous linear maps of $E$ into $F$. Assume that for each $x \in E$ the limit*

$$Tx = \lim_{n\to\infty} T_n x$$

*exists. Then $T$ is a continuous linear map of $E$ into $F$, and*

$$\lim_{x\to 0} T_n x = 0$$

*uniformly in $n$.*

*Proof.* It is clear that $T$ is linear. For each $x \in E$, the sequence $\{T_n x\}$ converges and hence is bounded, so that we can apply the theorem, and our corollary follows at once since we see that $T$ is continuous at 0.

The next two theorems provide one type of generalization of the inverse mapping theorem, to surjective mapping theorems.

**Theorem 3.4.** *Let $E$, $F$ be Banach spaces. The subset of $L(E, F)$ consisting of surjective maps is open in $L(E, F)$.*

*Proof.* Let $\lambda: E \to F$ be a continuous linear map and assume that $\lambda$ is surjective. By the open mapping theorem, and homogeneity, there exists $C > 0$ having the following property. Given $y \in F$ with $|y| \leqq 1$, there exists $x \in E$ such that $\lambda x = y$ and $|x| \leqq C|y|$. Changing the norm on $F$ to an equivalent norm, we may assume without loss of generality that $C = 1$. Let $0 < r < 1$. Let $\varphi \in L(E, F)$ be such that $|\lambda - \varphi| \leqq r$. We shall prove that $\varphi$ is surjective, and it will suffice to prove that $\varphi$ maps the ball of radius $\dfrac{1}{1-r}$ in $E$ onto the ball of radius 1 in $F$. Let $y = y_1 \in F$ and $|y_1| \leqq 1$. By what we have just remarked, there exists $x_1 \in E$ such that $\lambda x_1 = y_1$ and $|x_1| \leqq 1$. Let

$$y_2 = \lambda x_1 - \varphi x_1.$$

Then

$$|y_2| = |\lambda x_1 - \varphi x_1| \leqq |\lambda - \varphi||x_1| \leqq r.$$

There exists $x_2 \in E$ such that $\lambda x_2 = y_2$ and $|x_2| \leqq r$. Let

$$y_3 = \lambda x_2 - \varphi x_2.$$

Then

$$|y_3| = |\lambda x_2 - \varphi x_2| \leqq r|x_2| \leqq r^2.$$

There exists $x_3 \in E$ such that $\lambda x_3 = y_3$ and $|x_3| \leqq r^2$. Continuing inductively, we find $x_n$ such that $\lambda x_n = y_n$ and

$$y_{n+1} = \lambda x_n - \varphi x_n, \qquad |y_{n+1}| \leqq r^n, \qquad |x_{n+1}| \leqq r^n.$$

Then

$$y_1 = \varphi x_1 + \cdots + \varphi x_n + y_{n+1}.$$

If we let

$$x = \sum_{n=1}^{\infty} x_n,$$

then $|x| \leqq \dfrac{1}{1-r}$, and $\varphi x = y_1$, thus proving our theorem.

**Theorem 3.5** **(Surjective Mapping Theorem (Graves)).** *Let $U$ be open in a Banach space $E$. Let $f\colon U \to F$ be a $C^1$ map into a Banach space $F$. Let $x_0 \in E$. If $f'(x_0)$ is surjective, then $f$ is locally open in a neighborhood of $x_0$. More precisely, there exists an open neighborhood $V$ of $x_0$ contained in $U$ having the following property. For each $x \in V$ and open ball $B_x$ centered at $x$, contained in $V$, the image $f(B_x)$ contains an open neighborhood of $f(x)$.*

*Proof.* After a translation, we may assume that $x_0 = 0$ and $f(x_0) = 0$. Also by the preceding theorem, it will suffice to prove that if $B$ is an open ball centered at 0 in $E$, then $f(B)$ contains an open neighborhood of 0 in $F$. Let $\lambda = f'(0)$. By the open mapping theorem and homogeneity, there exists $C > 0$ having the following property. Given $y \in F$ with $|y| \leqq 1$, there exists $x \in E$ such that $\lambda x = y$ and $|x| \leqq C|y|$. Changing the norm on $F$ to an equivalent norm, we may assume without loss of generality that $C = 1$. Let $0 < r < 1$. Taking $B$ with sufficiently small radius, by the mean value theorem we have for $x, z \in \dfrac{1}{1-r}B$:

$$(*) \qquad |f(x) - f(z) - \lambda(x - z)| \leqq r|x - z|.$$

It will suffice to prove that

$$f\left(\frac{1}{1-r}B\right) \supset \lambda(B),$$

and we now prove this. Let $x_1 \in B$ and $\lambda x_1 = y_1$. Let $y_2 = \lambda x_1 - f(x_1)$. By (*) we find

$$|y_2| = |\lambda x_1 - f(x_1)| \leqq r|x_1|.$$

There exists $x_2$ with $|x_2| \leqq r|x_1|$ such that $\lambda x_2 = y_2$. Then

$$x_1 + x_2 \in (1 + r)B,$$

and by (*) we find

$$\begin{aligned} |\lambda x_1 - f(x_1 + x_2)| &= |f(x_1) - f(x_1 + x_2) + \lambda x_2| \\ &\leqq r|x_2| \leqq r^2|x_1|. \end{aligned}$$

Let $y_3 = \lambda x_1 - f(x_1 + x_2)$. There exists $x_3$ with $|x_3| \leqq |y_3| \leqq r^2|x_1|$, such that $\lambda x_3 = y_3$. Then

$$x_1 + x_2 + x_3 \in (1 + r + r^2)B.$$

We have

$$\begin{aligned} &|y_1 - f(x_1 + x_2 + x_3)| \\ &\quad = |\lambda x_1 - f(x_1 + x_2) + f(x_1 + x_2) - f(x_1 + x_2 + x_3)|, \\ &\quad = |\lambda x_3 + f(x_1 + x_2) - f(x_1 + x_2 + x_3)|. \end{aligned}$$

But by (*),

$$f(x_1 + x_2) - f(x_1 + x_2 + x_3) = \lambda x_3 + v_3, \qquad \text{with} \quad |v_3| \leqq r|x_3|,$$

so that we get

$$|y_1 - f(x_1 + x_2 + x_3)| \leqq r^3|x_1|.$$

Inductively, we find $x_n$ such that $\lambda x_n = y_n$, $|x_n| \leqq r^{n-1}|x_1|$, and

$$|y_1 - f(x_1 + \cdots + x_n)| \leqq r^n|x_1|.$$

We can then find $x_{n+1}$ such that $|x_{n+1}| \leqq r^n|x_1|$ and

$$\lambda x_{n+1} = y_1 - f(x_1 + \cdots + x_n).$$

Then

$$\begin{aligned} &|y_1 - f(x_1 + \cdots + x_{n+1})| \\ &\quad = |y - f(x_1 + \cdots + x_n) + f(x_1 + \cdots + x_n) - f(x_1 + \cdots + x_{n+1})| \\ &\quad \leqq r|x_{n+1}| \leqq r^{n+1}|x_1|. \end{aligned}$$

We let

$$x = \sum_{n=1}^{\infty} x_n,$$

and we see that $f(x) = y_1$. Furthermore, $x \in \frac{1}{1-r} B$, thus proving our theorem.

CHAPTER XVI

# The Spectrum

In this chapter we give basic facts about the spectrum of an element in a Banach algebra. Under certain circumstances, we represent such an algebra as the algebra of continuous functions on its spectrum, which is defined as the space of its maximal ideals (or the space of characters), to be given the weak topology. In the next chapters, we shall deal with spectral theorems corresponding to more specific examples of Banach algebras. The proofs in the later chapters are independent of those in the present chapter, except for Theorem 1.2 and its corollaries. Thus the rest of this chapter may be bypassed. On the other hand, the general representation of a Banach algebra as an algebra of continuous functions on the spectrum gives a nice application of the Stone–Weierstrass theorem, and is useful in other contexts besides the spectral theorems which form the remainder of the book, so I have included the basic results to provide a suitable background for applications not included in this book.

## XVI, §1. THE GELFAND–MAZUR THEOREM

Let $A$ be a Banach algebra over the complex numbers. We assume that $A$ has a unit element $e$. Let $v \in A$. The set of complex numbers $z$ such that $v - ze$ is not invertible is called the **spectrum** of $v$ and is denoted by $\sigma(v)$. We shall investigate special cases in Chapters XVII and XVIII. Here we shall prove that the spectrum is not empty. Before doing that, we make a simple remark concerning the spectrum.

**Theorem 1.1.** *The spectrum of an element $v \in A$ is a closed and bounded set in* **C**. *In fact, if $z$ is in the spectrum, then $|z| \leqq |v|$.*

*Proof.* We show that the complement is open. Let $z_0$ be a complex number such that $v - z_0 e$ is invertible. If $z$ is sufficiently close to $z_0$, then $v - ze$ is invertible because the set of invertible elements is open, so the spectrum is closed. Furthermore, if $|z| > |v|$, then $|v/z| < 1$, and hence $e - v/z$ is invertible by Theorem 2.1 of Chapter IV, §2, so that $v - ze$ is also invertible, as contended.

**Theorem 1.2.** *Let $A$ be a commutative normed algebra over the real numbers, with unit element $e$. Assume that there exists an element $j \in A$ such that $j^2 = -e$. Let $\mathbf{C} = \mathbf{R} + \mathbf{R}j$. Given $v \in A$, $v \neq 0$, there exists an element $c \in \mathbf{C}$ such that $v - ce$ is not invertible in $A$.*

*Proof* (Tornheim). Assume that $v - ze$ is invertible for all $z \in \mathbf{C}$. Consider the mapping $f: \mathbf{C} \to A$ defined by

$$f(z) = (v - ze)^{-1}.$$

Then $f$ is continuous, and for $z \neq 0$ we have

$$f(z) = z^{-1}(z^{-1}v - e)^{-1} = \frac{1}{z}\left(\frac{1}{(v/z) - e}\right).$$

From this we see that $f(z)$ approaches 0 when $z$ goes to infinity in $\mathbf{C}$. Hence the map $z \mapsto |f(z)|$ is a continuous map of $\mathbf{C}$ into the real numbers $\geqq 0$, is bounded, and is small outside some large circle. Hence it has a maximum, say $M$. Let $D$ be the set of elements $z \in \mathbf{C}$ such that $|f(z)| = M$. Then $D$ is not empty, $D$ is bounded, and is closed. We shall prove that $D$ is open, whence a contradiction.

Let $c_0$ be a point of $D$, which, after a translation, we may assume to be the origin. We shall see that if $r$ is real $> 0$, then all points on the circle of radius $r$ lie in $D$. Indeed, consider the sum

$$S(n) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{v - \alpha^k r},$$

where $\alpha$ is a primitive $n$-th root of unity, say $\alpha = e^{2\pi i/n}$. Let $t$ be a variable. Taking formally the logarithmic derivative of

$$t^n - r^n = \prod_{k=1}^{n} (t - \alpha^k r)$$

shows that

$$\frac{nt^{n-1}}{t^n - r^n} = \sum_{k=1}^{n} \frac{1}{t - \alpha^k r},$$

and hence, dividing by $n$ and by $t^{n-1}$, and substituting $v$ for $t$, we obtain

$$S(n) = \frac{1}{v - r(r/v)^{n-1}}.$$

If $r$ is small (say $|r/v| < 1$), then we see that

$$\lim_{n\to\infty} |S(n)| = |v^{-1}| = M.$$

Suppose that there exists a complex number $\xi$ of absolute value 1 such that

$$\left|\frac{1}{v - \xi r}\right| < M.$$

Then there exists an interval on the unit circle near $\xi$, and there exists $\varepsilon > 0$ such that for all roots of unity $\zeta$ lying in this interval, we have

$$\left|\frac{1}{v - \zeta r}\right| < M - \varepsilon.$$

(This is true by continuity.) Let us take $n$ very large. Let $b_n$ be the number of $n$-th roots of unity lying in our interval. Then $b_n/n$ is approximately equal to the length of the interval (over $2\pi$). We can express $S(n)$ as a sum

$$S(n) = \frac{1}{n}\left[\sum\nolimits_{\mathrm{I}} \frac{1}{v - \alpha^k r} + \sum\nolimits_{\mathrm{II}} \frac{1}{v - \alpha^k r}\right],$$

the first sum $\sum_{\mathrm{I}}$ being taken over those roots of unity $\alpha^k$ lying in our interval, and the second sum being taken over the others. Each term in the second sum has norm $\leqq M$ because $M$ is a maximum. Hence we obtain the estimate

$$\begin{aligned} |S(n)| &\leqq \frac{1}{n}\left[|\textstyle\sum_{\mathrm{I}}| + |\sum_{\mathrm{II}}|\right] \\ &\leqq \frac{1}{n}(b_n(M - \varepsilon) + (n - b_n)M) \\ &\leqq M - \frac{b_n}{n}\varepsilon. \end{aligned}$$

This contradicts the fact that the limit of $|S(n)|$ is equal to $M$, and proves our theorem.

**Corollary 1.3 (Gelfand–Mazur Theorem).** *Let $K$ be a normed field over the reals. Then $K = \mathbf{R}$ or $K = \mathbf{C}$.*

*Proof.* Assume first that $K$ contains $\mathbf{C}$. Then the theorem implies that $K = \mathbf{C}$. If $K$ does not contain $\mathbf{C}$, in other words does not contain a square root of $-1$, we let $E = K(j)$ where $j^2 = -1$. (One can give a formal definition of the field $E$ as one defines the complex numbers from ordered pairs of real numbers. Thus we let $E$ consist of pairs $(x, y)$ with $x, y \in K$, and define multiplication in $E$ as if $(x, y) = x + yj$. This makes $E$ into a field.) We can define a norm on $E$ by putting

$$|x + yj| = |x| + |y|$$

for $x, y \in K$. Then $E$ is a normed $\mathbf{R}$-space. Furthermore, if $z = x + yj$ and $z' = x' + y'j$ are in $E$, then

$$\begin{aligned} |zz'| &= |xx' - yy'| + |xy' + x'y| \\ &\leqq |xx'| + |yy'| + |xy'| + |x'y| \\ &\leqq |x||x'| + |y||y'| + |x||y'| + |x'||y| \\ &\leqq (|x| + |y|)(|x'| + |y'|) \\ &\leqq |z||z'|. \end{aligned}$$

Hence we have defined a norm on $E$. We can apply Theorem 1.2 to conclude the proof.

**Corollary 1.4.** *The spectrum of an element in any complex Banach algebra (commutative or not) with unit element is not empty.*

*Proof.* If $A$ is a Banach algebra with unit, and if $v \in A$, then the closure of the algebra generated by $e$ and $v$ is a commutative Banach algebra. Hence we can apply the theorem to it.

We shall see later in this chapter that under fairly general conditions, a Banach algebra is isomorphic to the algebra of continuous functions on a compact set. This set is obtained in a natural way, namely it is the maximal ideal space of $A$.

Finally, we remark that the fact that the spectrum is not empty can also be proved by quoting an elementary theorem about analytic functions of a complex variable, namely that a bounded analytic function is constant. The proof runs as follows. Suppose that we have an element $v$ in our algebra such that $(v - ze)^{-1}$ exists for all complex $z$. Then certainly the map

$$z \mapsto (v - ze)^{-1}$$

is not constant, and hence there exists a functional $\lambda$ on the algebra such that the map

$$z \mapsto \lambda[(v - ze)^{-1}] = f(z)$$

is not constant. However, it is immediately verified that this map is differentiable (i.e. complex differentiable), and since $(v - ze)^{-1} \to 0$ as $z \to \infty$, it follows that $f$ is bounded, a contradiction which proves what we wanted.

We will need to consider Banach-valued analytic functions somewhat further in the context of the remark above. Let $E$ be a Banach space, and let $U$ be open in $\mathbf{C}$. Let $f: U \to E$ be a mapping. We define $f$ to be **analytic** if given $z_0 \in U$ there exist elements $a_n \in E$ such that $f(z)$ is equal to a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for $z$ in some disc of positive radius centered at $z_0$. As usual in complex analysis, convergent here means absolutely convergent. Observe that for every functional $\lambda$ on $E$, we have

$$\lambda \circ f(z) = \sum_{n=0}^{\infty} \lambda(a_n)(z - z_0)^n.$$

Thus $\lambda \circ f$ is an analytic function in the usual sense of complex analysis. By the Hahn–Banach theorem, and the usual uniqueness of a power series expansion for complex valued analytic functions, we also get the uniqueness for Banach-valued analytic functions, and we get the uniqueness of analytic continuation. Furthermore, a number of theorems from complex analysis are valid in the Banach-valued case, and their proofs can be carried out by using the Hahn–Banach theorem. For instance for an analytic function on a closed disc of radius $R$, we have Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where $C$ is the circle of radius $R$ centered at the origin.

Indeed, the integral on the right is the simple-minded integral as defined in calculus, Chapter XIII, §1. To prove that both sides of the formula are equal, by the Hahn–Banach theorem it suffices to prove that applying every functional to one side is equal to the functional applied to the other side. We can use Proposition 1.1 of Chapter XIII to see this.

The Cauchy integral representation allows us to prove in the Banach-valued case that if $f$ is analytic on a closed disc of radius $R$, then the radius of convergence of its power series centered at 0 is at least equal to $R$. Indeed, the usual proof works by using the geometric series, writing

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \frac{1}{1 - z/\zeta} = \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n.$$

Thus we get the same integral formula for the coefficients of the power series for $f$ at 0 as in the complex valued cases.

We leave it to the reader to verify that if $A$ is a Banach algebra with unit $e$, and $x \in A$, then the function

$$f(z) = (x - ze)^{-1}$$

is analytic on the open set of complex numbers $z$ such that $x - ze$ is invertible.

The same arguments as in ordinary complex analysis show that the radius of convergence of a power series $\sum a_n z^n$ is $1/\limsup |a_n|^{1/n}$. We shall be concerned with a special power series as follows.

Let $A$ be a Banach algebra with unit element $e$. We denote by $\sigma(x)$ the spectrum of an element $x \in A$. We are interested in bounds for the spectrum. The essential structure will be derived from the power series

$$\sum x^n z^n$$

in the variable $z$, with coefficients in $A$. The next lemmas describe its radius of convergence more precisely.

**Lemma 1.5.** *Let $A$ be a normed algebra. Let $x \in A$. Then*

$$\lim_{n \to \infty} |x^n|^{1/n}$$

*exists, and is equal to* $\inf |x^n|^{1/n}$.

*Proof.* Without loss of generality, we may assume that $x^n \neq 0$ for all positive integers $n$. Let $c_n = |x^n| > 0$. We obviously have

$$c_{m+n} \leqq c_m c_n$$

for all positive integers $m$, $n$. We put $c_0 = 1$. Fix $m$. Then any integer $n$ can be written in the form $n = q(n)m + r(n)$ with $0 \leqq r(n) < m$. Then

$$c_n^{1/n} \leqq c_m^{q(n)/n} c_{r(n)}^{1/n}.$$

But $r(n)$ is bounded by $m$, and $\lim q(n)/n = 1/m$. Hence

$$\limsup_{n \to \infty} c_n^{1/n} \leqq c_m^{1/m}.$$

This inequality holds for all positive integers $m$, and hence

$$\limsup_{n \to \infty} c_n^{1/n} \leqq \inf_n c_n^{1/n} \leqq \liminf_{n \to \infty} c_n^{1/n}.$$

This proves the lemma.

In light of the lemma, we define the **spectral radius**

$$\rho(x) = \lim|x^n|^{1/n}.$$

The reason for the name will become clear from Theorem 1.7 below.

We are now ready to consider the power series

$$h_x(z) = \sum_{n=0}^{\infty} x^n z^n$$

with $x^n z^n \in A$ for all complex numbers $z$. We are interested in the domain of convergence of the series.

**Lemma 1.6.** *Suppose $A$ is a Banach algebra with unit. Then the radius of convergence of the series $h_x$ is $1/\rho(x)$.*

*Proof.* This is immediate from the usual arguments in the theory of a complex variable, which show in the present context that the radius of convergence is $1/\lim\sup|x^n|^{1/n}$. We can then apply Lemma 1.5.

In Theorem 1.1 we saw that the spectrum $\sigma(x)$ is compact and contained in the disc $|z| \leqq |x|$. We now prove a more precise inequality.

**Theorem 1.7.** *Let $A$ be a Banach algebra with unit $e \neq 0$. Then*

$$\rho(x) = \sup_{z \in \sigma(x)} |z|.$$

*Proof.* Suppose $|z| > \rho(x)$. Let

$$\rho(x) < r < |z|.$$

Then $|(z^{-1}x)^n| \leqq (r/|z|)^n$ for large $n$, so the series $\sum (z^{-1}x)^n$ converges and shows that $z \notin \sigma(x)$. Conversely, let $s = \sup|z|$ for $z \in \sigma(x)$, so $s$ is the smallest radius for a closed disc centered at the origin and containing the spectrum $\sigma(x)$. Suppose $w \in \mathbf{C}$ and $|w| > s$. Then $w \notin \sigma(x)$, so $x - we$ is invertible. Thus the function

$$h(z) = (e - zx)^{-1}$$

is analytic on the disc $|z| < s^{-1}$. As we remarked earlier in this section, this implies that the radius of convergence of the power series

$$\sum x^n z^n$$

is $\geqq s^{-1}$, and therefore

$$s^{-1} \leqq \rho(x)^{-1} \qquad \text{whence} \qquad \rho(x) \leqq s.$$

This concludes the proof of Theorem 1.7.

## XVI, §2. THE GELFAND TRANSFORM

*Throughout, we let $A$ be a commutative Banach algebra with unit element $e \neq 0$, over the complex numbers.*

We wish to represent elements of $A$ as continuous functions on a certain space, whose points are going to be the maximal ideals of $A$. We do this in a sequence of lemmas.

**Lemma 2.1.** *Let $M$ be a maximal ideal of $A$. Then $M$ is closed.*

*Proof.* By definition, $M \neq A$ and so $e \notin M$. By Theorem 2.1 of Chapter IV, there is an open neighborhood $U$ of $e$ consisting of invertible elements, and $U$ is contained in the complement of $M$, so $U$ is contained in the complement of the closure $\overline{M}$, thus proving the lemma.

**Lemma 2.2.** *An element $u \in A$ is invertible if and only if $u$ is not contained in any maximal ideal.*

*Proof.* It is clear that an invertible element is not contained in a maximal ideal (which, by definition, is $\neq A$). Conversely, suppose $u$ is not contained in any maximal ideal. Let $J = Au$ be the ideal generated by $u$. If $J = A$, then $u$ is invertible. If $J \neq A$, then by Zorn's lemma, $J$ is contained in a maximal ideal $M$. Indeed, let $S$ be the set of ideals $I$ containing $J$ and $\neq A$. Then $S$ is ordered by inclusion, and $S$ is inductively ordered, for let $\{I_k\}$ be a totally ordered subset. Let $I = \bigcup I_k$. Then it is immediately verified that $I$ is an ideal, and $I \neq A$, for otherwise $I$ contains the unit element, and this unit element must therefore lie in some ideal $I_k$, contrary to the assumption that $I_k \neq A$ for all $k$. This shows that if $J \neq A$ then $u$ is contained in some maximal ideal, a contradiction which proves the lemma.

Let $\lambda: A \to \mathbf{C}$ be a **functional**, i.e. a continuous linear map into the scalars $\mathbf{C}$. If in addition $\lambda$ satisfies the multiplicative conditions

$$\lambda(xy) = \lambda(x)\lambda(y) \qquad \text{and} \qquad \lambda(e) = 1,$$

then we call $\lambda$ a **character** of $A$. Thus a character is a Banach-algebra

homomorphism (rather than a Banach space homomorphism) into the scalars $\mathbf{C}$. The kernel of such a character is a maximal ideal, because its image is the field $\mathbf{C}$. Conversely, by the theorem of Gelfand–Mazur (Corollary 1.3) if $M$ is a maximal ideal, then $A/M$ is naturally isomorphic to $\mathbf{C}$, and therefore we obtain a character

$$\lambda_M : A \to \mathbf{C}$$

which to each element $x \in A$ associates its residue class $x + M$ in $A/M = \mathbf{C}$. We then obtain:

**Proposition 2.3.** *The association $M \mapsto \lambda_M$ is a bijection between the set of maximal ideals and the set of characters of $A$. For each character $\lambda$, we have $|\lambda| \leqq 1$.*

*Proof.* The first statement has been proved. As to the second, let us first prove that if $|x| \leqq 1$ then $|\lambda(x)| \leqq 1$. If $|\lambda(x)| > 1$, then $|\lambda(x^n)| \to \infty$ as $n \to \infty$, but $|x^n| \leqq 1$, which contradicts the continuity of $\lambda$. Thus $|\lambda(x)| \leqq 1$. Then for any $x \neq 0$, let $y = x/|x|$ so $|y| = 1$. Then $|\lambda(y)| \leqq 1$, so $|\lambda| \leqq 1$, thus proving the proposition.

In light of the fact that $\lambda = \lambda_M$ for some $M$, we see that we may rephrase Proposition 2.3 as follows. *If $x \in A$ and $c \in \mathbf{C}$ is such that* $x \equiv c \bmod M$, *and if* $|x| \leqq 1$, *then* $|c| \leqq 1$.

The Banach algebra $A$ being a Banach space has its dual $A'$. But it also has the set of characters, which we denote by $\hat{A}$, and $\hat{A} \subset A'_1$, where $A'_1$ is the unit ball in $A'$. In Chapter IV, §1 we defined the weak topology on $A'_1$ by embedding $A'_1$ in a product space

$$f : A'_1 \hookrightarrow \prod_{|x| \leqq 1} K_x \subset \prod_{x \in A} \mathbf{C}_x.$$

In particular, we obtain the weak topology on $\hat{A}$ because as we have just seen, we have an inclusion of $\hat{A}$ in the unit ball of $A'$:

$$\hat{A} \subset A'_1.$$

Let $\mathcal{M}$ be the set of maximal ideals of $A$. In light of the bijection of $\mathcal{M}$ with $\hat{A}$, we have a natural embedding (the restriction of $f$)

$$g : \hat{A} \text{ or } \mathcal{M} \hookrightarrow \prod_{|x| \leqq 1} K_x \qquad \text{by} \qquad \lambda \mapsto \prod g_x(\lambda),$$

where for each $x \in A$ and $\lambda \in \hat{A}$ we have $g_x(\lambda) = \lambda(x)$.

**Theorem 2.4.** *The image $g(\hat{A})$ is closed and therefore $\hat{A}$ is compact in the weak topology.*

*Proof.* The argument is basically the same as the argument used to prove Theorem 1.4 of Chapter IV. If $\gamma$ is in the closure of $g(\hat{A})$, then one proves as before by a continuity argument that $\gamma$ satisfies the condition

$$\gamma(xy) = \gamma(x)\gamma(y),$$

just as we proved $\gamma(x + y) = \gamma(x) + \gamma(y)$. We leave the routine to the reader.

Since each function $g_x$ ($x \in A$) is continuous on $\mathcal{M} \approx \hat{A}$ by definition of the weak topology, the map

$$x \mapsto g_x$$

is a mapping from $A$ to $C(\mathcal{M}, \mathbf{C}) \approx C(\hat{A}, \mathbf{C})$. It is immediately verified that this map is a ring homomorphism, i.e.

$$g_{x+y} = g_x + g_y \qquad \text{and} \qquad g_{xy} = g_x g_y.$$

Also $g_{cx} = cg_x$ for $c \in \mathbf{C}$, so we say that $x \mapsto g_x$ is an algebra homomorphism. Thus we have represented the elements of $A$ as continuous functions on the maximal ideal space, or on the character space depending on which notion one wishes to emphasize. This representation is called the **Gelfand transform**. The kernel of the homomorphism

$$A \to C(\mathcal{M}, \mathbf{C}) \approx C(\hat{A}, \mathbf{C})$$

is obviously the intersection of all maximal ideals, which is called the **radical** of $A$. For general Banach algebras, one cannot say anything further, but in the next section, we shall give a criterion when this kernel is 0. In such a case, $A$ "is" the algebra of continuous functions on a compact space, obtained from $A$ in a natural way.

## XVI, §3. $C^*$-ALGEBRAS

Let $A$ be a complex Banach algebra. By an **involution** on $A$ we mean a map $x \mapsto x^*$ of $A$ into itself satisfying:

$$(x + y)^* = x^* + y^*, \qquad (\alpha x)^* = \bar{\alpha} x^*,$$
$$(xy)^* = y^* x^*, \qquad x^{**} = x.$$

If $A$ has a unit element $e$, then the reader will verify at once that $e^* = e$. If further $x$ is invertible, then $(x^{-1})^* = (x^*)^{-1}$.

By a **$C^*$-algebra**, we mean a Banach algebra with an involution as above, satisfying the additional condition

$$|x^*x| = |x|^2 \qquad \text{for all } x \in A.$$

*We suppose $A$ is a $C^*$-algebra for the rest of this section.* Observe that from the defining condition, we get

$$|x|^2 \leqq |x^*||x| \qquad \text{whence} \qquad |x| \leqq |x^*|.$$

Since $x = x^{**}$, it follows that

$$|x| = |x^*|.$$

**Examples.** The standard example of a commutative $C^*$-algebra is that of the algebra of continuous functions on a compact Hausdorff space, with the sup norm. The involution is given by the complex conjugate, $f^* = \bar{f}$. Theorem 3.3 below shows that under certain conditions, there is no other.

Let $H$ be a Hilbert space, and let $A = \text{End}(H)$ be the algebra of bounded linear maps of $H$ into itself (operators). Then $A$ is a $C^*$-algebra, the star operation being the adjoint. In Chapter XVIII, we shall consider the commutative subalgebra generated by one hermitian operator and reprove the basic theorem independently of the result in this section.

For another example of an involution (which does not satisfy the condition of a $C^*$-algebra), see Exercise 9.

We have assumed the existence of a unit element in the algebra $A$ for simplicity. When there is no unit element a priori, one may embed $A$ in another algebra with unit, i.e. adjoin a unit element, and thereby reduce the study of an involution to the case when a unit element is present. We leave this construction as Exercises 5 through 8.

**Proposition 3.1.** *Let $A$ be a $C^*$-algebra with unit. Let $x \in A$. If $x = x^*$ then the spectrum $\sigma(x)$ is real and $\rho(x) = |x|$.*

*Proof.* Let $z \in \sigma(x)$. For all real $t$ it follows that $z + it \in \sigma(x + ite)$. By Theorem 1.7 we obtain:

$$\begin{aligned} |z + it|^2 \leqq |x + ite|^2 &\leqq |(x + ite)(x - ite)| \\ &\leqq |x^2 + t^2e| \\ &\leqq |x|^2 + t^2 \end{aligned}$$

using the condition of a $C^*$-algebra, and the convention that $|e| = 1$. Write $z = a + ib$ with $a$, $b$ real. Then for all real $t$ we find

$$a^2 + b^2 + 2bt \leqq |x|^2,$$

which is possible only if $b = 0$, so $z$ is real, and the spectrum is real.

As to the statement about the spectral radius, from the condition defining a $C^*$-algebra, we have

$$|x|^{2^n} = |x^{2^n}|,$$

and we simply use the definition of the spectral radius to conclude the proof.

**Proposition 3.2.** *Let $A$ be a commutative $C^*$-algebra with unit. Let $\lambda$ be a character. Then $\lambda(x^*) = \overline{\lambda(x)}$ for all $x \in A$.*

*Proof.* If $x = x^*$ then the assertion follows from Proposition 3.1 because $\lambda(x)$ is in the spectrum of $x$. For $x$ arbitrary, we decompose $x$ into a sum

$$x = u - iv \qquad \text{with} \qquad u = \tfrac{1}{2}(x + x^*) \quad \text{and} \quad v = \frac{1}{2i}(x - x^*).$$

Then $u$, $v$ satisfy $u = u^*$ and $v = v^*$. Furthermore

$$\lambda(x^*) = \lambda(u - iv) = \lambda(u) - i\lambda(v) = \overline{\lambda(u) + i\lambda(v)} = \overline{\lambda(x)},$$

which proves the proposition.

We now come to the main structure theorem.

**Theorem 3.3 (Gelfand–Naimark).** *Let $A$ be a commutative $C^*$-algebra with unit. Then the intersection of all maximal ideals of $A$ is 0, and the map $x \mapsto f_x$ gives a norm-preserving isomorphism of $A$ on $C(\mathcal{M}, \mathbf{C})$.*

*Proof.* It follows as an immediate consequence of the Stone–Weierstrass theorem that the image $g(A)$ in $C(\mathcal{M})$ is dense. Since $A$ is assumed complete, it will suffice to prove that the map $g$ is norm-preserving. Let $x \in A$ and write $g(x)$ instead of $g_x$. Then by Proposition 3.2,

$$g(x^*x) = g(x^*)g(x) = \overline{g(x)}g(x) = |g(x)|^2.$$

But the element $y = x^*x$ satisfies $y = y^*$. By Theorem 1.7 and Proposi-

tion 3.1, we find

$$|g(y)| = \rho(y) = |y|,$$

so $|g(x)| = |x|$ by the definition of a $C^*$-algebra, thus concluding the proof.

## XVI, §4. EXERCISES

1. Let $A$ be a complex Banach algebra with unit element, and let $u \in A$. Let $\sigma(u)$ be the spectrum of $u$. Let $p$ be a polynomial with complex coefficients. Show that the spectrum of $p(u)$ is equal to $p(\sigma(u))$, i.e. to the set of all numbers $p(\alpha)$, where $\alpha$ lies in the spectrum of $u$. [*Hint:* For one inclusion write

$$p(t) - p(\alpha) = (t - \alpha)q(t)$$

for some polynomial $q$, and for the other, write

$$p(t) - \alpha = (t - \alpha_1) \cdots (t - \alpha_n)$$

if $\alpha$ is in the spectrum of $p(u)$.] Of course, the result applies especially if $u$ is an operator on a Banach space $E$.

2. Let $E$ be a Banach space and let $F$, $G$ be two closed subspaces such that $E = F \oplus G$ is their direct sum. Let $A$ be an operator on $E$ and assume that $F$, $G$ are $A$-invariant. Let $A_F$ and $A_G$ denote the restrictions of $A$ to $F$ and $G$ respectively. Let $\sigma(A)$ denote the spectrum of $A$.
(a) Let $\alpha$ be a complex number. Show that $A - \alpha I$ is invertible if and only if $A_F - \alpha I_F$ and $A_G - \alpha I_G$ are invertible.
(b) Show that

$$\sigma(A) = \sigma(A_F) \cup \sigma(A_G).$$

3. Let $A$ be a complex Banach algebra with unit element $e$ and let $u \in A$. Show that the map

$$z \mapsto (u - ze)^{-1}$$

is (complex) differentiable and analytic on the complement of the spectrum of $u$. One calls $R(u, z) = (u - ze)^{-1}$ the **resolvent** of $u$.

Run through systematically some of the basic theorems of complex analysis and prove them in the Banach-valued case if they are true. (Liouville's theorem, Cauchy's theorem, etc.)

4. Let $\psi$ be the function defined for real $t$ by

$$\psi(t) = e^{it},$$

so that $\psi$ can be viewed as a function on the unit circle. Let $A$ be the set of

all functions which can be written as infinite sums

$$f = \sum_{-\infty}^{\infty} c_n \psi^n$$

with $c_n$ complex, satisfying

$$\sum_{-\infty}^{\infty} |c_n| < \infty.$$

For the norm defined by

$$\|f\| = \sum |c_n|,$$

show that $A$ is a Banach algebra under ordinary addition and multiplication of functions. Prove that if $f(t) \neq 0$ for all $t$, then $f$ is invertible in $A$. [*Hint*: It suffices to prove that for any character $\lambda$ of $A$ we have $\lambda(f) \neq 0$. Show that $|\lambda(\psi)| = 1$ so that $\lambda(\psi) = \psi(t_0)$ for some $t_0$.]

5. **Adjunction of a unit element.** Let $A$ be a normed algebra. We embed $A$ in a normed algebra with unit as follows. Let $\tilde{A} = \mathbf{C} \times A$, with addition and multiplication defined componentwise. Define

$$|(z, x)| = |z| + |x| \qquad \text{for} \quad z \in \mathbf{C} \text{ and } x \in A.$$

Prove that $A$ is a normed algebra, with unit $(1, 0)$, and containing an isomorphic image of $A$ as the subset of elements $(0, x)$ with $x \in A$. Show that $A$ is an ideal in $A$. *Warning*: If $A$ happened to have a unit element, then this unit is not the same one as the unit in $\tilde{A}$.

6. Suppose $A$ is a $C^*$-algebra but without our assuming that $A$ has a unit element. Prove that there exists on $\tilde{A}$ a norm extending the norm on $A$ which makes $A$ into a $C^*$-algebra with unit. (*Warning*: It is not the norm of Exercise 5.) [*Hint*: Observe that for $x \in A$, we have

$$|x| = \sup_{y \in A} \frac{|xy|}{|y|}$$

and so for $x \in \tilde{A}$ define $|x|$ by the same formula. Since $A$ is an ideal in $\tilde{A}$, it follows that $xy \in A$ for $x \in \tilde{A}$ and $y \in A$. Prove that this definition gives a norm on $A$, and that this norm satisfies the condition for a $C^*$-algebra.]

7. Let $A$ be a Banach algebra with an involution, and let $B$ be a $C^*$-algebra. Let $h: A \to B$ be an algebraic homomorphism (no condition is put with respect with the norm) such that $h(x^*) = h(x)^*$. Prove that

$$|h(x)| \leqq |x| \qquad \text{for all} \quad x \in A.$$

8. Using Exercise 7, prove that the norm on $\tilde{A}$ defined in Exercise 6 is the unique norm extending the norm on $A$, and making $\tilde{A}$ into a $C^*$-algebra.

9. With the same notation as Exercise 10 of Chapter XII, if $m \in M^1(G)$, define

$$m^* = \overline{m^\vee} = \overline{m}^\vee.$$

Show that this star operation is an involution, and that $\|m\| = \|m^*\|$.

10. Let $A$ be a $C^*$-algebra with unit, and let $x \in A$ be such that $x^* = x^{-1}$. Show that the spectrum of $x$ lies on the unit circle.

11. Let $A$ be a Banach algebra with unit $e$. Let $v \in A$. Show that there exists a unique $C^\infty$ mapping $K: \mathbf{R} \to A$ such that:

**H 1.** We have $\dfrac{d}{dt} K(t) = K(t)v$.

**H 2.** $K(0) = e$.

Show that the image of $K$ lies in the multiplicative group of invertible elements of $A$.

CHAPTER XVII

# Compact and Fredholm Operators

The operators in infinite dimensional spaces closest to operators in finite dimensional spaces are the compact operators, which will now be studied systematically. A large number of examples of compact operators are given in the exercises.

## XVII, §1. COMPACT OPERATORS

We recall that a subset of a topological space is said to be **relatively compact** if its closure is compact. We had proved a convenient criterion for this (Corollary 3.9 of Chapter II), namely:

*Let $X$ be a subset of a complete normed vector space. Assume that given $r > 0$ there exists a finite covering of $X$ by balls of radius $r$. Then $X$ is relatively compact.*

This criterion will be used frequently in this chapter.

Let $E$, $F$ be normed vector spaces (not necessarily complete) and let

$$u: E \to F$$

be a linear map. We say that $u$ is **compact** if $u$ maps bounded sets in $E$ into relatively compact sets in $F$. Equivalently, we can say that $u$ maps the unit ball in $E$ into a relatively compact set in $F$. It is then clear that $u$ must be continuous, because if $B$ is the unit ball in $E$, then $u(B)$ has compact closure, whence is bounded. It is also clear that our definition

is equivalent to saying that if $\{x_n\}$ is a bounded sequence in $E$, then $\{ux_n\}$ has a convergent subsequence.

**Examples.** If $E$ or $F$ is finite dimensional, then $u$ is compact. Consequently, if just the image of $u$ is finite dimensional, then $u$ is compact.

Since a locally compact Banach space is finite dimensional, by Corollary 3.15 of Chapter II, it follows that the identity map of an infinite dimensional Banach space is not compact.

In a later section, we shall prove that the following type of operator is compact. Let $K(x, y)$ be a continuous function on the rectangle $a \leqq x \leqq b$ and $c \leqq y \leqq d$. If $f$ is continuous on $[a, b]$, we define

$$Sf(y) = \int_a^b K(x, y)f(x)\,dx.$$

It will be shown later that $S$ is compact. Thus our theory applies to the study of this type of integral equation.

We denote by $K(E, F)$ the set of compact linear maps of $E$ into $F$.

**Theorem 1.1.** *The compact linear mappings from E to F form a vector space. If F is complete, then $K(E, F)$ is a closed subspace of $L(E, F)$.*

*Proof.* If $X$, $Y$ are compact in $F$ then $X + Y$ is compact, being the continuous image of the compact set $X \times Y$ under the map $(x, y) \mapsto x + y$. If $B$ is the unit ball in $E$, then it follows that for $u, v \in K(E, F)$ the set $\overline{u(B)} + \overline{v(B)}$ is compact. But then

$$\overline{u(B) + v(B)} \subset \overline{u(B)} + \overline{v(B)}.$$

Since $\overline{u(cB)} = \overline{cu(B)}$ for any scalar $c$, it follows that $K(E, F)$ is a vector space. To show it is closed in $L(E, F)$ when $F$ is complete, let $u$ be in its closure. It will suffice to prove that $u(B)$ is covered by a finite number of open balls of given radius $r$. Let $v \in K(E, F)$ be such that $|u - v| < r/2$. Since $v$ is compact, we know that $v(B)$ is covered by a finite number of open balls of radius $r/2$, centered say at points $y_1, \ldots, y_n$. For each $x \in B$ we then have

$$|u(x) - v(x)| < r/2 \qquad \text{and} \qquad |v(x) - y_i| < r/2$$

for some $i$. This implies that $|u(x) - y_i| < r$, and hence that $u(B)$ is covered by a finite number of balls of radius $r$, as was to be shown.

**Remark 1.** Let $F$ be a Banach space. It follows from Theorem 1 that if $\{u_n\}$ is a sequence of elements of $L(E, F)$ such that the image of $u_n$ is

finite dimensional for all $n$, and if $\{u_n\}$ converges to an element $u$ of $L(E, F)$, then $u$ is compact. It is not known, however, if a compact operator can always be expressed as the limit of such a sequence. It does hold for compact operators in Hilbert space.

**Remark 2.** We gave the definition of compact mappings on spaces which are not necessarily complete. Note that if $u: E \to F$ is compact, and if $\bar{E}$, $\bar{F}$ denote the completions of $E$ and $F$ respectively, then the linear continuous extension

$$\bar{u}: \bar{E} \to \bar{F}$$

is also compact. This is immediate. Furthermore, if $E_1$ is any subspace of $E$ and $F_1 \supset \overline{u(E_1)}$, then the restriction

$$u|E_1: E_1 \to F_1$$

is also compact.

**Theorem 1.2.** *Let $E$, $F$, $G$, $H$ be normed vector spaces and let*

$$f: E \to F, \qquad u: F \to G, \qquad g: G \to H$$

*be continuous linear maps. If $u$ is compact then $u \circ f$ and $g \circ u$ are compact. In particular, $K(E, E)$ is a two-sided ideal of $L(E, E)$.*

*Proof.* The first relation follows from the fact that a continuous image of a compact set is compact. The second is obvious. The third comes from the definitions.

**Theorem 1.3.** *Let $E$, $F$ be Banach spaces and $u: E \to F$ a compact linear map. Then $u': F' \to E'$ is compact.*

*Proof.* One can give a direct simple proof, but the reader will note that our assertion is an immediate consequence of the Ascoli theorem. We shall make no use of Theorem 1.3 in this book, and thus we leave the details to the reader.

## XVII, §2. FREDHOLM OPERATORS AND THE INDEX

Let $E$, $F$ be normed vector spaces. A continuous linear map

$$T: E \to F$$

is said to be **Fredholm** if:

(i) Ker $T$ is finite dimensional.
(ii) Im $T$ is closed and finite codimensional.

**Example (The Shift Operator).** Let $E$ be the Hilbert space of all sequences $\alpha = \{a_n\}$ such that $\sum |a_n|^2$ converges. (This is essentially the space of Fourier series.) We define

$$T: E \to E$$

by

$$T\alpha = (a_2, a_3, \ldots)$$

if $\alpha = (a_1, a_2, \ldots)$. The kernel of $T$ is 1-dimensional, and $T$ is surjective. There are variants of this operator, for instance the operator such that

$$(a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots),$$

which has 0 kernel, and whose image has codimension 1.

We shall show in Theorem 2.1 that if $u$ is compact, then $I - u$ is Fredholm. In a later section, we shall give other examples of Fredholm operators, as integral or differential operators. The reader may look at these now to see the concrete applications of our algebra to analysis.

We shall use constantly the corollaries of Theorem 1.3, Chapter XV, which the reader is advised to review carefully. The results expressed in these corollaries, most of which depend on the open mapping theorem, will be quoted without further specific reference. We note in particular that as a consequence of these corollaries, when $E$, $F$ are Banach spaces, then the hypothesis for Fredholm maps $T$ that $\operatorname{Im} T$ is closed follows from the finite codimensionality, and could thus be omitted from the definition of a Fredholm map in this case.

We shall also use the fact that a finite dimensional subspace of a Banach space admits a closed complement. This was an exercise using the Hahn–Banach theorem.

**Theorem 2.1.** *Let $E$ be a Banach space, and $u: E \to E$ a compact operator. Then $I - u$ is Fredholm.*

*Proof.* The identity $I$ restricted to the kernel of $I - u$ is equal to $u$, and is consequently compact. Hence this kernel is finite dimensional, because a locally compact normed vector space is finite dimensional.

Now we show that the image of $I - u$ is closed. Let $T = I - u$. Let $G$ be a closed complement for Ker $T$, so that

$$E = \operatorname{Ker}(I - u) \oplus G.$$

We obtain continuous linear maps

$$T|G: G \to E \qquad \text{and} \qquad u|G: G \to E,$$

the restrictions of $T$ and $u$ to $G$. Furthermore, the kernel of $T|G$ is $\{0\}$.

It will suffice to prove that $TG = TE$ is closed, and for this it will suffice to prove that the inverse map

$$(T|G)^{-1}: TG \to G$$

is continuous. (Indeed, in that case, $TG$ is complete, so closed.) It even suffices to prove that $(T|G)^{-1}$ is continuous at 0, by linearity. Suppose that this is not the case. Then we can find a sequence $\{x_n\}$ in $G$ such that $Tx_n \to 0$, but $\{x_n\}$ does not converge to 0. Selecting a suitable subsequence, we can assume without loss of generality that $|x_n| \geqq r > 0$ for all $n$. Then $1/|x_n| \leqq 1/r$ for all $n$, and consequently $T(x_n/|x_n|)$ also converges to 0. Furthermore, $x_n/|x_n|$ has norm 1, and hence some subsequence of

$$u\left(\frac{x_n}{|x_n|}\right)$$

converges. Since

$$T\left(\frac{x_n}{|x_n|}\right) = \frac{x_n}{|x_n|} - u\left(\frac{x_n}{|x_n|}\right)$$

it follows that a subsequence of $\{x_n/|x_n|\}$ converges to some element $z$ in $G$, also having norm 1. But then $0 = z - u(z)$, and $Tz = 0$. This contradicts the fact that $G \cap \operatorname{Ker} T = \{0\}$, and thus proves that $TE = TG$ is closed.

Finally we have to show that $TE$ has finite codimension. We shall need the following lemma, which will also be used later in the spectral theorem.

**Lemma 2.2.** *Given $\varepsilon$. Let $F$ be a closed subspace of a normed vector space $H$, and assume that $F \neq H$. Then there exists $x \in H$ with $|x| = 1$ such that*

$$d(x, F) = \inf_{y \in F} |x - y| \geqq 1 - \varepsilon.$$

*Proof.* Let $z \in H$ and $z \notin F$. Select $y_0 \in F$ such that

$$|z - y_0| \leqq \left(\inf_{y \in F} |z - y|\right)(1 + \varepsilon).$$

We let

$$x = \frac{z - y_0}{|z - y_0|}.$$

Then for $y \in F$ we have

$$|x - y| = \left|\frac{z - y_0}{|z - y_0|} - y\right| = \frac{|z - y_0 - |z - y_0|y|}{|z - y_0|} \geqq \frac{|z - y_0|}{|z - y_0|(1 + \varepsilon)},$$

which proves our lemma.

To apply the lemma, suppose that $TE$ does not have finite codimension. We can find a sequence of closed subspaces

$$TE = H_0 \subset H_1 \subset \cdots \subset H_n \subset \cdots$$

such that each $H_n$ is closed and of codimension 1 in $H_{n+1}$ just by adding one-dimensional spaces to $TE$ inductively. By the lemma, we can find in each $H_n$ an element $x_n$ such that $|x_n| = 1$ and $|x_n - y| \geqq 1 - \varepsilon$ for all $y \in H_{n-1}$. Then for all $k < n$:

$$\begin{aligned} |ux_n - ux_k| &= |x_n - Tx_n - x_k + Tx_k| \\ &\geqq 1 - \varepsilon \end{aligned}$$

because $-Tx_n - x_k + Tx_k$ lies in $H_{n-1}$. This shows that the sequence $\{ux_n\}$ cannot have a convergent subsequence, and contradicts the compactness of $u$, thus proving Theorem 2.1.

We denote by $\text{Fred}(E, F)$ the set of Fredholm operators from $E$ into $F$. If $T \in \text{Fred}(E, F)$, then we define the **index** of $T$ to be

$$\text{ind } T = \dim \text{Ker } T - \dim F/TE.$$

In the language of linear algebra, the factor space $F/TE$ is also called the **cokernel** of $T$, and thus

$$\text{ind } T = \dim \text{Ker } T - \dim \text{coker } T.$$

**Theorem 2.3.** *Let $E$, $F$ be Banach spaces. Then* $\text{Fred}(E, F)$ *is open in* $L(E, F)$, *and the function* $T \mapsto \text{ind } T$ *is continuous on* $\text{Fred}(E, F)$, *hence constant on connected components.*

*Proof.* Let $S: E \to F$ be a Fredholm operator. We wish to prove that if $T \in L(E, F)$ is close to $S$, then $T$ itself is Fredholm. Let $N$ be the kernel of $S$, and let $G$ be a closed complement for $N$, that is $E = N \oplus G$. Then $S$ induces a toplinear isomorphism of $G$ on its image $SG$ (by the open mapping theorem), and we can write $F = SG \oplus H$ for some finite dimensional subspace $H$. The map

$$G \times H \to SG \oplus H = F$$

given by

$$(x, y) \mapsto Sx + y$$

is a toplinear isomorphism. We know that the set of toplinear isomorphisms of one Banach space into another is open in the space of all

continuous linear maps. If $T$ is close to $S$, then the map

$$G \times H \to TG \oplus H = F$$

given by

$$(x, y) \mapsto Tx + y$$

is therefore also a toplinear isomorphism. Hence the kernel of $T$ is finite dimensional, since $G \cap \operatorname{Ker} T = \{0\}$, and in fact $\dim \operatorname{Ker} T$ is at most equal to the codimension of $G$ in $E$. The image of $T$ has finite codimension (at most equal to the dimension of $H$), and is consequently closed, by Corollary 1.8 of Chapter XV. This proves that $T$ is Fredholm, and proves our first assertion.

Now concerning the index, we observe that $G \oplus \operatorname{Ker} T$ is a direct sum of two closed subspaces, and there is some finite dimensional subspace $M$ such that

$$E = G \oplus \operatorname{Ker} T \oplus M.$$

Then $T$ induces a toplinear isomorphism

$$G \oplus M \to T(G \oplus M) = TG \oplus TM,$$

and

$$\dim M = \dim TM.$$

Hence we get

$$\begin{aligned}
\operatorname{ind} T &= \dim \operatorname{Ker} T - (\dim H - \dim TM) \\
&= \dim \operatorname{Ker} T + \dim M - \dim H \\
&= \dim \operatorname{Ker} S - \dim H \\
&= \operatorname{ind} S.
\end{aligned}$$

This proves our theorem.

**Corollary 2.4.** *Let $E$ be a Banach space and $u$ a compact operator on $E$. If $I - u$ is injective* (i.e. $\operatorname{Ker} I - u = \{0\}$), *then $I - u$ is a toplinear automorphism.*

*Proof.* For each real $t$, the operator $tu$ is compact. The map $t \mapsto tu$ is continuous, and so is the map

$$t \mapsto \operatorname{ind}(I - tu).$$

Hence this map is constant. Letting $t = 0$ and $t = 1$ shows that

$$\operatorname{ind}(I - u) = 0.$$

Hence $I - u$ is surjective, whence a toplinear isomorphism by the open mapping theorem.

**Note.** Examples of compact operators $u$ furnish immediately examples of Fredholm operators $I - u$. For other examples, cf. for instance Smale's paper [Sm 3]. One obtains Fredholm linear maps by taking the derivative of certain "Fredholm" non-linear maps, which are of interest in differential equations. Thus one sees the linearization provided by the derivative as a first step in analyzing non-linear problems.

Let $T$, $S$ be continuous linear maps $E \to F$. We say that $T$ is **congruent to $S$ modulo compact operators** if $T - S$ is compact, and we write

$$T \equiv S \bmod K(E, F).$$

This congruence is an equivalence relation, and if $T \equiv S$, $T_1 \equiv S_1$, then $TT_1 \equiv SS_1$. This is immediately verified as a consequence of Theorem 1.2. Of course, the composition $TT_1$ (or $SS_1$) must make sense. It means that we compose $T_1 : E_1 \to E$ with $T$ as above, and similarly with $SS_1$. Similar congruence statements hold for sums.

We say that $T: E \to F$ is **invertible modulo compact operators** if there exists a continuous linear map $T_1 : F \to E$ such that

$$TT_1 \equiv I_F \bmod K(F, F) \qquad \text{and} \qquad T_1 T \equiv I_E \bmod K(E, E).$$

Thus we call $T_1$ an **inverse** of $T$ modulo compact operators.

**Theorem 2.5.** *Let $E$, $F$ be Banach spaces and let $T: E \to F$ be a continuous linear map. Then $T$ is Fredholm if and only if $T$ is invertible modulo compact operators $K(E, F)$. We can select an inverse of $T$ modulo compact operators, having finite codimensional image.*

*Proof.* Let $T$ be Fredholm, and write direct sum decompositions

$$E = \operatorname{Ker} T \oplus G, \qquad F = \operatorname{Im} T \oplus H$$

with closed subspaces $G$, $H$. We let $S$ be the composite

$$F = \operatorname{Im} T \oplus H \xrightarrow{\text{pr}} \operatorname{Im} T \xrightarrow{T^{-1}} G \xrightarrow{\text{inc.}} E$$

where pr is the production, and inc. is the inclusion. Then $I_F - TS$ is the projection on $H$, and $I_E - ST$ is the projection on Ker $T$. This proves that $T$ has an inverse modulo compact operators. Conversely, suppose that $S$ is such an inverse; then we have

$$\operatorname{Ker} T \subset \operatorname{Ker} ST,$$

so $T$ has finite dimensional kernel. Also

$$\operatorname{Im} T \supset \operatorname{Im} TS$$

and $TS$ has a closed image of finite codimension by Theorem 2.1. Hence Im $T$ has a closed image of finite codimension, so that $T$ is Fredholm. This proves our theorem.

**Note.** As an exercise, prove the usual uniqueness of an inverse, that is, suppose that there exist continuous linear maps $T_1$, $T_2$ such that

$$TT_1 \equiv I_F \bmod K(F, F)$$

and

$$T_2 T \equiv I_E \bmod K(E, E).$$

Show that $T_1 \equiv T_2 \bmod K(F, E)$, and that $T_1$ or $T_2$ is thus an inverse for $T$ modulo compact operators.

**Corollary 2.6.** *The composite of Fredholm maps is Fredholm. If $T$ is Fredholm and $u$ is compact, then $T + u$ is Fredholm.*

*Proof.* Clear.

**Corollary 2.7.** *If $T$ is Fredholm and $u$ is compact, then*

$$\operatorname{ind}(T + u) = \operatorname{ind} T.$$

*Proof.* The same proof works as for the corollary of Theorem 2.3, namely we connect $T + u$ with $T$ by the segment.

$$T + tu, \qquad 0 \leqq t \leqq 1.$$

The next theorem will not be used later in a significant way and thus its proof can be omitted if the reader is allergic to formal algebra.

**Theorem 2.8.** *Let $E$, $F$, $G$ be Banach spaces, and let*

$$S\colon E \to F \qquad \textit{and} \qquad T\colon F \to G$$

*be Fredholm. Then*

$$\operatorname{ind} TS = \operatorname{ind} T + \operatorname{ind} S.$$

*Proof.* To do this proof properly, we need an algebraic lemma. Let $V$ be a vector space and $W$ a subspace. Let

$$f\colon V \to f(V)$$

be a linear map, with image $f(V)$, which we also write $fV$ for simplicity of notation. If the factor space $V/W$ is finite dimensional, we denote by $(V : W)$ the dimension of the factor space $V/W$. We denote by $V_f$ the kernel of $f$ in $V$, and by $W_f$ the kernel of $f$ in $W$, that is $W \cap V_f$.

**Lemma 2.9.** *Let $V$ be a vector space and $W$ a subspace. Let $f: V \to fV$ be a linear map. Then*

$$(V : W) = (fV : fW) + (V_f : W_f)$$

*in the sense that if two of these induces are finite, then so is the third, and the stated relation holds.*

*Proof.* Consider the composite of linear maps

$$V \to fV \to fV/fW.$$

The kernel certainly contains $V_f + W$. If $x \in V$ lies in the kernel, this means that there exists some $y \in W$ such that $f(x) = f(y)$, and then $f(x - y) = 0$, so $x - y$ lies in $V_f$. Hence the kernel is precisely equal to $V_f + W$. Hence we obtain an isomorphism

$$V/(V_f + W) \xrightarrow{\approx} fV/fW. \tag{1}$$

We have inclusions of subspaces

$$W \subset V_f + W \subset V. \tag{2}$$

We consider the linear map

$$V_f \to (V_f + W)/W$$

given by

$$x \mapsto \text{class of } x \text{ modulo } W.$$

An element of the kernel is such that it also lies in $W$, so that we obtain an isomorphism

$$V_f/W_f \xrightarrow{\approx} (V_f + W)/W. \tag{3}$$

From this we see at once that if two of our indices are finite, so is the third. Indeed, suppose that $(V : W)$ and $(fV : fW)$ are finite. Then $(V : V_f + W)$ is finite by (1) and hence $(V_f : W_f)$ is finite by (3). The others are proved similarly. As for the relation concerning the dimensions, we see that whenever our indices are finite, then

$$(V : W) = (V : V_f + W) + (V_f + W : W). \tag{4}$$

If we now use (1) and (3), we get the relation stated in the lemma, as was to be shown.

We return to the proof of Theorem 2.8.

For simplicity of notation, we use our notation $E_S$ for the kernel of $S$ in $E$. We write down the definitions of the index for $T$, $S$ and $TS$:

$$\operatorname{ind} S = (E_S : 0) - (F : SE),$$
$$\operatorname{ind} T = (F_T : 0) - (G : TF),$$
$$\operatorname{ind} TS = (E_{TS} : 0) - (G : TSE).$$

We have the following inclusions:

$$\{0\} \subset \operatorname{Ker} S \subset \operatorname{Ker} TS$$

because $Sx = 0$ implies $TSx = 0$, and also

$$TSE \subset TF \subset G.$$

Hence

(5) $$(E_{TS} : 0) = (E_{TS} : E_S) + (E_S : 0)$$

and

(6) $$(G : TSE) = (G : TF) + (TF : TSE).$$

We apply our lemma to the spaces $SE \subset F$, and to the map $T$. We then get

(7) $$(F : SE) = (TF : TSE) + (F_T : SE \cap F_T).$$

From the inclusions

$$\{0\} \subset SE \cap F_T \subset F_T$$

we obtain

(8) $$(F_T : 0) = (F_T : SE \cap F_T) + (SE \cap F_T : 0).$$

If we now substitute the values of (5), (6), (7), (8) into the expression for

$$\operatorname{ind} TS - \operatorname{ind} T - \operatorname{ind} S,$$

we obtain

$$(E_{TS} : E_S) - (SE \cap F_T : 0),$$

and we have to show that this is equal to 0. Let us write $E_{TS}$ as a direct

sum

$$E_{TS} = E_S \oplus W$$

for some finite dimensional $W$. Then we can write $E$ as a direct sum

$$E = E_{TS} \oplus U = E_S \oplus W \oplus U.$$

Then

$$SE = S(W \oplus U) = SW \oplus SU.$$

We contend that $SE \cap F_T = SW$. Indeed, it is clear that $SW \subset F_T$, and conversely, if $y \in SE$, $y = Sx$ and $TSx = 0$, then $x = x_1 + x_2$ with $x_1 \in E_{TS}$ and $x_2 \in W$, and $Sx = Sx_2 \in SW$ whence $SE \cap F_T = SW$. But then

$$(E_{TS} : E_S) = (W : 0) = (SW : 0)$$

because $S$ is an isomorphism on $W$. This concludes the proof of our theorem.

## XVII, §3. SPECTRAL THEOREM FOR COMPACT OPERATORS

*Throughout this section, we let $E$ be a Banach space, and let $u: E \to E$ be a compact operator.*

We are interested in the spectrum of $u$. We recall that a number $\alpha$ is called an **eigenvalue** for $u$ if there exists a non-zero vector $x \in E$ such that

$$ux = \alpha x.$$

In that case we call $x$ an **eigenvector** for $u$, belonging to $\alpha$.

If $\alpha$ is a number $\neq 0$, then $\alpha u$ is compact and so is $\alpha^{-1}u$. Hence $u - \alpha I$ and $\alpha I - u$ are Fredholm, by Theorem 2.1. Furthermore, for every positive integer $n$, the operator $(I - u)^n$ can be written as

$$(I - u)^n = I - u_1$$

for some compact $u_1$, because we expand with the binomial expansion, and use Theorem 1.2. Hence $(u - \alpha I)^n$ is also Fredholm for $\alpha \neq 0$.

By Corollary 2.4, we know that for $\alpha \neq 0$,

$$\operatorname{ind}(u - \alpha I)^n = 0,$$

in other words,

$$\dim \operatorname{Ker}(u - \alpha I)^n = \dim \operatorname{coker}(u - \alpha I)^n.$$

**Theorem 3.1.** *Let $\alpha$ be a number $\neq 0$. Either $u - \alpha I$ is invertible, or $\alpha$ is an eigenvalue of $u$. In other words, every element of the spectrum of $u$ is an eigenvalue, except possibly 0. If $E$ is infinite dimensional, then 0 is in the spectrum.*

*Proof.* By Corollary 2.4 we know that if $u - \alpha I$ has kernel $\{0\}$, then $u - \alpha I$ is invertible. Thus our first statement is essentially merely a reformulation of this corollary. If 0 is not in the spectrum, then $u$ is invertible, and then the image of the closed unit ball by $u$ is homeomorphic to this unit ball and is compact, so that $E$ is locally compact, and hence finite dimensional, thereby proving Theorem 3.1.

In the theory of a finite dimensional vector space $V$, with an endomorphism $u: V \to V$, one knows that we can decompose $V$ into a direct sum

$$V = N_1 \oplus \cdots \oplus N_m$$

such that each $N_i$ corresponds to an eigenvalue $\alpha_i$ of $u$, and such that for each $i$, there exists an integer $r_i$ having the property that

$$(u - \alpha_i I)^{r_i} N_i = 0.$$

As one says, $u - \alpha_i I$ is nilpotent on $N_i$. When that is the case, a theorem like the Jordan normal form theorem gives a canonical matrix representing $u$ with respect to a suitable basis, namely blocks consisting of triangular matrices of type

$$\begin{pmatrix} \alpha_i & 1 & 0 & \cdots & 0 \\ 0 & \alpha_i & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \alpha_i \end{pmatrix}$$

Thus the decomposition of $V$ into subspaces as above yields complete information concerning $u$. We shall now do this for compact operators. Of course, we get infinitely many subspaces in the decomposition, corresponding to possibly infinitely many eigenvalues.

**Lemma 3.2.** *Let $\alpha$ be a non-zero eigenvalue for $u$. Then there exists an integer $r > 0$ such that*

$$\operatorname{Ker}(u - \alpha I)^r = \operatorname{Ker}(u - \alpha I)^n$$

*for all $n \geqq r$.*

*Proof.* It suffices to prove that $\operatorname{Ker}(I-u)^r = \operatorname{Ker}(I-u)^n$, for all $n \geqq$ some $r$. Suppose that this is not the case. Then we have a strictly ascending chain of subspaces

$$\operatorname{Ker}(I-u) \subsetneqq \operatorname{Ker}(I-u)^2 \subsetneqq \cdots \subsetneqq \operatorname{Ker}(I-u)^n \subsetneqq \cdots.$$

By Lemma 2.2, we can find an element

$$x_n \in \operatorname{Ker}(I-u)^n$$

such that $|x_n| = 1$ and $x_n$ is at distance $\geqq 1-\varepsilon$ from $\operatorname{Ker}(I-u)^{n-1}$. Let $T = I - u$. Then just as in this lemma, we find for $k < n$:

$$\begin{aligned}|ux_n - ux_k| &= |x_n - Tx_n - x_k + Tx_k| \\ &\geqq 1-\varepsilon\end{aligned}$$

because $Tx_n$ lies in $\operatorname{Ker}(I-u)^{n-1}$. This contradicts the compactness of $u$ and proves the lemma.

It is clear that if $\operatorname{Ker}(u-\alpha I)^r = \operatorname{Ker}(u-\alpha I)^{r+1}$ for some $r$, then

$$\operatorname{Ker}(u-\alpha I)^r = \operatorname{Ker}(u-\alpha I)^n$$

for all $n \geqq r$. We call the smallest integer $r$ for which this is true the **exponent** of $\alpha$.

**Theorem 3.3.** *Let $\alpha$ be a non-zero eigenvalue of $u$, and let $r$ be its exponent. Then we have a direct sum decomposition*

$$E = \operatorname{Ker}(u-\alpha I)^r \oplus \operatorname{Im}(u-\alpha I)^r,$$

*and each of the spaces occurring in this direct sum is a closed invariant subspace of $u$. If $\beta \neq \alpha$ is another non-zero eigenvalue of $u$, and $s$ is its exponent, then*

$$\operatorname{Ker}(u-\beta I)^s \subset \operatorname{Im}(u-\alpha I)^r.$$

*Proof.* Let $T = (u-\alpha I)^r$. Then $T$ is Fredholm. Both $\operatorname{Ker} T$ and $\operatorname{Im} T$ are $u$-invariant closed subspaces. Furthermore, we have

$$\operatorname{Ker} T \cap \operatorname{Im} T = \{0\}.$$

Indeed, suppose that $x \in \operatorname{Ker} T \cap \operatorname{Im} T$. We can write $x = Ty$ for some $y \in E$. Since $Tx = 0$ we get $T^2 y = (u-\alpha I)^{2r} y = 0$. Since

$$\operatorname{Ker} T = \operatorname{Ker} T^2$$

by the lemma, we conclude that $y \in \operatorname{Ker} T$, and therefore $x = 0$. Finally, since the index of $T$ is equal to 0, it follows that codim Im $T$ = dim Ker $T$. Since we have already a direct sum decomposition $\operatorname{Ker} T \oplus \operatorname{Im} T$ for some subspace of $E$, it follows that this subspace must be all of $E$. This proves our first assertion. Let now $\beta \neq \alpha$ be another eigenvalue for $u$, and let $S = (u - \beta I)^s$. Then $ST = TS$, so Ker $T$ and Im $T$ are $S$-invariant subspaces. Let $x \in \operatorname{Ker} S$. We can write

$$x = y + z$$

uniquely with $y \in \operatorname{Ker} T$ and $z \in \operatorname{Im} T$. Then

$$0 = Sx = Sy + Sz,$$

and since $Sy \in \operatorname{Ker} T$, $Sz \in \operatorname{Im} T$, it follows from the uniqueness of the decomposition that $Sy = 0$. But $S$, $T$ are obtained as relatively prime polynomials in $u$, and hence there exist polynomials in $u$, namely $P$ and $Q$, such that

$$PS + QT = I.$$

(We recall the proof below.) Applying this to $y$ shows that $Iy = 0$ so that $y = 0$ and hence $x = z \in \operatorname{Im} T$, thus proving our theorem.

Now to recall the proof of the existence of $P$, $Q$, let $A = u - \alpha I$ and $B = u - \beta I$. There exist constants $a$, $b$ such that

$$aA + bB = I.$$

We take $n$ sufficiently large, and raise both sides to the $n$-th power. We obtain

$$\sum_{j=0}^{n} c_j (aA)^j (bB)^{n-j} = I.$$

If we take $n$ sufficiently large, then $j \geqq r$ or $n - j \geqq s$, and thus the existence of $P$, $Q$ follows as desired.

**Theorem 3.4.** *Assume that there are infinitely many eigenvalues. Then the eigenvalues $\neq 0$ of $u$ form a denumerable set, and if we order them as $\alpha_1, \alpha_2, \ldots$ such that*

$$|\alpha_i| \geqq |\alpha_{i+1}|,$$

*then*

$$\lim_{i \to \infty} \alpha_i = 0.$$

*Proof.* Given $c > 0$ we first show that there is only a finite number of eigenvalues $\alpha$ such that $|\alpha| \geqq c$. If this is not true, then we can find a sequence of eigenvectors $\{w_n\}$ belonging to *distinct* eigenvalues $\{\alpha_n\}$ such that $|w_n| = 1$ and $|\alpha_n| \geqq c > 0$ for all $n$. The vectors $w_1, \ldots, w_n$ are linearly independent, for otherwise, if $n \geqq 2$ and

$$c_1 w_1 + \cdots + c_n w_n = 0,$$

then we apply $u$ to this relation, and get

$$c_1 \alpha_1 w_1 + \cdots + c_n \alpha_n w_n = 0.$$

We divide by $\alpha_1$ and subtract, obtaining

$$c_2(1 - \alpha_2/\alpha_1)w_2 + \cdots + c_n(1 - \alpha_n/\alpha_1)w_n = 0.$$

By induction, we could assume $w_2, \ldots, w_n$ linearly independent, and hence $c_2 = \cdots = c_n = 0$, whence $c_1 = 0$, as was to be shown. We let $H_n$ be the space generated by $w_1, \ldots, w_n$. By Lemma 2.2 we can find $x_n \in H_n$ such that $|x_n| = 1$ and

$$|x_n - y| \geqq 1 - \varepsilon$$

for all $y \in H_{n-1}$. Then for $k < n$ we get for some $y \in H_{n-1}$:

$$|ux_n - ux_k| = |\alpha_n x_n - y| \geqq c(1 - \varepsilon).$$

This contradicts the compactness of $u$, and proves that the number of eigenvalues $\alpha$ such that $|\alpha| \geqq c$ is finite.

Thus we can order the eigenvalues in a sequence $\{\alpha_1, \alpha_2, \ldots\}$ such that

$$|\alpha_i| \geqq |\alpha_{i+1}|$$

and we get $\lim_{i \to \infty} \alpha_i = 0$. This proves our theorem.

Let $\{\alpha_i\}$ be the sequence of eigenvalues of $u$, ordered such that $|\alpha_i| \geqq |\alpha_{i+1}|$. Let $r_i$ be the exponent of $\alpha_i$. We can form the subspaces

$$\boxed{F_n = \sum_{i=1}^{n} \operatorname{Ker}(u - \alpha_i I)^{r_i},}$$

and the sum is direct since each $\operatorname{Ker}(u - \alpha_i)^{r_i}$ is finite dimensional. Then we get an ascending sequence of subspaces

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots.$$

Similarly, each one of these subspaces has a complementary closed subspace $H_n$ which can be described in various ways. For instance, we can proceed by induction, assuming that we have already found such a closed $u$-invariant subspace $H_n$. Then $u|H_n$ is a compact operator, whose eigenvalues are precisely $\alpha_j$ for $j > n$, and we can decompose $H_n$ as in Theorem 3.3. Assuming inductively that

$$\boxed{H_n = \operatorname{Im} \prod_{i=1}^{n} (u - \alpha_i I)^{r_i},}$$

we conclude that the same relation holds when $n$ is replaced by $n + 1$. We get a decreasing sequence

$$H_1 \supset H_2 \supset \cdots \supset H_n \supset \cdots$$

and a direct sum decomposition

$$\boxed{E = F_n \oplus H_n.}$$

This decomposition of $E$ as a direct sum is what we call the **spectral theorem** for compact operators. Our method of proof is due to Riesz. Cf. [Di] and [R–N].

We conclude this section with some remarks in case we are given a compact operator on a normed vector space which is not complete. This is often the case, when we are given an operator say on $C^\infty$ functions, and we extend it to the completion of the space of $C^\infty$ functions with respect to some norm.

**Theorem 3.5.** *Let $E$ be a normed vector space and $u: E \to E$ a compact operator. Let $\bar{E}$ be the completion of $E$, and let*

$$\bar{u}: \bar{E} \to \bar{E}$$

*be the continuous linear extension of $u$. Then $\bar{u}$ maps $\bar{E}$ into $E$ itself. If $\alpha \neq 0$ is an eigenvalue of $\bar{u}$ of exponent $r$, and $N_\alpha = \operatorname{Ker}(\bar{u} - \alpha I)^r$, then $N_\alpha$ is contained in $E$.*

*Proof.* Let $x \in \bar{E}$ and let $\{x_n\}$ be a sequence in $E$ approaching $x$. Then $\{ux_n\}$ has a subsequence which converges in $E$, and hence $ux$ lies in $E$, thus proving our first assertion. As to the second, let $x$ be an eigenvector of $\bar{u}$ in $\bar{E}$ belonging to the eigenvalue $\alpha \neq 0$. Then

$$x = \alpha^{-1}\bar{u}(x),$$

whence $x$ lies in $E$. Inductively, suppose that the kernel of $(\bar{u} - \alpha I)^k$ is contained in $E$, and suppose that $(\bar{u} - \alpha I)^{k+1}x = 0$ for some $x \in \bar{E}$. Then $(\bar{u} - \alpha I)x = y$ lies in the kernel of $(\bar{u} - \alpha I)^k$ and hence lies in $E$. Therefore

$$x = \alpha^{-1}(\bar{u}x - y)$$

lies in $E$, thus proving our theorem.

## XVII, §4. APPLICATION TO INTEGRAL EQUATIONS

We consider a continuous function $K(x, y)$ of two variables ranging over a square $[a, b] \times [a, b]$. Then we obtain an operator $S_K$ such that

$$S_K f(x) = \int_a^b K(x, y) f(y)\, dy. \tag{1}$$

We shall consider this operator with respect to two norms on the space $E$ of continuous functions on $[a, b]$.

**Case 1.** *We take $E$ with the sup norm, so that $E$ is a Banach space. Then $S_K$ is compact.*

Indeed, this follows trivially from Ascoli's theorem, because $K$ is uniformly continuous, and if $\Phi$ is a subset of $E$, bounded by $C > 0$, then estimating the integral in the usual way shows that $S_K(\Phi)$ is bounded, and for $f \in \Phi$ we have

$$\begin{aligned} |S_K f(x) - S_K f(x_0)| &\leqq \int_a^b |K(x, y) - K(x_0, y)|\,|f(y)|\, dy \\ &< C(b - a)\varepsilon \end{aligned} \tag{2}$$

whenever $|x - x_0| < \delta$. Hence $S_K(\Phi)$ is equicontinuous, and our assertion is proved.

Thus we can apply the spectral theorem for compact operators.

**Case 2.** *We take $E$ with the $L^2$-norm, arising from the hermitian product*

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\, dt.$$

*Then again $S_K$ is compact, even as a linear map of $E$ into itself (even though $E$ is not yet complete with respect to the $L^2$-norm!).*

*Proof.* Similar to the proof in the preceding case, except that we estimate the integral by the Schwarz inequality, namely

$$|\langle f, g\rangle| \leqq \|f\|_2 \|g\|_2.$$

If $M$ is a bound for $K$ on the square, and $K_x$ is the function such that $K_x(y) = K(x, y)$, then $\|K_x\|_2 \leqq M(b-a)^{1/2}$, and hence by Schwarz, from (1) we even get a bound for the **sup norm**:

$$\|S_K f\| \leqq M(b-a)^{1/2}\|f\|_2,$$

so that if $f$ lies in an $L^2$-bounded set, then $S_K f$ lies in a $C^0$-bounded set. Similarly, estimating (2) with the Schwarz inequality shows that if $\Phi$ is an $L^2$-bounded set, then $S_K(\Phi)$ is equicontinuous, so that we can apply Ascoli's theorem to $S_K(\Phi)$. Note that $S_K(\Phi)$ in this case is relatively compact with respect to the sup norm, let alone the $L^2$-norm, even though we started with a set $\Phi$ which was only $L^2$-bounded.

The spectral theorem applies therefore in the present case, and so does Theorem 3.5, which showed that the finite dimensional spaces corresponding to the eigenvalues $\neq 0$ actually had bases with elements in $E$ rather than in the $L^2$-completion of $E$.

If we take $K$ to be hermitian, for instance real valued and such that $K(x, y) = K(y, x)$, then we have a Fourier expansion of any function in $E$ as an $L^2$-convergent series. One can then start playing the same game as in the ordinary theory of Fourier series, and ask for uniform or pointwise convergence. We leave this to look up for readers who have a more direct interest in integral equations.

Several more examples of compact integral operators will be given in the exercises.

## XVII, §5. EXERCISES

1. Let $E$ be the space of $C^\infty$ functions of one variable, periodic of period $2\pi$. Let $D$ be the derivative. Denote by $E_0$ the space $E$ together with the norm arising from the hermitian product

$$\langle f, g\rangle_0 = \int_0^{2\pi} f\bar{g},$$

and for each positive integer $p$ denote by $E_p$ the same space but with the product

$$\langle f, g\rangle_p = \langle f, g\rangle_0 + \langle Df, Dg\rangle_0 + \cdots + \langle D^p f, D^p g\rangle_0.$$

(a) If $f \in E$ and $c_k$ is the Fourier coefficient of $f$ with respect to the function $e^{ikx}$ ($k$ integer), show that $c_k$ goes to 0 like $1/k^2$, for $k \to \infty$. Use integration by parts.

(b) Show that $T = I - D^2$ is a toplinear isomorphism

$$T\colon E_2 \to E_0$$

by constructing an inverse $S$, using term by term integration of the Fourier series.

(c) Show that this inverse is compact.

2. (a) Let $E$ be the normed vector space of continuous functions on $[0, 1]$ with the sup norm. Let $S\colon E \to E$ be the linear map such that

$$Sf(x) = \int_0^x f(t)\,dt.$$

Show that $S$ is continuous, and that $|S^n|^{1/n} \to 0$ as $n \to \infty$. [*Hint:* Show that $|(S^n f)(x)| \leqq \|f\| x^n/n!$ by induction. You will need some inequality like $n! \geqq n^n e^{-n}$.]

(b) Show that 0 is the only element in the spectrum of $S$, and that $S$ is compact.

(c) For each $\alpha \neq 0$, given a continuous function $g \in E$, show that there exists a continuous function $f \in E$ such that

$$Sf - \alpha f = g.$$

Express $f$ explicitly as an integral involving $g$ and the exponential function.

3. Let $J = [0, 1]$ and let $E$ be the vector space of all $C^1$ paths $\alpha\colon J \to \mathbf{R}^n$. Let $\|\ \|$ be the sup norm and $|\ |$ the euclidean norm on $\mathbf{R}^n$. Given two paths $\alpha, \beta$ define

$$\langle \alpha, \beta \rangle_0 = \int_0^1 \langle \alpha(t), \beta(t) \rangle\,dt,$$

the product $\langle \alpha(t), \beta(t) \rangle$ being the dot product. Its associated norm is called the $H^0$-norm on $E$, and will be denoted by $\|\ \|_0$. Define

$$\langle \alpha, \beta \rangle_1 = \langle \alpha(0), \beta(0) \rangle + \langle D\alpha, D\beta \rangle_0.$$

(a) Show that this is a positive definite scalar product, and that its norm, which we call the $H^1$-norm and denote by $\|\ \|_1$, is equivalent to the norm arising from the scalar product

$$(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle_0 + \langle D\alpha, D\beta \rangle_0.$$

(b) Show that for $\alpha \in E$ and $s, t \in J$, we have

$$|\alpha(s) - \alpha(t)| \leqq |t - s|^{1/2} \|D\alpha\|_0,$$

and

$$\|\alpha\| \leqq 2\|\alpha\|_1.$$

(c) Let $H^0(J, \mathbf{R}^n)$ be the completion of $E$ with respect to the $H^0$-norm, and let $H^1(J, \mathbf{R}^n)$ be its completion with respect to the $H^1$-norm. Show that the identity mapping on $E$ induces *injective* continuous linear maps

$$H^1(J, \mathbf{R}^n) \to C^0(J, \mathbf{R}^n) \quad \text{and} \quad H^1(J, \mathbf{R}^n) \to H^0(J, \mathbf{R}^n).$$

Show that both these maps are compact.

4. If you have not already done them, do Exercise 18 of Chapter IV and Exercise 7 of Chapter V.

5. Let $U$ be a bounded open set in a Banach space $E$ and let $F$ be a finite dimensional space. Let $p \geqq 1$. Let $BC^p(U, F)$ be the space of $C^p$ maps $f: U \to F$ such that $D^k f$ is bounded for $k = 1, \ldots, p$. Show that the identity map of $BC^{p+1}(U, F)$ into $BC^p(U, F)$ is a compact operator. [*Hint:* Use Ascoli's theorem and the mean value theorem.]

6. Let $H$, $E$ be Hilbert spaces, and let $A: H \to E$ be an operator. Show that $A$ is compact if and only if $A$ maps weakly convergent sequences into strongly convergent sequences. (A sequence is said to **converge weakly** if the sequence obtained by applying any functional converges. It **converges strongly** if it converges in the usual norm of the Hilbert space.) Show also that if $A$ is compact and $v_n \to 0$ weakly in $H$, then $Av_n \to 0$ strongly in $E$. [*Hint:* You may use the principle of uniform boundedness, and also the fact that the unit ball is closed in the weak topology, see Exercise 10 of Chapter IV. Cf. $SL_2(\mathbf{R})$, Appendix 4, for details of proof.]

7. Let $H$, $E$ be Hilbert spaces and let $A: H \to E$ be a compact linear map. Let $\{e_i\}$ $(i = 1, 2, \ldots)$ be an orthonormal basis in $H$. Let $H(N)$ be the closed subspace generated by the $e_i$ with $i \geqq N$. Show that given $\varepsilon$, there exists $N$ such that for all $v \in H(N)$, we have

$$|Av|_E \leqq \varepsilon |v|_H.$$

[*Hint:* Use the preceding exercise. If the conclusion is false, pick a sequence $v_n \in H(n)$ of unit vectors such that $|Av_n| > \varepsilon$.]

8. Let $H_r$ be the Hilbert space defined in Exercise 6 of Chapter VII. Following Exercise 8 of that chapter, if $r < s$, prove that the inclusion $H_s \hookrightarrow H_r$ is a compact linear map.

The next three exercises give examples of compact operators which are called Hilbert–Schmidt. For a systematic treatment of such operators in general Hilbert spaces, see Exercises 17 and 18 of Chapter XVIII.

9. **Hilbert–Schmidt operators in $L^2$.** Let $(X, \mathcal{M}, dx)$ and $(Y, \mathcal{N}, dy)$ be measured spaces. Assume that $L^2(X)$ and $L^2(Y)$ have countable Hilbert bases.
   (a) Show that if $\{\varphi_i\}$ and $\{\psi_j\}$ are Hilbert bases for $L^2(X)$ and $L^2(Y)$, respectively, then $\{\varphi_i \otimes \psi_j\}$ is a Hilbert basis for $L^2(dx \otimes dy)$.

(b) Let $K \in L^2(dx \otimes dy)$. Prove that the operator

$$f \mapsto S_K f \qquad \text{from} \qquad L^2(Y) \to L^2(X)$$

given by

$$S_K f(x) = \int_Y K(x, y) f(y)\, dy$$

is compact. [*Hint*: Prove first that it is bounded, with bound $\|K\|_2$. Using partial sums for the Fourier expansion of $K$, show that $S_K$ can be approximated by operators with finite dimensional images. Cf. Theorem 2 of $SL_2(\mathbf{R})$, Chapter I, §3.]

Let $H$ be a Hilbert space with countable Hilbert basis. An operator $A$ on $H$ is said to be **Hilbert–Schmidt** if there exists a Hilbert basis $\{\varphi_i\}$ such that

$$\sum_i |A\varphi_i|^2 < \infty.$$

(c) If $Y = X$, show that $S_K$ is a Hilbert–Schmidt operator.

(d) Conversely, let $T\colon L^2(X) \to L^2(X)$ be a Hilbert–Schmidt operator. Show that there exists $K \in L^2(X \times X)$ such that $T = S_K$.
[*Hint*: Let $T\varphi_i = \sum_j t_{ij}\varphi_j$. Show that $\sum |t_{ij}|^2 < \infty$, and let

$$K = \sum_{i,j} t_{ij}(\varphi_i \otimes \overline{\varphi}_j).$$

For a more subtle result along these lines, cf. $SL_2(\mathbf{R})$, Theorem 6 of Chapter XII, §3.]

10. Assume that $(X, \mathcal{M}, dx) = (Y, \mathcal{N}, dy)$ in the preceding exercise, and that $X$ has finite measure. Let

$$K = \sum_{m,n} c_{m,n}\varphi_m \otimes \overline{\varphi}_n$$

be the Fourier series for $K$. Let $P_{m,n}$ be the integral operator defined by the function $\varphi_m \otimes \overline{\varphi}_n$.

(a) Show that $P_{m,n}\varphi_n = \varphi_m$ and $P_{m,n}\varphi_j = 0$ if $h \neq n$.

(b) Assume that the coefficients $c_{m,n}$ tend to 0 sufficiently rapidly. Show that $K$ is in $L^1$ and that

$$\int_X K(x, x)\, dx = \sum_n c_{n,n}.$$

(c) Again assume that the coefficients $c_{m,n}$ tend to 0 sufficiently rapidly. Show that

$$\sum_n \langle S_K\varphi_n, \varphi_n\rangle = \sum_n c_{n,n},$$

Under suitable convergence conditions, this gives an integral expression for the "**trace**" of $S_K$.

(d) If the Fourier coefficients of $K$ tend to 0 sufficiently rapidly, show that $S_K$ is the product of two Hilbert–Schmidt operators. By definition, this means $S_K$ is of "trace class". See Exercise 23 of Chapter XVIII. [*Hint*: Look at the technique of the next exercise.]

11. Let $\mathbf{T}$ again be the circle, viewing functions on $\mathbf{T}$ as functions on the reals which are periodic of period 1. Let $K$ be a $C^\infty$ function on $\mathbf{T} \times \mathbf{T}$.
(a) Show that the integral operator $S_K$ given by

$$S_K f(x) = \int_0^1 f(y)K(x, y)\,dy$$

is the product of two Hilbert–Schmidt operators.
(b) Show that with the notation of Exercise 10, we have

$$\int_0^1 K(x, x)\,dx = \sum_n c_{n,n}.$$

[*Hint*: Let $\{\varphi_n\}$ be the Hilbert basis given by $\varphi_n(x) = e^{2\pi i n x}$. Let

$$B = \sum_{m,n} c_{m,n}(1 + n^2)P_{m,n} \qquad \text{and} \qquad C = \sum_j \frac{1}{1 + j^2} P_j.$$

Show that $BC = S_K$.]

12. Let $\mathbf{T}$ be the circle as before, and for $f \in L^2(\mathbf{T})$ define

$$Cf(x) = \int_0^1 e^{2\pi i(x-t)} f(t)\,dt.$$

(a) Prove that $C$ is a compact operator on $L^2(\mathbf{T})$.
(b) Prove that $C^* = C$ (so $C$ is self-adjoint).
(c) Describe the spectrum of $C$.

13. Prove the following theorem (worked out in $SL_2(\mathbf{R})$, Chapter XII, §3).

**Theorem.** *Let $X$ be a locally compact space with a finite positive measure $\mu$. Let $H$ be a closed subspace of $L^2(X, \mu) = L^2(X)$, and let $T$ be a linear map of $H$ into the vector space $BC(X)$ of bounded continuous functions on $X$. Assume that there exists $C > 0$ such that*

$$\|Tf\| \leqq C\|f\|_2 \qquad \text{for all} \quad f \in H,$$

*where $\|\ \ \|$ is the sup norm. Then*

$$T\colon H \to L^2(X)$$

*is a compact operator, which can be represented by a kernel in $L^2(X \times X)$, as in Exercise 9.*

CHAPTER XVIII

# Spectral Theorem for Bounded Hermitian Operators

This chapter may be viewed as a direct continuation of the linear algebra in the context of Hilbert space first discussed in Chapter V.

## XVIII, §1. HERMITIAN AND UNITARY OPERATORS

Let $E$ be a Hilbert space. We recall that an operator $A: E \to E$ is a continuous (or bounded) linear map. We defined the **adjoint** $A^*$ in Chapter V, §2. An operator $A$ such that $A = A^*$ is called **hermitian**, or **self-adjoint**. If $E$ is a real Hilbert space, then instead of hermitian we also say that $A$ is **symmetric**. If $A$ is invertible, then one sees at once that

$$(A^{-1})^* = (A^*)^{-1}.$$

The case when $A = A^*$ is the main one studied in this chapter. *For a complex Hilbert space*, the following properties are equivalent, concerning an operator $A$:

*We have $A = A^*$.*

*The form $\varphi_A: (x, y) \mapsto \langle Ax, y\rangle$ is hermitian.*

*The numbers $\langle Ax, x\rangle$ are real for all $x \in E$.*

The equivalence between the first two is left to the reader. As to the third, suppose that $A = A^*$. Then

$$\langle Ax, x\rangle = \langle x, A^*x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$$

so $\langle Ax, x\rangle$ is real. Conversely, assume that this is the case. Then

$$\langle Ax, x\rangle = \langle x, Ax\rangle = \langle A^*x, x\rangle$$

for all $x$, whence $\langle (A - A^*)x, x\rangle = 0$ for all $x$, and $A = A^*$ by polarization (Theorem 2.4 of Chapter V).

Let $E$ be a Hilbert space, and $A$ an invertible operator. Then the following conditions are equivalent:

**UN 1.** $A^* = A^{-1}$.

**UN 2.** $|Ax| = |x|$ for all $x \in E$.

**UN 3.** $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in E$.

**UN 4.** $|Ax| = 1$ for every unit vector $x \in E$.

An invertible operator satisfying these conditions is said to be **hilbertian**, or **unitary**. The equivalence between the four conditions is very simple to establish and will be left to the reader (Exercise 3). The set of unitary (or hilbertian) operators is a group, denoted by Hilb($E$).

In §7 you will see how to decompose an arbitrary operator into a product of hermitian and unitary operators. Thus hermitian and unitary operators are the basic ones. We shall now study especially the hermitian ones.

## XVIII, §2. POSITIVE HERMITIAN OPERATORS

We wish to see how much information on the norm of $A$ can be derived from knowing the values of the quadratic form $\langle Ax, x\rangle$.

**Lemma 2.1.** *Let $A$ be an operator, and $c$ a number such that*

$$|\langle Ax, x\rangle| \leqq c|x|^2$$

*for all $x \in E$. Then for all $x, y$ we have*

$$|\langle Ax, y\rangle| + |\langle x, Ay\rangle| \leqq 2c|x||y|.$$

*Proof.* By the polarization identity,

$$2|\langle Ax, y\rangle + \langle Ay, x\rangle| \leqq c|x + y|^2 + c|x - y|^2 = 2c(|x|^2 + |y|^2).$$

Hence

$$|\langle Ax, y\rangle + \langle Ay, x\rangle| \leqq c(|x|^2 + |y|^2).$$

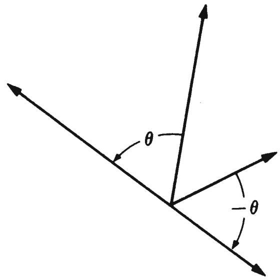



Figure 18.1

We multiply $y$ by $e^{i\theta}$ and thus get on the left-hand side

$$|e^{-i\theta}\langle Ax, y\rangle + e^{i\theta}\langle Ay, x\rangle|.$$

The right-hand side remains unchanged, and for suitable $\theta$, the left-hand side becomes

$$|\langle Ax, y\rangle| + |\langle Ay, x\rangle|.$$

(In other words, we are lining up two complex numbers by rotating one by $\theta$ and the other by $-\theta$.) Next we replace $x$ by $tx$ and $y$ by $y/t$ for $t$ real and $t > 0$. Then the left-hand side remains unchanged, while the right-hand side becomes

$$g(t) = t^2|x|^2 + \frac{1}{t^2}|y|^2.$$

The point at which $g'(t) = 0$ is the unique minimum, and at this point $t_0$ we find that

$$g(t_0) = |x||y|.$$

This proves our lemma.

**Theorem 2.2.** *Let $A$ be a hermitian operator. Then $|A|$ is the greatest lower bound of all values $c$ such that*

$$|\langle Ax, x\rangle| \leqq c|x|^2$$

*for all $x$, or equivalently, the sup of all values $|\langle Ax, x\rangle|$ taken for $x$ on the unit sphere in $E$.*

*Proof.* When $A$ is hermitian we obtain

$$|\langle Ax, y\rangle| \leqq c|x||y|$$

for all $x, y \in E$, so that we get $|A| \leqq c$ in the lemma. On the other hand, $c = |A|$ is certainly a possible value for $c$ by the Schwarz inequality. This proves our theorem.

Theorem 2.2 allows us to define an ordering in the space of hermitian operators. If $A$ is hermitian, we define $A \geqq O$ and say that $A$ is **positive** if $\langle Ax, x \rangle \geqq 0$ for all $x \in E$. If $A$, $B$ are hermitian we define $A \geqq B$ if $A - B \geqq O$. This is indeed an ordering, and the usual rules hold: if $A_1 \geqq B_1$ and $A_2 \geqq B_2$, then

$$A_1 + A_2 \geqq B_1 + B_2.$$

If $c$ is a real number $\geqq 0$ and $A \geqq O$, then $cA \geqq O$. So far, however, we have said nothing about a product of positive hermitian operators $AB$, even if $AB = BA$. We shall deal with this question later.

Let $c$ be a bound for $A$. Then $|\langle Ax, x \rangle| \leqq c|x|^2$ and consequently

$$-cI \leqq A \leqq cI.$$

For simplicity, if $\alpha$ is real, we sometimes write $\alpha \leqq A$ instead of $\alpha I \leqq A$, and similarly we write $A \leqq \beta$ instead of $A \leqq \beta I$. If we let

$$\alpha = \inf_{|x|=1} \langle Ax, x \rangle \qquad \text{and} \qquad \beta = \sup_{|x|=1} \langle Ax, x \rangle,$$

then we have

$$\alpha \leqq A \leqq \beta,$$

and from Theorem 2.2,

$$|A| = \max(|\alpha|, |\beta|).$$

The next two sections are devoted to generalizing to Hilbert space the spectral theorem in the finite dimensional case. These two sections are logically independent of each other. In the finite dimensional case, the spectral theorem for hermitian operators asserts that there exists a basis consisting of eigenvectors. We recall that an **eigenvector** for an operator $A$ is a vector $w \neq 0$ such that there exists a number $c$ for which $Aw = cw$. We then call $c$ an **eigenvalue**, and say that $w$, $c$ belong to each other. In the next section, we describe a special type of hermitian operator for which the generalization to Hilbert space has the same statement as in the finite dimensional case. Afterwards, we give a theorem which holds in the general case, and can be used as a substitute for the "basis" statement in many applications. Some of these applications are described in subsequent sections.

## XVIII, §3. THE SPECTRAL THEOREM FOR COMPACT HERMITIAN OPERATORS

An operator $A: E \to E$ is said to be **compact** if given a bounded sequence $\{x_n\}$ in $E$, the sequence $\{Ax_n\}$ has a convergent subsequence. It is precisely this condition which will allow us to get an orthogonal basis for a hermitian operator. It is clear that if $E$ is finite dimensional, every operator is compact.

*Throughout this section, we let $E$ be a complex Hilbert space, and $A: E \to E$ a compact hermitian operator.*

A subspace $V$ of $E$ is called $A$-**invariant** if $AV \subset V$, i.e. if $x \in V$, then $Ax \in V$. If $V$ is $A$-invariant, then its closure is also $A$-invariant. Furthermore $V^\perp$ is $A$-invariant because if $x \in V^\perp$, then for all $y \in V$ we get

$$\langle y, Ax \rangle = \langle Ay, x \rangle = 0.$$

We recall that the **spectrum** of $A$ is the set of numbers $c$ such that $A - cI$ is not invertible. We note that an eigenvalue $c$ of a hermitian operator is real, because if $w$ is an eigenvector belonging to $c$, then

$$c\langle w, w \rangle = \langle Aw, w \rangle = \langle w, Aw \rangle = \bar{c}\langle w, w \rangle,$$

so $c = \bar{c}$. Since the 1-dimensional space generated by an eigenvector is $A$-invariant, it follows that the orthogonal complement of this space is $A$-invariant.

For each eigenvalue $c$ let $E_c$ be the space generated by all eigenvectors having this eigenvalue, and call it the $c$-**eigenspace**. Then for every $x \in E_c$ we have $Ax = cx$. We note that $E_c$ is a closed subspace, and $E_0$ is the kernel of $A$.

Let $w_1$ and $w_2$ be eigenvectors belonging to eigenvalues $c_1$, $c_2$ respectively, such that $c_1 \neq c_2$. Then $w_1 \perp w_2$, because

$$c_1\langle w_1, w_2 \rangle = \langle Aw_1, w_2 \rangle = \langle w_1, Aw_2 \rangle = c_2\langle w_1, w_2 \rangle.$$

Consequently $E_{c_1}$ is orthogonal to $E_{c_2}$.

*Suppose that $V$ is a non-zero finite dimensional $A$-invariant subspace of $E$. Then $A$ restricted to $V$ induces a self-adjoint operator on $V$, and there exists an orthogonal basis of $V$ consisting of eigenvectors for $A$.*

This is a trivial fact of linear algebra, which we reprove here. Let $w$ be a non-zero eigenvector for $A$ in $V$, and let $W$ be the orthogonal

complement of $w$ in $V$. Then $W$ is $A$-invariant, and we can complete the proof by induction.

**Theorem 3.1 (Spectral Theorem).** *Let $A$ be a compact hermitian operator on the Hilbert space $E$. Then the family of eigenspaces $\{E_c\}$, where $c$ ranges over all eigenvalues (including 0), is an orthogonal decomposition of $E$.*

*Proof.* Let $F$ be the closure of the subspace generated by all $E_c$ (as in Corollary 1.9 of Chapter V), and let $H$ be the orthogonal complement of $F$. Then $H$ is $A$-invariant, and $A$ induces a compact hermitian operator on $H$, which has no eigenvalue. We must show that $H = \{0\}$. This will follow from the next lemma.

**Lemma 3.2.** *Let $A$ be a compact hermitian operator on the Hilbert space $H \neq \{0\}$. Let $c = |A|$. Then $c$ or $-c$ is an eigenvalue for $A$.*

*Proof.* There exists a sequence $\{x_n\}$ in $H$ such that $|x_n| = 1$ and

$$|\langle Ax_n, x_n\rangle| \to |A|.$$

Selecting a subsequence if necessary, we may assume that

$$\langle Ax_n, x_n\rangle \to \alpha$$

for some number $\alpha$, and $\alpha = \pm|A|$. Then

$$\begin{aligned} 0 \leqq |Ax_n - \alpha x_n|^2 &= \langle Ax_n - \alpha x_n, Ax_n - \alpha x_n\rangle \\ &= |Ax_n|^2 - 2\alpha\langle Ax_n, x_n\rangle + \alpha^2|x_n|^2 \\ &\leqq \alpha^2 - 2\alpha\langle Ax_n, x_n\rangle + \alpha^2. \end{aligned}$$

The right-hand side approaches 0 as $n$ tends to infinity. Since $A$ is compact, after selecting a subsequence, we may assume that $\{Ax_n\}$ converges to some vector $y$, and then $\{\alpha x_n\}$ must converge to $y$ also. If $\alpha = 0$, then $|A| = 0$ and $A = O$, so we are done. If $\alpha \neq 0$, then $\{x_n\}$ itself must converge to some vector $x$, and then $Ax = \alpha x$ so that $\alpha$ is the desired eigenvalue for $A$, thus proving our lemma, and the theorem.

We observe that each $E_c$ has a Hilbert basis consisting of eigenvectors, namely any Hilbert basis of $E_c$, because all non-zero elements of $E_c$ are eigenvectors. Hence $E$ itself has a Hilbert basis consisting of eigenvectors. Thus we recover precisely the analog of the theorem in the finite dimensional case. Furthermore, we have some additional information, which follows trivially:

*If $c \neq 0$, each $E_c$ is finite dimensional*; otherwise a denumerable subset from a Hilbert basis would provide a sequence contradicting the compactness of $A$. For a similar reason, *given $r > 0$, there is only a finite number of eigenvalues $c$ such that $|c| \geqq r$.* Thus 0 is a limit of the sequence of eigenvalues if $E$ is infinite dimensional.

## XVIII, §4. THE SPECTRAL THEOREM FOR HERMITIAN OPERATORS

Let $p$ be a polynomial with real coefficients, and let $A$ be a hermitian operator. Write

$$p(t) = a_n t^n + \cdots + a_0.$$

As in Chapter IV, §5, we define

$$p(A) = a_n A^n + \cdots + a_0 I.$$

We let $\mathbf{R}[A]$ be the algebra generated over $\mathbf{R}$ by $A$, that is the algebra of all operators $p(A)$, where $p(t) \in \mathbf{R}[t]$. We wish to investigate the closure of $\mathbf{R}[A]$ in the (real) Banach space of all operators. We shall show how to represent this closure as a ring of continuous functions on some compact subset of the reals. First, we observe that the hermitian operators form a closed subspace of $\operatorname{End}(E)$, and that $\overline{\mathbf{R}[A]}$ is a closed subspace of the space of hermitian operators.

As observed at the end of §2, we can find real numbers $\alpha$, $\beta$ such that

$$\alpha I \leqq A \leqq \beta I.$$

We shall prove that if $p$ is a real polynomial which takes on positive values on the interval $[\alpha, \beta]$, then $p(A)$ is a positive operator. For this we need a purely algebraic lemma.

**Lemma 4.1.** *Let $p$ be a real polynomial such that $p(t) \geqq 0$ for all $t \in [\alpha, \beta]$. Then we can express $p$ in the form*

$$p(t) = c[\sum Q_i(t)^2 + \sum (t - \alpha)Q_j(t)^2 + \sum (\beta - t)Q_k(t)^2]$$

*where $Q_i$, $Q_j$, $Q_k$ are real polynomials, and $c \geqq 0$.*

*Proof.* We first factor $p$ into linear and irreducible quadratic factors over the real numbers. If $p$ has a root $\gamma$ such that $\alpha < \gamma < \beta$, then the multiplicity of $\gamma$ is even (otherwise $p$ changes sign near $\gamma$, which is impossible), and then $(t - \gamma)$ occurs in an even power. If a root $\gamma$ is $\leqq \alpha$ we

have a linear factor $t - \gamma$ which we write

$$t - \gamma = (t - \alpha) + (\alpha - \gamma)$$

and note that $\alpha - \gamma$ is a real square. If $\gamma$ is a root $\geqq \beta$, then we write the linear factor as

$$\gamma - t = (\gamma - \beta) + (\beta - t)$$

and note that $\gamma - \beta$ is a real square. In a factorization of $p$ we can take the factors to be of type $(t - \gamma)^{2m(\gamma)}$ if $\gamma$ is root such that $\alpha < \gamma < \beta$, and otherwise to be of type $t - \gamma$ or $\gamma - t$ according as $\gamma < \alpha$ or $\gamma > \beta$. The quadratic factors are of type $(t - a)^2 + b^2$. The constant $c$ (which can be taken as a constant factor) is then $\geqq 0$ since $p$ is positive on the interval. Multiplying out all these factors, and noting that a sum of squares times a sum of squares is a sum of squares, we conclude that $p$ has an expression as stated in the lemma, except that there still appear terms of type

$$(t - \alpha)(\beta - t)Q(t)^2$$

where $Q$ is a real polynomial. However, such terms can be reduced to terms of the other types by using the identity

$$(t - \alpha)(\beta - t) = \frac{(t - \alpha)^2(\beta - t) + (t - \alpha)(\beta - t)^2}{\beta - \alpha}.$$

This proves our lemma.

Now to study $\overline{\mathbf{R}[A]}$, we observe that the map

$$p \mapsto p(A)$$

is a ring-homomorphism of $\mathbf{R}[t]$ onto the ring $\mathbf{R}[A]$. Furthermore, if $B$, $C$ are hermitian operators such that $BC = CB$ and $B \geqq O$, then trivially, $BC^2$ is positive because

$$\langle BC^2x, x\rangle = \langle CBCx, x\rangle = \langle BCx, Cx\rangle \geqq 0.$$

The sum of two positive hermitian operators is positive. Hence from the expression of $p$ in the lemma, we obtain

**Lemma 4.2.** *If $p$ is positive on $[\alpha, \beta]$, then $p(A)$ is a positive operator. If $p$, $q$ are polynomials such that $p \leqq q$ on $[\alpha, \beta]$, then $p(A) \leqq q(A)$. Finally,*

$$|p(A)| \leqq \|p\|,$$

*the sup norm being taken on $[\alpha, \beta]$.*

*Proof.* The first assertion comes from the remarks preceding our lemma. The second follows at once by considering $q - p$. Finally, if we let

$$q(t) = \|p\| \pm p(t)$$

then $q \geqq 0$ on $[\alpha, \beta]$ and hence $q(A) \geqq O$, whence the last assertion follows from Theorem 2.2.

We conclude that the map

$$p \mapsto p(A)$$

is a continuous linear map from the space of polynomial functions on $[\alpha, \beta]$ into $\mathbf{R}[A]$. By the linear extension theorem, we can extend this map to the Banach space of continuous functions on $[\alpha, \beta]$ by continuity, and thus we can define $f(A)$ for any continuous function $f$ on $[\alpha, \beta]$, by the Stone–Weierstrass theorem. If $\{p_n\}$ is a sequence of polynomials converging uniformly to $f$ on $[\alpha, \beta]$, then by definition,

$$f(A) = \lim p_n(A).$$

Furthermore, again by continuity, we have

$$|f(A)| \leqq \|f\|,$$

the sup norm being taken on $[\alpha, \beta]$. If $p_n \to f$ and $q_n \to g$, then $p_n q_n \to fg$. Hence we obtain $(fg)(A) = f(A)g(A)$ for any continuous functions, $f$, $g$. In other words, our map is also a ring-homomorphism.

**Theorem 4.3.** *If $A \geqq O$, then there exists $B \in \overline{\mathbf{R}[A]}$ such that $B^2 = A$. The product of two commuting positive hermitian operators is again positive.*

*Proof.* The continuous function $t^{1/2}$ maps on a square root of $A$ in $\overline{\mathbf{R}[A]}$, and it is clear that any element of $\overline{\mathbf{R}[A]}$ commutes with $A$. If $A$, $C$ commute and we write $A = B^2$ with $B$ in $\overline{\mathbf{R}[A]}$, then $B$ and $C$ also commute because $C$ commutes with $p(A)$ for all real polynomials $p$, and hence $C$ commutes with all elements of $\overline{\mathbf{R}[A]}$. But as we have seen, if $C \geqq O$, then $B^2 C \geqq O$. This proves our theorem.

The kernel of our map $f \mapsto f(A)$ is a closed ideal in the ring of continuous functions on $[\alpha, \beta]$. We forget for a moment our definition of the spectrum given in Chapter XVI, §1, and here define the **spectrum** $\sigma(A)$ to be the closed set of zeros of this ideal. We use Theorem 2.1 of Chapter III.

If $f$ is any continuous function on $\sigma(A)$, we extend $f$ to a continuous

function on $[\alpha, \beta]$ having the same sup norm, say $f_1$, and define

$$f(A) = f_1(A).$$

If $g$ is another extension of $f$ to $[\alpha, \beta]$, then $g - f_1$ vanishes on $\sigma(A)$, and hence $g(A) = f_1(A)$. Hence $f(A)$ is well defined, independently of the particular extension of $f$ to $[\alpha, \beta]$. We denote by $\| \ \|_A$ the sup norm with respect to $\sigma(A)$; thus

$$\|f\|_A = \sup_{t \in \sigma(A)} |f(t)|.$$

We then obtain a ring-homomorphism from the ring of continuous functions on $\sigma(A)$ into $\overline{\mathbf{R}[A]}$, and we have

$$|f(A)| \leqq \|f\|_A.$$

We now state the **spectral theorem.**

**Theorem 4.4.** *The map $f \mapsto f(A)$ is a Banach-isomorphism from the algebra of continuous functions on $\sigma(A)$ onto the Banach algebra $\overline{\mathbf{R}[A]}$. A continuous function $f$ is $\geqq 0$ on $\sigma(A)$ if and only if $f(A) \geqq O$.*

*Proof.* We had derived the norm inequality previously from the positivity statement. We do this again in the opposite direction. Thus we assume first that $f(A) \geqq O$ and prove that $f$ is $\geqq 0$ on the spectrum of $A$. Assume that this is not the case. Then $f$ is negative at some point $c$ of the spectrum. Let $g$ be a continuous function whose graph is as follows:

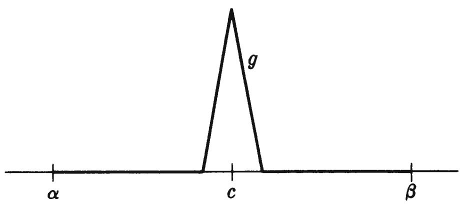



Figure 18.2

Thus $g$ is $\geqq 0$, and has a positive peak at $c$. Then $fg$ is $\leqq 0$ and $fg$ is negative at the point $c$ of the spectrum. Hence $-fg \geqq 0$, and hence $-f(A)g(A) \geqq O$. But $f(A) \geqq O$ and $g(A) \geqq O$, so that by Theorem 4.3 we also have $f(A)g(A) \geqq O$. This implies that $f(A)g(A) = O$, which is impossible since $fg$ does not vanish on the spectrum. We conclude that $f \geqq 0$ on $\sigma(A)$, and in view of our previous result this proves the positivity statement of the theorem.

Now for the norm, let $b = |f(A)|$. Then $bI \pm f(A) \geqq O$, whence $b \pm f(t) \geqq 0$ on the spectrum. This proves that

$$\|f\|_A \leqq |f(A)|,$$

and hence a sequence $\{f_n(A)\}$ converges if and only if the sequence of continuous functions $\{f_n\}$ converges uniformly on the spectrum. This concludes the proof of the spectral theorem.

There remains to identify the spectrum as we have defined it in this section, and the spectrum of Chapter XVI, §1, which we shall call the general spectrum.

**Corollary 4.5.** *If $A$ is hermitian, then the spectrum $\sigma(A)$ is equal to the set of complex numbers $z$ such that $A - zI$ is not invertible.*

*Proof.* Let $z$ be complex and such that $A - zI$ is not invertible. Then $z$ is real, for otherwise let

$$g(t) = (t - z)(t - \bar{z}).$$

Then $g(t) \neq 0$ on $\sigma(A)$, and hence $h(t) = 1/g(t)$ is its inverse. Then $h(A)(A - \bar{z}I)$ would be an inverse for $A - zI$, a contradiction. This proves that $z$ is real.

Let $\xi$ be real and not in the spectrum $\sigma(A)$. Then $t - \xi$ is invertible on $\sigma(A)$, and hence so is $A - \xi I$.

Suppose that $\xi$ is in the spectrum $\sigma(A)$. Let $g$ be the continuous function whose graph is as follows.

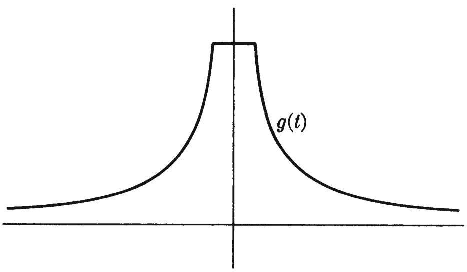



Figure 18.3

That is,

$$g(t) = \begin{cases} 1/|t - \xi| & \text{if } |t - \xi| \geqq 1/N, \\ N & \text{if } |t - \xi| \leqq 1/N. \end{cases}$$

If $A - \xi I$ is invertible, let $B$ be an inverse,

$$B(A - \xi I) = (A - \xi I)B = I.$$

Since $|(t - \xi)g(t)| \leqq 1$ we get $|(A - \xi I)g(A)| \leqq 1$, whence

$$|g(A)| = |B(A - \xi I)g(A)| \leqq |B|.$$

But $g(t)$ has a large sup on the spectrum if we take $N$ large, and hence $|g(A)|$ is equally large, a contradiction. Theorem 4.4 is proved.

The main idea to use the positivity to get the spectral theorem is due to F. Riesz. However, most treatments go from the positivity statement to an integral representation of $A$ which we give in Chapter XX. Von Neumann always emphasized that it is much more efficient to prove at once the statement of Theorem 4.4, which suffices for many applications, and can be obtained quite simply from the positivity statement. In fact, the arguments used to derive Theorem 4.4 from the positivity statement are taken from a seminar of Von Neumann around 1950.

**Example.** Let $A$ be hermitian as above. Given a real number $s$, consider the continuous function

$$f_s(t) = e^{-st}.$$

Then the operator $f_s(A) = e^{-sA}$ is defined. It is an easy matter to show that the association $s \mapsto e^{-sA}$ is a $C^\infty$ map from $\mathbf{R}$ into Laut($E$), that it is also a homomorphism, and satisfies the conditions:

**H 1.** $\dfrac{d}{ds}e^{-sA} = -Ae^{-sA}.$

**H 2.** For every $v \in E$, we have

$$\lim_{s \to 0} e^{-sA}v = v.$$

One calls $e^{-sA}$ the **Heat operator** associated with $A$.

For examples of uses of Theorem 4.3, see §7.

## XVIII, §5. ORTHOGONAL PROJECTIONS

Corollary 1.8 of Chapter V shows that we have orthogonal decompositions in Hilbert space similar to those in euclidean spaces. A standard criterion for such decompositions in algebra generalizes to Hilbert spaces, namely:

**Theorem 5.1.** *Let $P$, $Q$ be hermitian operators such that*

$$P^2 = P, \qquad PQ = QP = O, \qquad P + Q = I.$$

*Then $Q^2 = Q$, and we have*

$$\text{Ker } P = \text{Im } Q = (\text{Ker } Q)^{\perp}.$$

*In particular, we have the orthogonal decomposition*

$$E = \text{Ker } P + \text{Im } P.$$

*Proof.* This proof is independent of the spectral theorem, and uses only basic definitions, together with Corollary 1.8 of Chapter V. Let $F = \text{Ker } P$. If $x \in F$, we have

$$x = Ix = Px + Qx = Qx$$

so that $x$ is in the image of $Q$. Since $PQ = QP = O$, it follows that the image of $Q$ is in the kernel of $P$, whence $\text{Ker } P = \text{Im } Q$. We obviously have

$$Q^2 = (I - P)^2 = I - P = Q$$

so that our relations between $P$ and $Q$ are symmetric. We still must show that $F^{\perp} = \text{Ker } Q$. Suppose that $\langle F, x \rangle = 0$. Then from

$$\langle E, Qx \rangle = \langle QE, x \rangle = \langle F, x \rangle$$

we conclude that $Qx = 0$ so $F^{\perp} \subset \text{Ker } Q$. The converse inclusion follows from these same equalities, and our theorem is proved.

Let $A$ be an operator and let $F$ be a subspace of $E$. We say that $F$ is **invariant** for $A$ if $AF \subset F$ (that is $Ax \in F$ for all $x \in F$). If this is the case, then it is clear that the closure $\bar{F}$ is also an invariant subspace.

*Let $A$, $B$ be operators such that $AB = BA$. Then* Ker $B$ *and* Im $B$ *are invariant subspaces for $A$.* Indeed, if $Bx = 0$, then $BAx = ABx = 0$, so Ker $B$ is invariant. If $y = Bx$, then $Ay = ABx = BAx$, so Im $B$ is invariant.

An operator $A$ which is hermitian is said to be **positive definite** if $A \geqq cI > O$ for some $c > 0$. If $F$ is a closed invariant subspace for $A$, we say that $A$ is positive definite on $F$ if the restriction of $A$ to $F$ is positive definite. (This restriction is clearly hermitian.) We say that $A$ is **negative definite** if $-A$ is positive definite.

**Corollary 5.2.** *Let $A$ be an invertible hermitian operator. Then there exists an orthogonal decomposition $E = F + F^{\perp}$ such that $F$, $F^{\perp}$ are*

*A-invariant closed subspaces, and such that $A$ is positive definite on $F$ and negative definite on $F^\perp$.*

*Proof.* We use the spectral theorem. Let $g$ be the function such that $g(t) = 1$ if $t \geqq 0$ and $g(t) = 0$ if $t < 0$. Since $A$ is invertible, it follows that 0 is not in the spectrum of $A$. Hence $g$ is continuous on the spectrum, and $g^2 = g$ on the spectrum. Hence $g(A) = P$ satisfies $P^2 = P$. Let $F = \operatorname{Im} P$. Then $P$ is an orthogonal projection on $F$ by Theorem 5.1. Since $A$ commutes with $g(A)$, and since $tg(t) \leqq 0$ on the spectrum of $A$, it follows that $AP = PA$ is a positive operator. Furthermore, $A$ maps $F$ into itself, and since $A^{-1}$ exists on $E$, and also maps $F$ into itself, let $A^+$ be the restriction of $A$ to $F$. Then $A^+$ is positive, invertible on $F$, whence positive definite (because the spectrum is closed, and 0 is not in the spectrum). Similarly, let $h(t) = 1 - g(t)$ and $Q = h(A)$. Then $th(t) \leqq 0$ on the spectrum of $A$, and by similar arguments, letting $A^-$ be the restriction of $-A$ to $F^\perp$, we conclude that $A^-$ is positive definite on $F^\perp$. This proves what we wanted.

**Corollary 5.3.** *Let $A$ be an invertible hermitian operator. Then there exist an orthogonal decomposition $E = F + F^\perp$ and positive definite operators $A^+$ on $F$, $A^-$ on $F^\perp$ such that if we write $x = y + z$ with $y \in F$ and $z \in F^\perp$, then*

$$\langle Ax, x\rangle = \langle A^+y, y\rangle - \langle A^-z, z\rangle.$$

*Proof.* This is a rephrasing of the preceding result.

Finally, for a positive operator, we can go one step further in our normalization. Namely, if $A \geqq O$, then we can write $A = B^2$ for some $B$ in $\overline{\mathbf{R}[A]}$, and hence if $A \geqq O$, then the quadratic form $x \mapsto \langle Ax, x\rangle$ can be written

$$\langle Ax, x\rangle = \langle A^{1/2}x, A^{1/2}x\rangle.$$

If $A$ is invertible, so is $A^{1/2}$. This corresponds to the diagonalization of quadratic (or symmetric bilinear) forms in the finite dimensional case. Indeed, in that case, a positive form can be written as

$$a_1y_1^2 + \cdots + a_ny_n^2 \qquad (a_i \geqq 0)$$

and a negative form can be written as

$$-(b_1z_1^2 + \cdots + b_sz_s^2) \qquad (b_j \geqq 0)$$

with respect to a suitable orthonormal basis of the given positive definite hermitian product $\langle\ ,\ \rangle$ on euclidean space.

## XVIII, §6. SCHUR'S LEMMA

Let $S$ be a set of operators on the Hilbert space $E$, and let $F$ be a subspace of $E$. We say that $F$ is **invariant** under $S$ if for every $A \in S$ and $x \in F$ we have $Ax \in F$. In other words, $AF \subset F$ for every $A \in S$. If $F$ is invariant under $S$, we observe that its closure $\bar{F}$ is also invariant under $S$.

**Theorem 6.1.** *Let $S$ be a set of operators on the Hilbert space $E$, leaving no closed subspace invariant except $\{0\}$ and $E$ itself. Let $A$ be a hermitian operator such that $AB = BA$ for all $B \in S$. Then $A = cI$ for some real number $c$.*

*Proof.* It will suffice to prove that there is only one element in the spectrum of $A$. Suppose that there are two, $c_1 \neq c_2$. There exist continuous functions $f$, $g$ on the spectrum such that neither is 0 on the spectrum, but $fg$ is 0 on the spectrum. For instance, we can take for $f$, $g$ the functions whose graphs are indicated on the next figure.

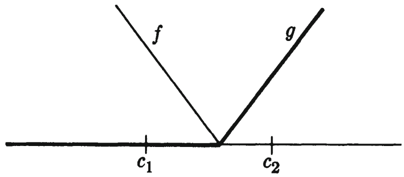



Figure 18.4

We have $f(A)B = Bf(A)$ for all $B \in S$ (because $B$ commutes with real polynomials in $A$, hence with their limits). Hence $f(A)E$ is invariant under $S$ because

$$Bf(A)E = f(A)BE \subset f(A)E.$$

Let $F$ be the closure of $f(A)E$. Then $F \neq \{0\}$ because $f(A) \neq O$. Furthermore, $F \neq E$ because $g(A)f(A)E = \{0\}$ and hence $g(A)F = \{0\}$. Since $F$ is invariant under $S$, we have a contradiction, thus proving our theorem.

**Corollary 6.2.** *Let $S$ be a set of operators of the Hilbert space $E$, leaving no closed subspace invariant except $\{0\}$ and $E$ itself. Let $A$ be an operator such that $AA^* = A^*A$, $AT = TA$, and $A^*T = TA^*$ for all $T \in S$. Then $A = cI$ for some complex number $c$.*

*Proof.* Write $A = B + iC$ where $B$, $C$ are hermitian and commute (e.g. $B = (A + A^*)/2$ and $C = (A - A^*)/2i$). Apply the theorem to each one of $B$ and $C$ to prove the corollary.

**Remark.** Schur's lemma is used, among other places, in the representation theory of groups. Let $G$ be a group, and suppose that we have a homomorphism (called a representation)

$$\rho: G \to \mathrm{Laut}(E)$$

of $G$ into the toplinear automorphisms of a Hilbert space $E$. Assume that $G$ is commutative, and that the image $\rho(G)$ satisfies the hypotheses of the set $S$ in the corollary, and also is such that if $A \in \rho(G)$, then $A^* \in \rho(G)$. Then we conclude that for each $\sigma \in G$, the image $\rho(\sigma)$ is equal to $c_\sigma I$. Thus $\sigma \mapsto c_\sigma$ is a homomorphism of $G$ into the multiplicative group of complex numbers. In the terminology of representations, one says that an irreducible unitary representation of $G$ is one dimensional, because $E$ must then be of dimension 1.

## XVIII, §7. POLAR DECOMPOSITION OF ENDOMORPHISMS

Among other things, this section shows some ways how Theorem 4.3 is used. Namely, for any operator $T$ on a Hilbert space $E$, the operator $T^*T$ is hermitian positive, and so has a square root by Theorem 4.3. We start with a special case of the polar decomposition which is of interest for its own sake.

**Theorem 7.1.** *Let $T: E \to E$ be an operator on the Hilbert space $E$. Assume that* $\mathrm{Ker}\, T = 0$, *and that $TE$ is dense in $E$. Then* $\mathrm{Im}(T^*T)^{1/2}$ *is dense, and there exists a continuous linear map $U$ defined on this image such that*

$$U(T^*T)^{1/2}x = Tx.$$

*This operator $U$ is norm preserving, and its kernel is $\{0\}$.*

*Proof.* To show that $U$ is well defined, it suffices to prove that if

$$(T^*T)^{1/2}x = (T^*T)^{1/2}y,$$

then $Tx = Ty$. But applying $(T^*T)^{1/2}$ yields $T^*Tx = T^*Ty$. On the other hand, the kernel of $T^*$ is $\{0\}$, because if $T^*u = 0$ then for any $v \in E$ we have

$$0 = \langle T^*u, v\rangle = \langle u, Tv\rangle,$$

and since the image of $T$ is dense, this implies that $u$ is orthogonal to all of $E$, whence $u = 0$. Hence we get $Tx = Ty$, thus defining $U$ uniquely by our given formula. Then we find

$$\langle (T^*T)^{1/2}x, (T^*T)^{1/2}y\rangle = \langle x, T^*Ty\rangle = \langle Tx, Ty\rangle.$$

whence it follows that $U$ preserves lengths on the domain of definition. The fact that $\operatorname{Ker} U = \{0\}$ is a consequence of the norm-preserving property.

The image of $(T^*T)^{1/2}$ is dense, for suppose that $y$ is orthogonal to this image. Let $x = (T^*T)^{1/2}y$. Then $(T^*T)^{1/2}x = T^*Ty$, and we get

$$0 = \langle T^*Ty, y\rangle = \langle Ty, Ty\rangle = |Ty|^2.$$

Since $\operatorname{Ker} T = \{0\}$, we get $y = 0$ whence $\operatorname{Im}(T^*T)^{1/2}$ is dense. This concludes the proof of the theorem.

The norm-preserving linear map $U$ in the theorem can then be extended by continuity to all of $H$ (which is the closure of its domain of definition), and this extension is a unitary automorphism of $H$.

This result is useful in the theory of group representations. Indeed, suppose that we are given a group $G$ and two homomorphisms

$$R\colon G \to \operatorname{Laut}(E) \qquad \text{and} \qquad S\colon G \to \operatorname{Laut}(E)$$

of $G$ into the group of invertible operators on $E$. Assume that $T$ and $T^*$ commute with $(R, S)$ in the sense that for all $\sigma \in G$ we have

$$TR(\sigma) = S(\sigma)T \qquad \text{and} \qquad T^*S(\sigma) = R(\sigma)T^*.$$

Verify that $U$ satisfies similar relationships. This shows that our two representations are "isomorphic".

Next we deal with the general polar decomposition. Let $H$ be a Hilbert space. Let $U\colon H \to E$ be a bounded linear map into some other Hilbert space $E$. We say that $U$ is a **partial isometry** if there exists an orthogonal decomposition

$$H = H_1 \perp H_2$$

such that the restriction of $U$ to $H_1$ is an isometry (norm-preserving linear map) onto the image of $H_1$, and the restriction of $U$ to $H_2$ is equal to 0.

The purpose of this section is to prove:

**Theorem 7.2.** *Let $A\colon H \to H$ be an arbitrary operator. Then there exists a unique decomposition*

$$A = UP,$$

*where $U$ is a partial isometry and $P$ is hermitian positive.*

*Proof.* We shall give the essential steps of the proof, and leave certain routine details as Exercise 15.

First note that $A^*A$ is symmetric positive and has a unique symmetric positive square root, denoted by

$$P_A = (A^*A)^{1/2}.$$

(a) We can define a linear map $U = U_A$ on $\operatorname{Im} P_A$ by the formula

$$U(A^*A)^{1/2}v = Av,$$

namely this is well defined. (If $(A^*A)^{1/2}v = 0$ then $Av = 0$.)

(b) The map $U\colon \operatorname{Im} P_A \to \operatorname{Im} A$ is a unitary map, which can therefore be extended by continuity to the closure of $\operatorname{Im} P_A$. Define $U$ to be 0 on the orthogonal complement of $\operatorname{Im} P_A$. Then $U$ is a partial isometry.

(c) The decomposition $A = UP$ into a partial isometry $U$ (relative to $\operatorname{Im} P$) and a positive operator $P$ is unique, i.e. if $A = WQ$, then $P = Q$, $U = W$.

(d) We have $A^* = U_{A^*}P_{A^*}$, where $U_{A^*} = U^*$ and $P_{A^*} = UP_AU^*$.

The decomposition $A = UP$ is called the **polar decomposition** of $A$, and (d) gives the polar decomposition of $A^*$ in terms of the polar decomposition of $A$.

## XVIII, §8. THE MORSE–PALAIS LEMMA

Let $U$ be an open set in some (real) Hilbert space $E$, and let $f$ be a $C^{p+2}$ function on $U$, with $p \geqq 1$. We say that $x_0$ is a **critical point** for $f$ if $Df(x_0) = 0$. We wish to investigate the behavior of $f$ at a critical point. After translations, we can assume that $x_0 = 0$ and that $f(x_0) = 0$. We observe that the second derivative $D^2f(0)$ is a continuous bilinear form on $E$. Let $\lambda = D^2f(0)$, and for each $x \in E$ let $\lambda_x$ be the functional $y \mapsto \lambda(x, y)$. If the map $x \mapsto \lambda_x$ is a toplinear isomorphism of $E$ with its dual space $E'$, then we say that $\lambda$ is **non-singular**, and we say that the critical point is **non-degenerate**.

We recall that a local $C^p$-isomorphism $\varphi$ at 0 is a $C^p$-invertible map defined on an open set containing 0.

**Theorem 8.1.** *Let $f$ be a $C^{p+2}$ function defined on an open neighborhood of 0 in the Hilbert space $E$, with $p \geqq 1$. Assume that $f(0) = 0$, and that 0 is a non-degenerate critical point of $f$. Then there exists a local $C^p$-isomorphism at 0, say $\varphi$, and an invertible symmetric operator $A$ such*

*that*

$$f(x) = \langle A\varphi(x), \varphi(x)\rangle.$$

*Proof.* We may assume that $U$ is a ball around 0. We have

$$f(x) = f(x) - f(0) = \int_0^1 Df(tx)x\,dt,$$

and applying the same formula to $Df$ instead of $f$, we get

$$f(x) = \int_0^1 \int_0^1 D^2f(stx)tx \cdot x\,ds\,dt = g(x)(x, x)$$

where

$$g(x) = \int_0^1 \int_0^1 D^2f(stx)t\,ds\,dt.$$

Then $g$ is a $C^p$ map into the Banach space of continuous bilinear maps on $E$, and even the space of symmetric such maps by Theorem 5.3 of Chapter XIII. We know that this Banach space is toplinearly isomorphic to the space of symmetric operators on $E$, and thus we can write

$$f(x) = \langle A(x)x, x\rangle$$

where $A: U \to \mathrm{Sym}(E)$ is a $C^p$ map of $U$ into the space of symmetric operators on $E$. A straightforward computation shows that

$$\tfrac{1}{2}D^2f(0)(v, w) = \langle A(0)v, w\rangle.$$

Since we assumed that $D^2f(0)$ is non-singular, this means that $A(0)$ is invertible, and hence $A(x)$ is invertible for all $x$ sufficiently near 0.

We want to define $\varphi(x)$ to be $C(x)x$ where $C$ is a suitable $C^p$ map from a neighborhood of 0 into the open set of invertible operators, and in such a way that we have

$$\langle A(x)x, x\rangle = \langle A(0)\varphi(x), \varphi(x)\rangle = \langle A(0)C(x)x, C(x)x\rangle.$$

This means that we must seek a map $C$ such that

$$C(x)^*A(0)C(x) = A(x).$$

If we let $B(x) = A(0)^{-1}A(x)$, then $B(x)$ is close to the identity $I$ for small $x$. The square root function has a power series expansion near 1, which is a uniform limit of polynomials, and is $C^\infty$ on a neighborhood of $I$ (cf. Exercise 2 of Chapter XIII), and we can therefore take the square root of

$B(x)$, so that we let

$$C(x) = B(x)^{1/2}.$$

We contend that this $C(x)$ does what we want. Indeed, since both $A(0)$ and $A(x)$ (or $A(x)^{-1}$) are self adjoint, we find that

$$B(x)^* = A(x)A(0)^{-1},$$

whence

$$B(x)^*A(0) = A(0)B(x).$$

But $C(x)$ is a power series in $I - B(x)$, and $C(x)^*$ is the same power series in $I - B(x)^*$. The preceding relation holds if we replace $B(x)$ by any power of $B(x)$ (by induction), hence it holds if we replace $B(x)$ by any polynomial in $I - B(x)$, and hence finally, it holds if we replace $B(x)$ by $C(x)$, and thus

$$C(x)^*A(0)C(x) = A(0)C(x)C(x) = A(0)B(x) = A(x),$$

which is the desired relation.

All that remains to be shown is that $\varphi$ is a local $C^p$-isomorphism at 0. But one verifies that in fact, $D\varphi(0) = C(0)$, so that what we need follows from the inverse mapping theorem. This concludes the proof of Theorem 8.1.

**Corollary 8.2.** *Let $f$ be a $C^{p+2}$ function near 0 on the Hilbert space $E$, such that 0 is a non-degenerate critical point. Then there exists a local $C^p$-isomorphism $\psi$ at 0, and an orthogonal decomposition $E = F + F^\perp$, such that if we write $x = y + z$ with $y \in F$ and $z \in F^\perp$, then*

$$f(\psi(x)) = \langle y, y \rangle - \langle z, z \rangle.$$

*Proof.* The theorem reduces the problem to the case discussed in Corollaries 5.2 and 5.3. In that case, on a space where $A$ is positive definite, we can always make the toplinear isomorphism $x \mapsto A^{1/2}x$ to get the quadratic form to become the given hermitian product $\langle\ ,\ \rangle$, and similarly on the space where $A$ is negative definite.

**Note.** The Morse–Palais lemma was proved originally by Morse in the finite dimensional case, using the Gram–Schmidt orthogonalization process. The elegant generalization and its proof in the Hilbert space case is due to Palais. Cf. [Pa 2]. It shows (in the language of coordinate systems) that a function near a critical point can be expressed as a quadratic form after a suitable change of coordinate system (satisfying requirements of differentiability). It comes up naturally in the calculus of

variations, [Pa 1] and [Sm 1]. For instance, one considers a space of paths (of various smoothness) $\sigma: [a, b] \to E$ where $E$ is a Hilbert space. One then defines a function on these paths, essentially related to the length

$$f(\sigma) = \int_a^b \langle \sigma'(t), \sigma'(t) \rangle \, dt$$

and one investigates the critical points of this function, especially its minimum values. These turn out to be the solutions of the variational problem, by definition of what one means by a variational problem. Even if $E$ is finite dimensional, so a euclidean space, the space of paths is infinite dimensional, so that we need an infinite dimensional theory to deal with this question.

## XVIII, §9. EXERCISES

1. Let $E$ be a Hilbert space.
   (a) Let $P$ be a hermitian operator such that $P^2 = P$. Show that $P$ is an orthogonal projection on a closed subspace.
   (b) Conversely, let $E = F + F^\perp$ be an orthogonal decomposition, where $F$ is closed subspace. Let $P$ be the orthogonal projection on $F$, and assume that $F \neq \{0\}$. (i) Show that $|P| = 1$, and that $P^2 = P$. (ii) Show that $P$ is hermitian.

2. Let $A$ be hermitian and positive. Show that for all $x$, $y$ we have

$$|\langle Ax, y \rangle|^2 \leqq |\langle Ax, x \rangle| |\langle Ay, y \rangle|.$$

3. Prove that the four conditions **UN 1** through **UN 4** defining a unitary operator are equivalent.

4. If $A$ is hermitian, show that $I + iA$ is invertible.

5. Let $A$ be an operator and let $\sigma(A)$ be its spectrum (we assume that our Hilbert space is complex). Show that

$$\sigma(A^*) = \overline{\sigma(A)}.$$

6. Prove the statement made in the text that if $A$ is compact, hermitian, then $\langle Ax, x \rangle$ takes on a maximum or minimum on the unit sphere.

7. Let $E$ be the space of real valued continuous functions on $[0, 1]$ and let $M: E \to E$ be the linear map given by

$$(Mf)(x) = xf(x).$$

   We take the $L^2$-norm on $E$, arising from the scalar product

$$\langle f, g \rangle = \int_0^1 fg,$$

and we let $E_2$ denote the completion of $E$ with respect to this norm. Show that $M$ is self-adjoint, and that for any real $\alpha$, the operator $M - \alpha I$ is not invertible on $E_2$, for otherwise, it would be invertible on $E$. Show that $\alpha$ is not an eigenvalue of $M$. *Note:* $M$ is obviously injective on $E$, but you will have to prove that, for instance, it is injective on $E_2$, so deal with $L^2$-Cauchy sequences in $E$.

8. Let $E$ be a complex Hilbert space with a denumerable orthonormal basis $\{x_n\}$ $(n = 1, 2, \ldots)$. Let $S$ be a compact infinite subset of the complex numbers. Show that there exists a denumerable dense subset $\{\alpha_n\}$ of $S$. Show that there exists a unique operator $A$ on $E$ such that $Ax_n = \alpha_n x_n$ for all $n$, and that the spectrum of $A$ is equal to $S$. Show that the eigenvalues of $A$ are precisely equal to the numbers $\alpha_n$. Show that if $\alpha$ is in $S$ and not equal to any $\alpha_n$, then the image of $A - \alpha I$ is dense in $E$ but not equal to $E$.

9. Let $l^2$ be the Hilbert space of sequences $\alpha = \{a_n\}$, $n \geqq 1$, such that $\sum |a_n|^2$ converges, with the hermitian product

$$\langle \alpha, \beta \rangle = \sum a_n \bar{b}_n$$

if $\beta = \{b_n\}$. Let $T$ be the shift operator, that is

$$T\alpha = (0, a_1, a_2, a_3, \ldots).$$

Show that the spectrum of $T$ is the unit disc and that $T$ has no eigenvalue.

10. **Irreducible representations of compact groups.** Let $G$ be a compact group, and $E$ a complex Hilbert space. A **(unitary) representation** $R$ of $G$ in $E$ is a continuous homomorphism,

$$R \colon G \to \mathrm{Aut}(E)$$

of $G$ into the group of (unitary) automorphisms of $E$. We let $dx$ be a Haar measure on $G$ such that $G$ has measure 1. We say that a representation $R$ is irreducible if there is no closed subspace of $E$ invariant under $R(G)$ other than 0 and $E$ itself. The basic result is:

**Theorem.** *If $R \colon G \to \mathrm{Aut}(E)$ is a unitary irreducible representation of a compact group, then $E$ is finite dimensional.*

Prove this by the following steps. Let $\{v_i\}$ be an orthonormal basis of $E$. Let $P$ be the projection on the one-dimensional space generated by $v_1$.

(a) Using Schur's lemma, prove that there exists a number $c$ such that

$$\int_G R(x) P R(x)^{-1} \, dx = cI.$$

In fact, show that the operator on the left is a positive operator, commuting with all $R(a)$, $a \in G$.

(b) Considering $\langle cv_1, v_1 \rangle$, show that $c > 0$.

(c) For any $x \in G$, $\{R(x)^{-1}v_i\}$ is an orthonormal basis $\{w_i\}$. Prove that for any $n$,

$$\sum_{i=1}^{n} \langle Pw_i, w_i \rangle \leqq 1.$$

(d) Conclude that $nc \leqq 1$, whence $n$ is bounded.

11. Let $A$ be a hermitian operator on the Hilbert space $E$, and assume that the spectrum of $A$ is the union of two disjoint closed sets $S$, $T$. Show that $E$ admits a direct sum decomposition into two closed subspaces $E_S$ and $E_T$ which are $A$-invariant, and such that, if we let $A_S$ and $A_T$ be the restriction of $A$ to $E_S$ and $E_T$ respectively, then the spectrum of $A_S$ is $S$ and the spectrum of $A_T$ is $T$. (Cf. Exercise 2 of Chapter XVI.)

12. Let $A$ be a hermitian operator on a Hilbert space. If $c$ is an isolated point of the spectrum, show that $c$ is an eigenvalue.

13. Show that an operator $A$ on a Hilbert space is hermitian positive if and only if there exists an operator $B$ such that $A = B^*B$.

14. Let $S$ be a non-zero Banach subalgebra of operators on a Hilbert space $E$. Assume that $S$ is *-closed (i.e. if $A \in S$ then $A^* \in S$), and that all elements of $S$ consist of compact operators. Prove that there exists an $S$-irreducible subspace (i.e. a subspace $\neq 0$ which has no $S$-invariant subspace other than 0 and itself), and that $E$ is the orthogonal sum of $S$-irreducible subspaces. [*Hint:* Writing $A = B + iC$, where $B$, $C$ are hermitian, you can find a hermitian element $A$ in $S$ such that $A \neq 0$. Let $\lambda$ be an eigenvalue for $A$, and among all $S$-invariant subspaces, let $M \neq 0$ be such that the eigenspace $M_\lambda$ for $A$ has minimal dimension. Let $v \in M$, $v \neq 0$. Prove that the closure of $Sv$ is irreducible.]

15. Give the details of the proofs in statements (a), (b), (c), (d) for Theorem 8.1.

The next exercise gives a complement and refinement of Theorem 8.1 when we deal with an automorphism of the Hilbert space.

## Polar Decomposition of an Automorphism

16. Let $A: E \to E$ be an operator on a Hilbert space.
  (a) If $A$ is unitary, then show that $A^{-1}$ is unitary. Also, $A$ is hilbertian if and only if $A^*A = I$. If $A$, $B$ are hilbertian, so is $AB$. In the language of algebra, hilbertian operators form a group, denoted by Hilb($E$).
  (b) An operator $A$ is said to be **skew-symmetric** if $A^* = -A$. Since we work in the real case, we shall say that an operator is symmetric instead of hermitian, and let Sym($E$) denote the space of symmetric operators on $E$. We let Sk($E$) denote the space of skew-symmetric operators. Show that End($E$) is a direct sum

$$L(E, E) = \text{Sym}(E) \oplus \text{Sk}(E).$$

(c) For all operators $A$, show that the series

$$\exp(A) = I + A + \frac{A^2}{2!} + \cdots$$

converges, and if $AB = BA$, then

$$\exp(A + B) = \exp(A)\exp(B).$$

For all operators $A$ sufficiently close to the identity $I$, the series

$$\log A = (A - I) - \frac{(A - I)^2}{2} + \cdots$$

converges, and if $AB = BA$, then

$$\log AB = \log A + \log B.$$

(d) If $A$ is symmetric (resp. skew-symmetric), then $\exp A$ is symmetric positive definite (resp. hilbertian). If $A$ is a toplinear automorphism sufficiently close to $I$ and is positive definite symmetric (resp. hilbertian), then $\log A$ is symmetric (resp. skew symmetric).

(e) Show that the exponential map gives a homeomorphism from the space $\mathrm{Sym}(E)$ of symmetric operators of $E$ to the space $\mathrm{Pos}(E)$ of symmetric positive definite automorphisms of $E$. Define its inverse.

(f) Show that the space of toplinear automorphisms of the Hilbert space $E$ is homeomorphic to the product

$$\mathrm{Hilb}(E) \times \mathrm{Pos}(E)$$

under the map given by

$$(H, P) \mapsto HP.$$

[*Hint:* To construct the inverse, given an invertible operator $A$, we must express it in a unique way as a product $A = HP$ where $H$ is hilbertian, $P$ is symmetric positive definite, and both $H$, $P$ depend continuously on $A$. Show that

$$P = (A^*A)^{1/2} \qquad \text{and} \qquad H = AP^{-1}$$

exist and satisfy our requirements.]

## Hilbert–Schmidt Operators

17. Assume that $H$ has a countable Hilbert basis. An operator $A$ is called **Hilbert–Schmidt** if there exists some Hilbert basis $\{u_i\}$ such that

$$\sum_i |Au_i|^2 < \infty.$$

(a) Prove that the same convergence holds for any other Hilbert basis $\{v_j\}$. For Hilbert–Schmidt operators $A$ and $B$, define their **scalar product**

$$\langle A, B\rangle = \sum \langle Au_i, Bu_i\rangle$$

with some Hilbert basis $\{u_i\}$.

(b) Show that the sum is absolutely convergent, and independent of the choice of Hilbert basis. Show that $B^*A$ is Hilbert–Schmidt.

(c) Show that the Hilbert–Schmidt operators form a vector space, which therefore has the scalar product defined above. Denote the corresponding **norm** by

$$N_2(A) \quad \text{or} \quad \|A\|_2,$$

so that

$$\|A\|_2^2 = \sum |Au_i|^2.$$

Prove the additional properties, where $A$, $B$ denote Hilbert–Schmidt operators, and $X$ denotes an arbitrary operator.

**HS 1.** $\|A^*\|_2 = \|A\|_2$.

**HS 2.** $XA$ and $AX$ are Hilbert–Schmidt, and

$$\|XA\|_2 \leqq |X|\,\|A\|_2, \qquad \|AX\|_2 \leqq |X|\,\|A\|_2.$$

**HS 3.** A Hilbert–Schmidt operator is compact.
[For **HS 3**, use the projection on the finite dimensional spaces generated by $u_1, \ldots, u_N$ for a finite number of $u_i$.]

**HS 4.** $$\|A + B\|_2^2 - \|A\|_2^2 - \|B\|_2^2 = 2 \cdot \mathrm{Re}\langle A, B\rangle.$$

**HS 5.** $$\mathrm{Re} \sum \langle Au_i, Bu_i\rangle = \mathrm{Re} \sum \langle A^*u_i, B^*u_i\rangle.$$

**HS 6.** $$\langle A^*, B^*\rangle = \langle A, B\rangle.$$

**HS 7.** $$\langle XA, B\rangle = \langle A, X^*B\rangle \quad \text{and} \quad \langle AX, B\rangle = \langle A, BX^*\rangle.$$

## Trace Class Operators

18. An operator is said to be of **trace class** if it is the product of two Hilbert–Schmidt operators, say $A = B^*C$ where $B$, $C$ are Hilbert–Schmidt. For such operators $A$, define the **trace** of $A$:

$$\mathrm{tr}(A) = \sum \langle Au_i, u_i\rangle = \sum \langle Cu_i, Bu_i\rangle = \langle C, B\rangle.$$

The first sum shows that the trace is independent of the choice of $B$, $C$. Show:

**TR 1.** $$|\mathrm{tr}(A)| \leqq \|B\|_2 \|C\|_2.$$

**TR 2.** If $A$ is of trace class, so are $AX$ and $XA$, and we have

$$\operatorname{tr}(AX) = \operatorname{tr}(XA).$$

**TR 3.** $A$ is of trace class if and only if $P_A$ is of trace class, and

$$\operatorname{tr}(P_A) = \operatorname{tr}(P_{A^*}).$$

**TR 4.** Let $P$ be a symmetric positive operator. Then $P$ is of trace class if and only if $\sum \langle Pu_i, u_i\rangle$ converges.

[*Hint*: Use $P^{1/2}$.] If $A$ is an operator of trace class, define

$$N_1(A) = \|A\|_1 = \operatorname{tr} P_A = \|P_A^{1/2}\|_2^2.$$

**TR 5.** The operators of trace class form a vector space; the function

$$A \mapsto \|A\|_1$$

is a norm, satisfying $\|A\|_1 = \|A^*\|_1$.

[*Hint*: Write $P_{A+B} = U^*(A + B)$ where $U$ is a partial isometry.]

**TR 6.** If $A$ is of trace class, so are $XA$, $AX$, and we have

$$\|XA\|_1 \leqq |X|\,\|A\|_1 \qquad \text{and} \qquad \|AX\|_1 \leqq |X|\,\|A\|_1.$$

**TR 7.** If $A$ is of trace class, then $|\operatorname{tr} A| \leqq \|A\|_1$.

**TR 8.** *Let $T_n$ be a sequence of operators on $H$ converging weakly to an operator $T$. In other words, for each $v, w \in H$, suppose that $\langle T_n v, w\rangle \to \langle Tv, w\rangle$. Let $A$ be of trace class. Then*

$$\operatorname{tr}(TA) = \lim \operatorname{tr}(T_n A).$$

[*Hint*: Assume first that $A = P$ is positive symmetric. Since $A$ is compact (because $A$ is Hilbert–Schmidt and **HS 3**), there is a Hilbert basis of $H$ consisting of eigenvectors $\{u_i\}$, with $Au_i = c_i u_i$. You will now need the uniform boundedness theorem to conclude that the norms $T_n$ are bounded. Then use the absolute convergence

$$\sum |c_i| < \infty$$

to prove the assertion in this case. In general write the polar decomposition $A = UP$, so $TA = (TU)P$ and you can apply the first part of the proof.]
*Note*: For complete proofs, cf. [La 3], Appendix of Chapter VII, and [Sh].

# CHAPTER XIX

# Further Spectral Theorems

In this chapter, we use the spectral theorem of Chapter XVIII to give a finer theory, making sense of the expression $f(A)$ when $f$ is not continuous. Ultimately, one wants to use very general functions $f$ in the context of measure theory, namely bounded measurable functions, as a corollary of what was done in Chapter XVIII. For our purposes here, we deal with an intermediate category of functions, essentially characteristic functions of intervals. These give rise to projection operators, whose formalism is important for its own sake. We also want to deal with unbounded operators as an application.

This chapter may be omitted, and is included only for those who want to go into spectral theory a little more deeply, as in the next chapter. The development of spectral measures gives a good example of how measure theory and the general functional analysis of this chapter can be put together.

## XIX, §1. PROJECTION FUNCTIONS OF OPERATORS

We need to extend the notion $f(A)$ to functions $f$ which are not continuous, to include at least characteristic functions of intervals. We follow Riesz–Nagy more or less. We let $H$ be a Hilbert space.

**Lemma 1.1.** *Let $\alpha$ be real, and let $\{A_n\}$ be a sequence of hermitian operators such that $A_n \geqq \alpha I$ for all $n$, and such that $A_n \geqq A_{n+1}$. Given $v \in H$, the sequence $\{A_n v\}$ converges to an element of $H$. If we denote this element by $Av$, then $v \mapsto Av$ is a bounded hermitian operator.*

*Proof.* From the inequality

$$\langle A_n v, v \rangle \geqq \alpha \langle v, v \rangle$$

we conclude that $\langle A_n v, v \rangle$ converges, for each $v \in H$. Since

$$\langle A_n v, w \rangle = \tfrac{1}{2}\langle A_n(v + w), v + w \rangle - \tfrac{1}{2}\langle A_n(v - w), v - w \rangle,$$

it follows that $\langle A_n v, w \rangle$ converges for each pair of elements $v$, $w \in H$. Define

$$\lambda_v(w) = \lim_{n \to \infty} \langle A_n v, w \rangle.$$

Then $\lambda_v$ is antilinear, and $|\langle A_n v, w \rangle| \leqq C|v||w|$ for some $C$ and all $v$, $w \in H$. Hence there exists an operator $A$ such that

$$\langle Av, w \rangle = \lim_{n \to \infty} \langle A_n v, w \rangle.$$

Since $\langle A_n v, w \rangle = \langle v, A_n w \rangle$, it follows that $A$ is hermitian.

**Lemma 1.2.** *Let $f$ be a function on the spectrum of $A$, bounded from below, and which can be expressed as a pointwise convergent limit of a decreasing sequence of continuous functions, say $\{h_n\}$. Then*

$$\lim_{h \to \infty} h_n(A)$$

*is independent of the sequence $\{h_n\}$.*

*Proof.* Say $g_n(t)$ decreases also to $f(t)$. Given $k$, for large $n$ we have

$$\max(g_n, h_k) \leqq h_k + \varepsilon,$$

by Dini's theorem, so for all $t$ we have $g_n(t) \leqq h_k(t) + \varepsilon$, and hence

$$g_n(A) \leqq h_k(A) + \varepsilon I.$$

This shows that

$$\lim g_n(A) \leqq h_k(A) + \varepsilon I,$$

and therefore that

$$\lim g_n(A) \leqq \lim h_k(A) + \varepsilon I.$$

This is true for all $\varepsilon$. Letting $\varepsilon \to 0$ and using symmetry, we have proved our lemma.

From Lemma 1.2, we see that the association

$$f \mapsto f(A)$$

can be extended to the linear space generated by functions which can be obtained as limits from above of decreasing sequences of continuous functions, and are bounded from below. The map is additive, order preserving, and clearly multiplicative, i.e.

$$(fg)(A) = f(A)g(A)$$

for $f$, $g$ in this vector space.

The most important functions to which we apply this extension are characteristic functions like the function $\psi_c(t)$ whose graph is drawn in Figure 19.1. It is a limit of the functions $h_n(t)$ drawn in Figure 19.2.

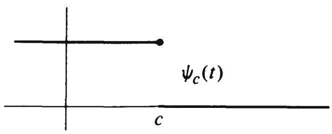



Figure 19.1

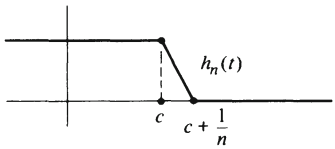



Figure 19.2

**Lemma 1.3.** *Let* $\psi_c(A) = P_c$. *If* $\alpha I \leqq A \leqq \beta I$, *then*:

(i) $P_c = 0$ *if* $c < \alpha$, *and* $P_c = I$ *if* $c \geqq \beta$.
(ii) *If* $c \leqq c'$, *then* $P_c \leqq P_{c'}$.

*Proof.* Clear from Lemma 1.2.

Observe that we also have $P_c^2 = P_c$, i.e. that $P_c$ is a projection. We call $\{P_c\}$ the **spectral family** associated with $A$.

We keep the same notation, and we shall make use of the two functions $f_c$, $g_c$ whose graph is drawn in Figure 19.3. Thus $f_c(t) + g_c(t) = |t - c|$. We have

$$(t - c)(1 - \psi_c(t)) = f_c(t).$$

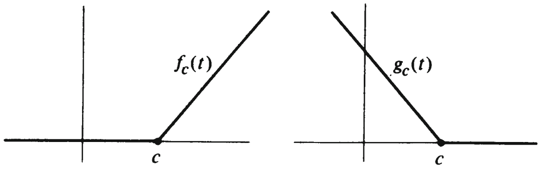



Figure 19.3

Hence

$$(1)\qquad (A - cI)(I - P_c) = f_c(A),$$

$$(2)\qquad A - cI = f_c(A) - g_c(A),$$

$$(3)\qquad (A - cI)P_c = -g_c(A)P_c = -g_c(A).$$

**Theorem 1.4.** *Let $P_c$ be the spectral family associated with $A$. If $b \leqq c$, then we have*

$$bI \leqq A \leqq cI, \qquad \text{on } \mathrm{Im}(P_c - P_b).$$

*Proof.* From (1) above, we have $A - bI = f_b(A)$ on the orthogonal complement of $P_b$, whence the inequality $bI \leqq A$ follows on this complement since $f_b \geqq 0$. From (3) above, we have

$$A - cI = -g_c(A)$$

on the image of $P_c$, and since $-g_c$ is $\leqq 0$, we get $A \leqq cI$ on this image. This proves our theorem.

**Theorem 1.5.** *The family $\{P_t\}$ is strongly continuous from the right.*

*Proof.* Let $v \in H$. Our assertion means that $P_{c+\varepsilon}v \to P_c v$ as $\varepsilon \to 0$. It suffices to prove that

$$\langle P_{c+\varepsilon}v, v\rangle \to \langle P_c v, v\rangle$$

because

$$\langle (P_{c+\varepsilon} - P_c)v, v\rangle = |(P_{c+\varepsilon} - P_c)v|^2.$$

Let $h_\varepsilon(t)$ be the function whose graph is shown in Figure 19.4. We have

$$\psi_c(t) \leqq \psi_{c+\varepsilon}(t) \leqq h_\varepsilon(t)$$

and

$$P_c \leqq P_{c+\varepsilon} \leqq h_\varepsilon(A).$$

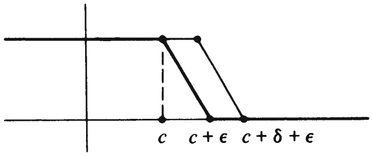



Figure 19.4

In other words, we have

$$\langle P_c v, v\rangle \leqq \langle P_{c+\varepsilon} v, v\rangle \leqq \langle h_\varepsilon(A)v, v\rangle.$$

We let $\varepsilon \to 0$. Then $h_\varepsilon(t)$ decreases to $\psi_c(t)$ and $\langle h_\varepsilon(A)v, v\rangle$ decreases to $\langle P_c v, v\rangle$, which completes the proof of the theorem.

**Theorem 1.6 (Lorch).** *From the left,*

$$\lim_{\varepsilon \to 0} (P_c - P_{c-\varepsilon}) = Q_c$$

*is the projection on the c-eigenspace of A.*

*Proof.* Using Theorem 1.4, we have

$$(c - \varepsilon)(P_c - P_{c-\varepsilon}) \leqq A(P_c - P_{c-\varepsilon}) \leqq c(P_c - P_{c-\varepsilon})$$

whence

$$|(A - cI)(P_c - P_{c-\varepsilon})| \leqq \varepsilon.$$

But for each $v$, $\lim_{\varepsilon\to 0}(P_c - P_{c-\varepsilon})v$ exists, say $= w$. It follows that $Aw = cw$, i.e. $Q_c$ maps $H$ into the $c$-eigenspace.

Conversely, if $\varphi$ is a continuous function, then for any $A$-invariant closed subspace $F$, we have

$$\varphi(A|f) = \varphi(A)|f.$$

We want to show that $Q_c$ is the identity on the $c$-eigenspace, and without loss of generality we may therefore assume that $H = H_c$ is the eigenspace. Then $P_c = 0$ because $f_c = 0$ on the spectrum of $A$. If $b < c$, then

$$f_b(A) = A - bI = (c - b)I$$

is invertible, and hence $P_b = 0$. This proves Lorch's theorem.

## XIX, §2. SELF-ADJOINT OPERATORS

Let $H$ be a Hilbert space and $A$ a linear map,

$$A: D_A \to H$$

defined on a dense subspace. Consider the set of vectors $v \in H$ such that there exists $w \in H$ such that

$$\langle u, w \rangle = \langle Au, v \rangle, \qquad \text{all } u \in D_A,$$

or in other words, $\langle u, w \rangle - \langle Au, v \rangle = 0$. The set of such $v$ is the projection on the first factor of the intersection of the kernels of

$$(v, w) \mapsto \langle u, w \rangle - \langle Au, v \rangle, \qquad u \in D_A.$$

It is a vector space. To each $v$ in this vector space there is exactly one $w$, if it exists, having the above property, because

$$u \mapsto \langle Au, v \rangle$$

is a functional on a dense subspace. Hence we can define an operator $A^*$ by the formula

$$A^*v = w,$$

on the space $D_{A^*}$ of such vectors $v$. We call the pair $(A^*, D_{A^*})$ the **adjoint** of $A$.

Let $J: H \times H \to H \times H$ be the operator such that $J(x, y) = (-y, x)$. Then $J^2 = -I$. We note that the graph $G_{A^*}$ of $A^*$ is given by the formula

$$\boxed{G_{A^*} = (JG_A)^{\perp},}$$

where $\perp$ denotes orthogonal complement, and hence the graph of $A^*$ is closed.

We say that $A$ is **closed** if its graph $G_A$ is closed.

*If $A$ is closed, then $D_{A^*}$ is dense in $H$.*

*Proof.* Let $h \in D_{A^*}^{\perp}$, so

$$(h, 0) \in (G_{A^*})^{\perp} = (JG_A)^{\perp\perp} = JG_A$$

because we assumed that $A$ is closed. We conclude that $(0, h) \in G_A$, and hence $h = 0$, proving our assertion.

*If $A$ is closed, then $A^{**} = A$.*

*Proof.* $G_{A^{**}} = (JG_{A^*})^\perp = (J(JG_A)^\perp)^\perp = G_A$.

*If $D_A$ and $D_{A^*}$ are dense, then $G_{A^{**}} =$ closure of $G_A$.*

*Proof.* $G_{A^{**}} = (JG_{A^*})^\perp = (J(JG_A)^\perp)^\perp = \overline{G_A}$.

If $A$ is defined on $D_A$ and $B$ is defined on $D_B$, if $D_A \subset D_B$, and if the restriction of $B$ to $D_A$ is $A$, then one usually says that $A$ is **contained in** $B$, and one writes $A \subset B$. The above assertion shows that $A \subset A^{**}$.

We say that $A$ is **symmetric** if $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in D_A$. We say that $A$ is **self-adjoint**, $A = A^*$, if in addition $D_A = D_{A^*}$.

*If $A$ is symmetric, then $A \subset A^*$.*

This is clear. Recall that we assumed $D_A$ dense in $H$.

*If $A$, $B$ are self-adjoint and $A \subset B$, then $A = B$.*

This is also clear, because in general $B^* \subset A^*$, so in the self-adjoint case, $B \subset A$, whence $A = B$.

Let $A$ be symmetric, defined on $D_A$ dense as above. Let $\lambda \in \mathbf{C}$ not be real. Then $A - \lambda I$ is injective on $D_A$, because from

$$Au = \lambda u \qquad \text{and} \qquad \langle Au, u\rangle = \langle u, Au\rangle$$

we conclude

$$\lambda\langle u, u\rangle = \langle \lambda u, u\rangle = \langle u, \lambda u\rangle = \bar{\lambda}\langle u, u\rangle,$$

so $u = 0$. Hence we can define an operator

$$U = U_{A,\lambda} = (A + \bar{\lambda}I)(A + \lambda I)^{-1}$$

on the image $(A + \lambda I)D_A$. *We contend that $U$ is unitary.* This amounts to verifying that for $u, v \in D_A$ we have

$$\langle Au + \bar{\lambda}u, Av + \bar{\lambda}v\rangle = \langle Au + \lambda u, Av + \lambda v\rangle,$$

which is obvious.

**Lemma 2.1.** *If $A$ is symmetric, closed, and $\lambda \in \mathbf{C}$ is not real, then $(A + \lambda I)D_A$ is closed.*

*Proof.* Let $\{u_n\}$ be a sequence in $D_A$ such that $\{(A + \lambda I)u_n\}$ is Cauchy. Since $U$ is unitary, it follows that

$$\{(A + \bar{\lambda} I)u_n\}$$

is also Cauchy, hence $\{(\lambda - \bar{\lambda})u_n\}$ is Cauchy, and $\{u_n\}$ is Cauchy, say converging to $u$. But

$$\{2Au_n + (\lambda + \bar{\lambda})u_n\}$$

is Cauchy, whence also $\{Au_n\}$ is Cauchy. Since the graph of $A$ is assumed closed, we conclude that $\{(u_n, Au_n)\}$ converges to an element $(u, Au)$ in the graph, and the sequence

$$\{(A + \lambda I)u_n\}$$

converges to $(A + \lambda I)u$. This proves that $(A + \lambda I)D_A$ is closed.

**Theorem 2.2.** *Let $A$ be symmetric, closed with dense domain. Let $\lambda \in \mathbf{C}$ be not real, and such that $(A + \lambda I)D_A$ and $(A + \bar{\lambda} I)D_A$ are dense (whence equal to $H$ by the lemma). Then $A$ is self-adjoint.*

*Proof.* Let $v \in D_{A^*}$. It suffices to show that $v \in D_A$. We have by definition

$$\langle Au, v\rangle = \langle u, A^*v\rangle, \qquad \text{all } u \in D_A.$$

Since $(A + \lambda I)D_A = H$, there exists $u_1 \in D_A$ such that

$$A^*v + \lambda v = Au_1 + \lambda u_1.$$

Then

$$\langle Au, v\rangle = \langle u, Au_1 + \lambda u_1 - \lambda v\rangle, \qquad \text{all } u \in D_A,$$

whence

$$\langle (A + \bar{\lambda} I)u, v\rangle = \langle (A + \bar{\lambda} I)u, u_1\rangle, \qquad \text{all } u \in D_A.$$

This proves that $v = u_1$, as was to be shown.

**Remark.** In the literature, you will find that the dimension of the cokernel of $(A + \lambda I)D_A$ is called a defect index. We are concerned here with a situation in which the defect indices are 0.

**Corollary 2.3.** *Let $A$ be symmetric with dense domain. Let $\lambda \in \mathbf{C}$ be not real, and such that $(A + \lambda I)D_A$ and $(A + \bar{\lambda} I)D_A$ are dense. Then the closure of $G_A$ is the graph of an operator which is self-adjoint.*

*Proof.* Since $A$ is symmetric, the domain of $A^*$ is also dense, and we have shown above that $G_{A^{**}}$ is the closure of $G_A$, so $A$ has a closure. It is immediate that this closure is also symmetric, and the theorem applies.

An operator $A$ defined on $D_A$ is called **essentially self-adjoint** if the closure of its graph is the graph of a self-adjoint operator. The corollary gives a sufficient condition for an operator to be essentially self-adjoint.

**Theorem 2.4.** *Let $A$ be a self-adjoint operator. Let $z \in \mathbf{C}$ and $z$ not real. Then $A - zI$ has kernel* 0. *There is a unique bounded operator*

$$R(z) = (A - zI)^{-1}: H \to D_A$$

*which establishes a bijection between $H$ and $D_A$, and is the inverse of $A - zI$. We have*

$$R(z)^* = R(\bar{z}).$$

*If* $\mathrm{Im}\, z, \mathrm{Im}\, w \neq 0$, *then we have the resolvent equation*

$$(z - w)R(z)R(w) = R(z) - R(w) = (z - w)R(w)R(z),$$

*so in particular, $R(z)$, $R(w)$ commute. We have* $|R(z)| \leqq 1/|\mathrm{Im}\, z|$.

*Proof.* Let $z = x + iy$. If $u$ is in the domain of $A$, then

$$|(A - zI)u|^2 = |(A - xI)u|^2 + y^2|u|^2 \geqq y^2|u|^2$$

because $A$ is symmetric, so the cross terms disappear. This proves that the kernel of $A - zI$ is 0, and that the inverse of $A - zI$ is continuous, when viewed as defined on the image of $A - zI$. If $v$ is orthogonal to this image, i.e.

$$\langle Au - zu, v \rangle = 0$$

for all $u \in D_A$, then $\langle Au, v \rangle = \langle u, \bar{z}v \rangle$, and since $A$ is self-adjoint, it follows that $v$ lies in the domain of $A$ and that $Av = \bar{z}v$. Since the kernel of $A - \bar{z}I$ is 0, we conclude that $v = 0$. Hence the image of $A - zI$ is dense, so that by Lemma 2.1 this image is all of $H$ and $R(z)$ is everywhere defined, equal to the inverse of $A - zI$. We then have

$$[(A - wI) - (A - zI)]R(w) = (z - w)R(w).$$

Multiplying this on the left by $R(z)$ yields the resolvent formula of the theorem, whose proof is concluded.

We write

$$R(i) = (A - iI)^{-1} = C + iB$$

where $B$, $C$ are bounded hermitian. From the resolvent equation between $R(i)$ and $R(-i)$ we conclude that $B$, $C$ commute. We may call $B$ the **imaginary part of** $(A - iI)^{-1}$, symbolically

$$B = \operatorname{Im}(A - iI)^{-1}.$$

**Lemma 2.5.** *With the above notation, we have $C = AB$ and $BA \subset AB$. The kernel of $B$ is 0, and $O \leqq B \leqq I$.*

*Proof.* We have from $R(i)^* = R(-i)$ that

$$(A - iI)^{-1} - (A + iI)^{-1} = 2iB.$$

We multiply this on the left with $A$, noting that

$$A(A - iI)^{-1} = i(A - iI)^{-1} + I$$

and

$$A(A + iI)^{-1} = -i(A + iI)^{-1} + I.$$

We then obtain $C = AB$. For $BA$ we multiply the first relation on the right by $A$, so that we use

$$(A - iI)^{-1}(A - iI) = I_{D_A}$$

and similarly for $A + iI$. The relation $BA \subset AB$ follows. The kernel of $B$ is 0, for any vector in the kernel is also in the kernel of $C = AB$, whence in the kernel of $(A - iI)^{-1}$, and therefore equal to 0. We leave the relation $B \geqq O$ to the reader. That $B \leqq I$ follows from $|R(i)| \leqq 1$, a special case of the last inequality in Theorem 2.4.

We now give an example of a self-adjoint operator. It will be shown later that any self-adjoint operator is of this nature.

**Theorem 2.6.** *Let $\{H_n\}$ be a sequence of Hilbert space. Let $A_n$ be a bounded self-adjoint operator on $H_n$. Let $H$ be the orthogonal direct sum of the $H_n$, so that $H$ consists of all series $\sum u_n$ with $\sum |u_n|^2 < \infty$. There exists a unique self-adjoint operator $A$ on $H$ such that each $H_n$ is contained in the domain $D_A$ and such that the restriction of $A$ to $H_n$ is $A_n$. Its domain is the vector space of series $u = \sum u_n$ such that*

$$\sum |A_n u_n|^2 < \infty,$$

*and $Au = \sum A_n u_n$.*

*Proof.* The uniqueness is clear from the property that if $A$, $B$ are self-adjoint and $A \subset B$, then $A = B$. It suffices now to prove that if we let $D_A$ be the domain described above, and define $Au$ by $\sum A_n u_n$, then $A$ is self-adjoint. It is clear that $A$ is symmetric. Let $v \in D_{A^*}$. Then

$$\langle u, A^*v \rangle = \langle Au, v \rangle, \qquad \text{all } u \in D_A.$$

Say $u = \sum u_n$. Then

$$\sum \langle u_n, A^*v \rangle = \sum \langle Au_n, v \rangle.$$

If $u \in H_n$, then

$$\langle u_n, A^*v \rangle = \langle Au_n, v \rangle,$$
$$\langle u_n, (A^*v)_n \rangle = \langle Au_n, v_n \rangle,$$

whence $(A^*v)_n = A_n v_n$. Then

$$\sum |A_n v_n|^2 = \sum |(A^*v)_n|^2 = |A^*v|^2,$$

whence $v \in D_A$, so $D_{A^*} \subset D_A$ and $A$ is therefore self-adjoint. This proves the theorem.

In the situation of Theorem 2.6, we use the notation

$$A = \hat{\bigoplus} A_n.$$

We deal with the converse of Theorem 2.6. Let $A$ be an arbitrary self-adjoint operator on the Hilbert space $H$, and let

$$(A - iI)^{-1} = C + iB$$

as above.

We are in a position to decompose our Hilbert space by means of $B$. Let $\theta_c$ be the function whose graph is given in Figure 19.5, and which gives rise to a projection operator.

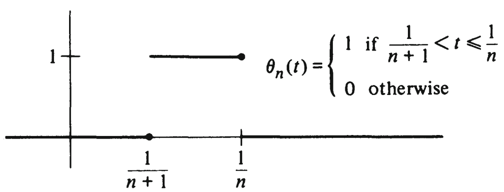



Figure 19.5

Let $\{P\}$ be the spectral family for $B$, and let

$$Q_n = \theta_n(B) = P_{1/n} - P_{1/(n+1)}.$$

Then $Q_n$ is a projection operator, and we let

$$H_n = Q_n H = \operatorname{Im} Q_n.$$

Then

$$H = \widehat{\bigoplus} H_n$$

is an orthogonal direct sum. In fact, let $\theta$ and $\eta$ be the functions whose graphs are shown in Figure 19.6(a) and (b) respectively.

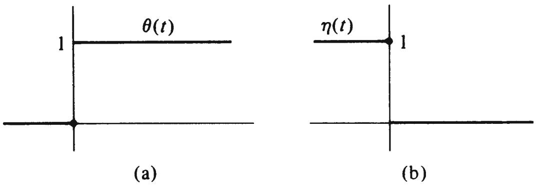



Figure 19.6

Then $1 - \theta = \eta$ and $\eta(B) = 0$ because the spectral family for $B$ is continuous at 0, in view of Lemma 2.5 (kernel $B = 0$) and Lorch's theorem (Theorem 1.6).

Let $s_n(t)$ be the function whose graph is shown in Figure 19.7. Then

$$Bs_n(B) = \theta_n(B) = Q_n.$$

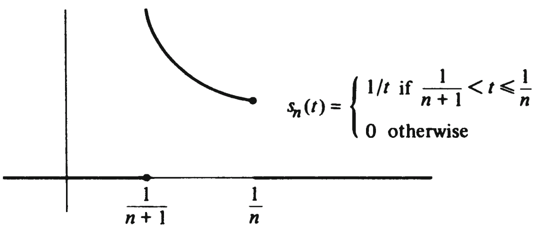



Figure 19.7

**Theorem 2.7.** *Let $A$ be a self-adjoint operator and let $B = \operatorname{Im}(A - iI)^{-1}$. Let $Q_n = \theta_n(B)$ be the projection operator defined by the function $\theta_n$ above. Then $A$ is defined on* $\operatorname{Im} Q_n$, *and*

$$Q_n A \subset AQ_n = s_n(B)C.$$

*Let $H_n = Q_n H$. Then $H$ is the orthogonal direct sum of the spaces $H_n$, the restriction of $A$ to $H_n$ is a bounded operator $A_n$, and*

$$A = \widehat{\bigoplus} A_n.$$

*Proof.* Since $ts_n(t) = \theta_n(t)$, we get $Bs_n(B) = \theta_n(B) = Q_n$. Then by Lemma 2.5

$$AQ_n = ABs_n(B) = Cs_n(B) = s_n(B)C.$$

In particular, $AQ_n$ is everywhere defined. On the other hand,

$$Q_n A = s_n(B)BA \subset s_n(B)AB \subset s_n(B)C.$$

This proves that $Q_n A \subset AQ_n$. It means that given $v \in D_A$, if

$$v = \sum v_n$$

is the decomposition of $v$ according to the space $H_n$, and if

$$Av = \sum w_n,$$

then

$$Q_n Av = w_n = AQ_n v = Av_n.$$

So $Av = \sum Av_n$, and the theorem is proved.

## XIX, §3. EXAMPLE: THE LAPLACE OPERATOR IN THE PLANE

We shall give a typical example of an unbounded symmetric operator. We shall assume that the reader is acquainted with some notions of advanced calculus, and in particular with Stokes' theorem in the plane. These notions are treated later in this book, but to give examples, one has to use something concrete, taken possibly from other courses. We let $(x, y)$ be the variables of $\mathbf{R}^2$ and we let

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

be the **Laplace operator**. We let

$$L = -\Delta$$

in order that $L$ turn out to be a positive operator.

If $U$ is a region in the plane with piecewise smooth boundary, and $F = (f, g)$ is a smooth vector field on $U$, we have the Stokes–Green theorm in dimension 2,

$$\iint_U \operatorname{div} F \, dx \, dy = \int_{\mathrm{Bd}\, U} F \cdot n \, ds,$$

where $\mathrm{Bd}\, U$ is the boundary of $U$ with the appropriate orientation. Letting

$$F = g \cdot \operatorname{grad} f = (gf_x, gf_y) \qquad \text{or} \qquad f \cdot \operatorname{grad} g = (fg_x, fg_y),$$

we obtain the formula

$$\iint_U (g\Delta f - f\Delta g) \, dx \, dy = \int_{\mathrm{Bd}\, U} \left( g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n} \right) ds.$$

If $f$, $g$ have compact support, and $U$ is the whole plane, then there is no boundary, and the term on the right in this last formula is equal to 0.

We shall use the scalar product for $f, g \in C_c^\infty(\mathbf{R})$ given by

$$\langle f, g \rangle = \iint_{\mathbf{R}^2} f(x, y)\overline{g(x, y)} \, dx \, dy.$$

Consider now $L$ or $\Delta$ to be operators on the space $C_c^\infty(\mathbf{R}^2)$ of $C^\infty$-functions with compact support. Then $L$ is of course not bounded, it is just a linear map. To avoid putting complex conjugates, assume that the functions $f$, $g$ are real valued in this space. Then we find:

**L 1.** *The operator $L$ or $\Delta$ is symmetric, that is*

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$

This comes from the Stokes–Green formula with no boundary, as we just observed. In addition, we have the property:

**L 2.** *The operator $L$ is positive, and in fact we have*

$$\langle Lf, f \rangle = \iint_{\mathbf{R}^2} \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right] dx \, dy.$$

To prove this second formula, consider the differential form

$$\omega(x, y) = \frac{\partial f}{\partial x} f \, dy - \frac{\partial f}{\partial y} f \, dx.$$

Then

$$d\omega = \left[\frac{\partial^2 f}{\partial x^2} f + \left(\frac{\partial f}{\partial x}\right)^2\right] dx \wedge dy - \left[\frac{\partial^2 f}{\partial y^2} f + \left(\frac{\partial f}{\partial y}\right)^2\right] dy \wedge dx$$

whence the second formula follows by the standard form of Stokes' theorem, namely

$$\iint_U d\omega = \int_{\mathrm{Bd}\,U} \omega.$$

Considering $L$ as an operator on functions on the unit disc, say, or on the plane with appropriate behavior at infinity to insure convergence, one can then show that $L$ is a self-adjoint operator. For the analogous theorem on the upper half plane, cf. $SL_2(\mathbf{R})$.

It is a technical and not trivial matter to give an explicit representation for the resolvent in terms of classical integrals. We don't go into this here. However, we do mention one other object associated with a situation like the above, namely a fundamental solution. Let $z = (x, y)$, and define

$$G(z, z') = \frac{1}{2\pi} \log |z - z'|.$$

Then $G$ is symmetric in $z$, $z'$ and is $C^\infty$ except on the diagonal $z = z'$. Furthermore, the (improper) integral of $\log|z|$ exists on any compact subset of the plane, because the function $\log r$ is locally integrable near $r = 0$, and

$$dx\, dy = r\, dr\, d\theta,$$

where $(r, \theta)$ are polar coordinates. (No fancy integration is needed here, but in the language of integration, we could say that $\log r$ is locally $L^1$.) Now we may view $G$ as defining an **integral operator**

$$S_G\colon C_c^\infty(\mathbf{R}^2) \to C^\infty(\mathbf{R}^2)$$

by the formula

$$S_G f(z') = \int_{\mathbf{R}^2} G(z, z') f(z)\, dz,$$

where $dz = dx\, dy$ for simplicity. This integral can also be written

$$\frac{1}{2\pi} \int_{\mathbf{R}^2} (\log|w|) f(w + z')\, dw.$$

Standard theorems from advanced calculus justify the fact that you can differentiate under the integral sign because $f$ is assumed to be smooth

with compact support, thus showing that $S_G f$ is $C^\infty$. (Cf. Lemma 2.2 of Chapter VIII.) If $D$ is the open disc of radius 1, then it is immediate that $S_G$ maps $C_c^\infty(D)$ into $BC^\infty(D)$, namely that $S_G f$ is bounded.

We now have the **fundamental formula**:

**L 3.**
$$S_G \circ \Delta = I,$$

where $I$ is the identity. Suppose we fix $z'$, and let

$$g(z) = G(z, z')$$

viewed as a function of $z$ only. Then the formula asserts that

$$\boxed{\iint_{\mathbf{R}^2} g\Delta f \, dx\, dy = f(z').}$$

For the proof, let $r = |z - z'|$, and view $f$ in terms of the polar coordinates $(r, \theta)$. Apply the Stokes–Green formula to the region $U(\varepsilon)$ outside a circle of radius $\varepsilon$, so that the boundary is the circle $S(\varepsilon)$ of radius $\varepsilon$ with reversed orientation. We have $ds = d\theta$ if we parametrize the circle of radius $\varepsilon$ by

$$\left(\varepsilon \cos\left(\frac{\theta}{\varepsilon}\right), \varepsilon \sin\left(\frac{\theta}{\varepsilon}\right)\right), \qquad 0 \leqq \theta \leqq 2\pi\varepsilon.$$

Also, $\Delta g = 0$. The right-hand side of Green's formula gives

$$\int_{S(\varepsilon)} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) ds = \int_0^{2\pi\varepsilon} f(\varepsilon, \theta)\frac{1}{2\pi\varepsilon}\, d\theta - \int_0^{2\pi\varepsilon} \frac{1}{2\pi}(\log \varepsilon)\frac{\partial f}{\partial r}\, d\theta.$$

As $\varepsilon \to 0$, the first term goes to $f(0, 0)$ and the second term goes to 0. Hence the desired formula follows.

We see that we have inverted the Laplace operator in some fashion, but only by a function with a logarithmic singularity, and definitely unbounded. Such a **fundamental solution** is, however, useful for constructing other solutions, or for constructing the resolvent. We don't go into this here. Cf. for instance, Folland [Fo]. The resolvent can be represented by a kernel in terms of Bessel functions.

CHAPTER XX

# Spectral Measures

In the spectral theorems of Chapters XVIII and XIX, we defined functions of an operator $f(A)$ with continuous functions first, and then essentially characteristic functions of an interval by a limiting process. If $v$ is a given vector, then the association

$$f \mapsto \langle f(A)v, v \rangle$$

defines a functional on $C_c(\mathbf{R})$. But from Chapter IX we know that such a functional determines a unique measure, which is thus associated with the operator $A$ and the vector $v$. This measure is called a **spectral measure**. The point of this chapter is to reformulate the results of previous chapters in terms of measures, and to extend the meaning of $f(A)$ to cases when $f$ is Borel measurable. In this way, we put together a lot of previous material, which is thus put to work: the spectral theorem of Chapter XIX; measures on locally compact spaces in Chapter IX; convolutions and Dirac sequences (families) in Chapter VIII.

## XX, §1. DEFINITION OF THE SPECTRAL MEASURE

We first state formally as a theorem the measure associated with the functional mentioned in the introduction.

**Theorem 1.1.** *Let $A$ be a bounded hermitian operator on a Hilbert space $H$. Let $v \in H$. There exists a unique positive measure $\mu_v = \mu_{A,v}$*

*on* **R** *such that for every* $\varphi \in C_c(\mathbf{R})$ *we have*

$$\langle \varphi(A)v, v \rangle = \int_{\mathbf{R}} \varphi \, d\mu_v.$$

*This measure is finite, and especially*

$$\|\mu_v\| = \mu_v(\mathbf{R}) \leqq |v|^2.$$

*Proof.* Let $\lambda_v$ be the functional on $C_c(\mathbf{R})$ defined by

$$\lambda_v(\varphi) = \langle \varphi(A)v, v \rangle.$$

If $\varphi \geqq 0$, then $\varphi(A) \geqq O$ by Theorem 4.4 of Chapter XIX. Hence $\lambda_v$ is a positive functional. The existence and uniqueness of the measure $\mu_v$ is then a special case of Theorem 2.7 of Chapter IX. Furthermore, by Theorem 4.4 of Chapter XIX we have

$$\langle \varphi(A)v, v \rangle \leqq \|\varphi\|_\infty |v|^2,$$

thus giving the desired bound for the measure $\mu_v$, and concluding the proof.

The measure $\mu_v$ is called the **spectral measure** associated with $A$ and $v$.
By polarization, for $v$, $w \in H$ we see the **existence and uniqueness of a complex measure** $\mu_{v,w}$ such that

$$\langle \varphi(A)v, w \rangle = \int_{\mathbf{R}} \varphi \, d\mu_{v,w} \qquad \text{for all} \quad \varphi \in C_c(\mathbf{R}).$$

It is clear that $\mu_{v,w}$ is **C**-linear in $v$ and anti-linear in $w$. Furthermore, we again have a bound

$$|\langle \varphi(A)v, w \rangle| \leqq \|\varphi\|_\infty |v| |w|,$$

whence by Theorem 4.2 of Chapter IX we also obtain the bound

$$\|\mu_{v,w}\| \leqq |v| |w|.$$

The measure $\mu_{v,w}$ is also called the **spectral measure** associated with $A$, $v$, $w$. Applying the defining formula for $\mu_{v,w}$ to a real valued function $\varphi$, we see immediately that

$$\overline{\mu_{v,w}} = \mu_{w,v}.$$

Let $BM(\mathbf{R})$ be the Banach space of bounded (Borel) measurable functions on $\mathbf{R}$. For each $f \in BM(\mathbf{R})$ the association

$$(v, w) \mapsto \int f \, d\mu_{v,w}$$

is linear in $v$ and anti-linear in $w$. Furthermore

$$\left| \int f \, d\mu_{v,w} \right| \leqq \|f\|_\infty |v| |w|,$$

as one sees by applying the dominated convergence theorem to a sequence $\{\varphi_n\}$ in $C_c(\mathbf{R})$ approaching $f$ pointwise almost everywhere with respect to the measure $|\mu_{v,w}|$, and such that $|\varphi_n| \leqq \|f\|_\infty$. Thus our association is continuous, and there exists a unique bounded operator, which we denote by $f(A)$, such that

$$\langle f(A)v, w \rangle = \int f \, d\mu_{v,w}.$$

The following properties are then satisfied for $f, g \in BM(\mathbf{R})$.

**SPEC 1.** $$(fg)(A) = f(A)g(A).$$

**SPEC 2.** $$f(A)^* = \bar{f}(A).$$

**SPEC 3.** *If $f_1$ is the function $f_1(t) = 1$, then $f_1(A) = I$.*

**SPEC 4.** *If the functions $f(t)$ and $g(t) = tf(t)$ are bounded measurable, then $g(A) = Af(A)$.*

**SPEC 5.** *We have $|f(A)| \leqq \|f\|_\infty$. Furthermore, if $\{f_n\}$ is a bounded sequence in $BM(\mathbf{R})$ converging pointwise to $f$, then $\{f_n(A)\}$ converges strongly to $f(A)$.*

Properties **SPEC 1** through **SPEC 4** are special cases of the spectral theorem, as formulated in Theorem 4.4 of Chapter XVIII, in case $f, g \in C_c(\mathbf{R})$, and so is the bound

$$|f(A)| \leqq \|f\|_\infty$$

in that case. The properties for $f \in BM(\mathbf{R})$ then follow by applying the dominated convergence theorem and taking limits. For instance, to prove **SPEC 1**, fix $\psi \in C_c(\mathbf{R})$ and let $\{\varphi_n\}$ be a bounded sequence in $C_c(\mathbf{R})$ converging to $f$ pointwise almost everywhere with respect to the positive measures

$$|\mu_{\psi(A)v,w}| \qquad \text{and} \qquad |\mu_{v,w}|.$$

We obtain

$$\langle \varphi_n(A)\psi(A)v, w\rangle = \int \varphi_n\psi \, d\mu_{v,w}$$
$$= \int \varphi_n \, d\mu_{\psi(A)v,w},$$

which converges to

$$\int f\psi \, d\mu_{v,w} = \langle (f\psi)(A)v, w\rangle$$

by the dominated convergence theorem if we use the first expression on the right, and also converges to

$$\int f \, d\mu_{\psi(A)v,w} = \langle f(A)\psi(A)v, w\rangle$$

if we use the second expression on the right. This takes care of one factor. We take care of the other by using a sequence $\{\psi_n\}$ converging to $g$ in the same manner as above. This proves **SPEC 1**, and also proves the equivalent formula

**SPEC 6.** $$\int fg \, d\mu_{v,w} = \int f \, d\mu_{g(A)v,w}.$$

We wish to extend the above results to unbounded operators.

**Theorem 1.2.** *Let $A$ be a self-adjoint operator. There exists a unique association $f \mapsto f(A)$ from $BM(\mathbf{R})$ into the bounded operators on $H$ satisfying* **SPEC 1** *through* **SPEC 5**.

*Proof.* By Theorem 2.7 of Chapter XIX there exists a direct sum decomposition

$$H = \hat{\bigoplus} H_n \quad \text{and} \quad A = \hat{\bigoplus} A_n$$

where $A_n = A|H_n$ is the restriction of $A$ to $H_n$ and is bounded self-adjoint. Let $f \in BM(\mathbf{R})$ and $v \in H$,

$$v = \sum v_n.$$

Since $|f(A_n)v_n| \leqq \|f\|_\infty |v_n|$, there is a unique bounded operator $f(A)$ such that

$$f(A)v = \sum f(A_n)v_n \quad \text{for all} \quad v \in H.$$

To each $H_n$ and $v_n \in H_n$ we can associate the measure $\mu_{v_n}^{(n)}$ as above. Since

$$|\langle f(A)v_n, v_n\rangle| \leqq \|f\|_\infty |v_n|^2,$$

the series

$$\sum_n \langle f(A)v_n, v_n\rangle$$

converges absolutely, and defines a positive functional on $C_c(\mathbf{R})$ (even on $BM(\mathbf{R})$). Therefore:

**Proposition 1.3.** *Given a direct sum decomposition as above, there exists a unique positive measure $\mu_v$ such that for all $f \in C_c(\mathbf{R})$ we have the formula*

$$\int f\, d\mu_v = \sum_n \langle f(A)v_n, v_n\rangle = \sum_n \int f\, d\mu_{v_n}^{(n)}.$$

The formalism of the five **SPEC** properties extends at once to the case of an unbounded operator $A$. For example, in the case of **SPEC 4**, note that

$$|Af(A)v_n| \leqq \|g\|_\infty |v_n|, \qquad \text{where} \quad g(t) = tf(t).$$

It follows that

$$\sum f(A)v_n \in D_A,$$

where $D_A$ is the domain of $A$, as in Chapter XIX, Theorems 2.6 and 2.7. Hence **SPEC 4** is valid.

Uniqueness will be proved in the next section, as an application of Dirac families.

We defined the measure $\mu_v$ non-canonically, in a way seemingly dependent on the decomposition of the Hilbert space into a direct sum such that $A$ restricts to a bounded operator on each summand. Of course, the measure can be characterized intrinsically as follows.

**Theorem 1.4.** *Let $A$ be a self-adjoint operator on $H$. Let $v \in H$. The measure $\mu_v$ is the unique positive measure $\mu$ such that for all $\varphi \in C_c(\mathbf{R})$ we have*

$$\langle \varphi(A)v, v\rangle = \int_{\mathbf{R}} \varphi\, d\mu.$$

*Proof.* Assuming the uniqueness in Theorem 1.2 to be proved in the next section, the present theorem is merely a special case of Theorem 2.7, Chapter IX (associating measures to functionals).

**Example.** This example is the unbounded analogue of the example given in Chapter XVIII, §4. Let $A$ be self-adjoint, and assume that there exists a positive number $c$ such that

$$\langle Av, v\rangle \geqq -c|v|^2 \qquad \text{for all} \quad v \in H.$$

Then there exists a unique differentiable mapping $K: \mathbf{R}^+ \to \mathrm{End}(H)$ satisfying the following conditions:

**H 1.** The image of $K$ is contained in the domain of $A$, and

$$\frac{d}{ds} K(s) = -AK(s).$$

**H 2.** For each $v \in H$ we have $\lim_{s \to 0} K(s)v = v$.

The proof is an exercise, which can be carried out by using the case for bounded operators, and the direct sum decomposition of Theorem 2.6 for unbounded operators. I took the statement from Faltings [Fa 1], Lemma 3.4. Readers can find another idea for the existence proof in this reference, which also proves the uniqueness.

Observe that for each positive $s$, the function $\varphi_s(t) = e^{-st}$ is bounded on the spectra of the bounded operators $A_n$ on the components $H_n$, and sufficiently uniformly so that there is no difficulty in handling the existence part of the proof by considering its effect on infinite sums $\sum v_n$. As in Proposition 1.3 the existence proof by this method is not invariant, but the uniqueness show that the end result $\varphi_s(A)$ is independent of the direct sum decomposition of $H$. One may write

$$K(s) = e^{-sA},$$

and $K(s)$ is called the **heat operator** associated with $A$.

## XX, §2. UNIQUENESS OF THE SPECTRAL MEASURE: THE TITCHMARSH–KODAIRA FORMULA

The uniqueness proof of this section provides a substantial example of the use of Dirac families, with weaker conditions than have been mentioned previously. For our purposes here, we define a **Dirac family** to be a family $\{\varphi_\varepsilon\}$ $(\varepsilon > 0)$ of $L^1$-functions on $\mathbf{R}$ satisfying the following properties:

**DIR 1.** We have $\varphi_\varepsilon \geqq 0$ for all $\varepsilon$.

**DIR 2.** For all $\varepsilon$, we have

$$\int_{\mathbf{R}} \varphi_\varepsilon(x)\,dx = 1.$$

**DIR 3.** Given $\delta > 0$ and $\delta' > 0$, we have

$$\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \varphi_\varepsilon < \delta'$$

for all $\varepsilon$ sufficiently close to 0.

**Theorem 2.1.** *Let $\{\varphi_\varepsilon\}$ be a Dirac family satisfying* **DIR 1, DIR 2, DIR 3**. *Let $h$ be bounded measurable on* **R**. *Then $\varphi_\varepsilon * h$ converges uniformly to $h$ as $\varepsilon \to 0$, on every compact set where $h$ is continuous.*

*Proof.* Same as in Chapter VIII, Theorem 3.1.

Suppose given an association $f \mapsto f(A)$ satisfying the five spectral properties. For each $v, w \in H$ there is a unique measure $\mu_{v,w}$ such that

$$\langle f(A)v, w\rangle = \int f\,d\mu_{v,w}.$$

Let $z$ be complex and not real. The function $f(t)$ such that

$$f(t) = \frac{1}{t - z}$$

is bounded measurable, and $tf(t)$ is bounded. Also $(t - z)f(t) = 1$. Hence

$$(A - zI)f(A) = I.$$

This means that the resolvent has the integral expression

$$\boxed{\langle (A - zI)^{-1}v, w\rangle = \int_{\mathbf{R}} \frac{1}{t - z}\,d\mu_{v,w}(t).}$$

We write $\mu_v$ instead of $\mu_{v,v}$. Note that $\mu_v$ is a positive measure.

**Theorem 2.2.** *Let $A$ be a self-adjoint operator on $H$ and let $v \in H$. Let $R(z) = (A - zI)^{-1}$ for $z$ not real. For any $\psi \in C_c(\mathbf{R})$ we have*

$$\int_{\mathbf{R}} \psi(\lambda)\,d\mu_v(\lambda) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\mathbf{R}} \langle [R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)]v, v\rangle \psi(\lambda)\,d\lambda.$$

*If $\lambda_1 < \lambda_2$ are real numbers which have $\mu_v$-measure 0, then*

$$\int_{\lambda_1}^{\lambda_2} d\mu_v(\lambda) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} \langle [R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)]v, v\rangle \, d\lambda.$$

The proof is based on the following lemma.

**Lemma 2.3.** *Let $\mu$ be a positive regular measure on $\mathbf{R}$ such that $\mu(\mathbf{R})$ is finite. Then for $\psi \in C_c(\mathbf{R})$ we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varepsilon}{(t-\lambda)^2 + \varepsilon^2} \psi(\lambda) \, d\mu(t) \, d\lambda = \int_{-\infty}^{\infty} \psi(\lambda) \, d\lambda.$$

*Furthermore, if $\lambda_1 < \lambda_2$ are real and such that the set $\{\lambda_1, \lambda_2\}$ has $\mu$-measure 0, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \int_{-\infty}^{\infty} \frac{\varepsilon}{(t-\lambda)^2 + \varepsilon^2} \, d\mu(t) \, d\lambda = \int_{\lambda_1}^{\lambda_2} d\mu(\lambda).$$

*Proof.* First observe that the family of functions

$$\varphi_\varepsilon(\lambda) = \frac{1}{\pi} \frac{\varepsilon}{\lambda^2 + \varepsilon^2}$$

is a Dirac family on $\mathbf{R}$ for $\varepsilon \to 0$. The left-hand side integrals in our lemma can be written

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_\varepsilon(t - \lambda) h(\lambda) \, d\mu(t) \, d\lambda$$

where $h$ is either $\psi$ or the characteristic function of the interval $[\lambda_1, \lambda_2]$. We apply Fubini's theorem to see that this expression is equal to

$$\int_{-\infty}^{\infty} \varphi_\varepsilon * h(t) \, d\mu(t).$$

Note that $\varphi_\varepsilon * h$ is bounded, and converges pointwise to $h$ if $h = \psi$, and pointwise to $h$ except at the end points $\lambda_1$, $\lambda_2$ in the other case. Since we picked our interval so that the end points have $\mu$-measure 0, we can apply the dominated convergence theorem to conclude the proof.

The lemma obviously proves Theorem 2.2, because

$$\frac{1}{t - \lambda - i\varepsilon} - \frac{1}{t - \lambda + i\varepsilon} = \frac{2i\varepsilon}{(t-\lambda)^2 + \varepsilon^2}.$$

Furthermore, Theorem 2.2 provides the desired uniqueness left hanging in the last section, because it gives the value of the measure entirely in terms of the resolvent and Lebesgue measure, as on the right-hand side of the first formula on elements of $C_c(\mathbf{R})$.

It is possible to develop the spectral theory by starting with a direct proof of Theorem 2.2, showing that the limit on the right-hand side exists. One then defines the spectral measure as that associated with the corresponding functional, and one proves the other properties from there. Cf. Akhiezer–Glazman, *Theory of Linear Operators in Hilbert Space*, Translated from the Russian, New York, Frederick Ungar, 1963, pp. 8 and 31.

## XX, §3. UNBOUNDED FUNCTIONS OF OPERATORS

In the first two sections, we studied bounded functions of an operator, and this operator could be bounded or unbounded. But the values $f(A)$ were bounded. We shall now extend this definition to arbitrary Borel measurable functions $f$, and in this way recover $A$ itself as an integral. If $A$ is unbounded, then we shall see that $A = f(A)$ where $f(t) = t$; and of course, $t$ is not a bounded function of $t$.

**Theorem 3.1.** *Let $A$ be a self-adjoint operator. Let $f$ be a real valued Borel measurable function on* $\mathbf{R}$. *Then there exists a unique self-adjoint operator $f(A)$ such that:*

(i) *The domain of $f(A)$ consists of those $v \in H$ for which*

$$f \in \mathscr{L}^2(\mu_v).$$

(ii) *For all $v$ in the domain of $f(A)$, we have*

$$\langle f(A)v, v\rangle = \int f\, d\mu_v.$$

(iii)
$$|f(A)v|^2 = \int f^2\, d\mu_v.$$

(iv) *If $f$ is bounded, then $f(A)$ has the previous meaning, and if $f(t) = t$, then $f(A) = A$.*

*Proof.* Observe first that the integral in (ii) exists by the Schwarz inequality. To prove the theorem, let

$$f_n(t) = \begin{cases} f(t) & \text{if } n < f(t) \leqq n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_n = f^{-1}((n, n+1])$. Thus $f_n$ is equal to $f$ on $Y_n$ and 0 outside $Y_n$. Let $\chi_n$ be the characteristic function of $Y_n$ and let $E_n = \chi_n(A)$. Then $E_n$ is a projection operator. Let

$$H_n = E_n H.$$

Then the $H_n$ are clearly mutually orthogonal, and we contend that we have a direct sum decomposition

$$H = \hat{\bigoplus} H_n$$

as in Theorem 2.6 of Chapter XIX.

To see this, note that

$$\sum_{n=-N}^{N} \chi_n \to 1$$

as $N \to \infty$, and hence

$$\sum_{-N}^{N} E_n \to I$$

strongly.

Let $B_n = f_n(A)$, so that $B_n$ is bounded, and operates on $H_n$ through the projection on $H_n$ because $f_n \chi_n = \chi_n f_n = f_n$, whence

$$f_n(A)\chi_n(A) = \chi_n(A) f_n(A) = f_n(A).$$

Let $f(A) = B$ be the self-adjoint operator whose domain is the usual one, consisting of $v = \sum v_n$ with $v_n \in H_n$ and $\sum |B_n v_n|^2 < \infty$. Then

$$\begin{aligned}\sum_n \langle f_n(A)v_n, f_n(A)v_n \rangle &= \sum_n \int f_n^2 \, d\mu_v \\ &= \int \sum_n f_n^2 \, d\mu_v\end{aligned}$$

by the monotone convergence theorem. It follows that $f \in \mathscr{L}^2(\mu_v)$. The converse is similarly clear. This proves (iii). Also we get

$$\begin{aligned}\langle f(A)v, v \rangle &= \sum \langle f_n(A)v_n, v_n \rangle \\ &= \sum \int f_n \, d\mu_v \\ &= \int f \, d\mu_v.\end{aligned}$$

This proves everything except the final assertion that if $f(t) = t$, then $f(A) = A$. But this follows from the fact that $f_n(A)$ is equal to $A$ re-

stricted to $H_n = E_n H$, since $f_n(A)$ and $\chi_n(A)$ have the usual meaning, as in §1. This concludes the proof of the theorem.

## XX, §4. SPECTRAL FAMILIES OF PROJECTIONS

By a **spectral family** in a Hilbert space $H$ we mean a family of orthogonal projections $\{P_t\}$, $t \in \mathbf{R}$, satisfying the following conditions:

**SF 1.** If $a \leqq b$, then $P_a \leqq P_b$.

**SF 2.** $\lim_{t \to -\infty} P_t = 0$ and $\lim_{t \to \infty} P_t = I$ **strongly**.

The first condition means that if $H_a$ and $H_b$ are the subspaces on which $P_a$ and $P_b$ project $H$, then $H_a \subset H_b$ and $P_a$ projects $H_b$ on $H_a$. The second means that for each vector $v \in H$ we have

$$\lim_{t \to -\infty} P_t v = 0 \qquad \text{and} \qquad \lim_{t \to \infty} P_t v = v.$$

In Chapter XIX we defined such a family for a bounded hermitian operator. In this case, we note that $P_a = 0$ for $a$ large negative, and $P_b = 1$ for $b$ large positive. A spectral family satisfying this additional condition is called **limited**.

Observe also that the spectral family associated with a bounded operator is continuous on the right by Theorem 1.5 of Chapter XIX. However, we do not assume right-continuity in our general definition of a spectral family. The spectral family associated with a bounded operator $A$ was defined as follows. For each $b \in \mathbf{R}$ we let $\psi_b$ be the function as shown in Figure 20.1.

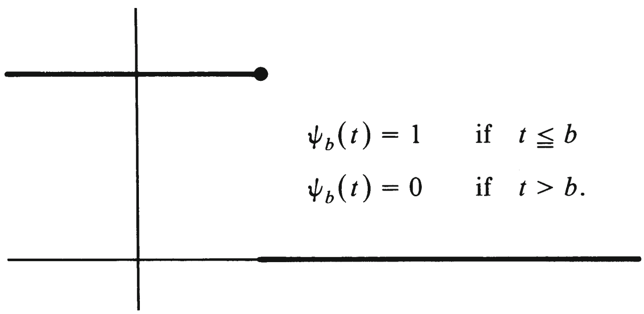



Figure 20.1

Then $P_t = \psi_t(A)$ defines the spectral family associated with $A$. But we have seen in the first section of this chapter how to make sense of $f(A)$ when $f$ is bounded measurable and $A$ is a self-adjoint operator, not necessarily bounded. This allows us to get:

**Theorem 4.1.** *Let $A$ be a self-adjoint operator on $H$. Let $P_t = \psi_t(A)$. Then $\{P_t\}$ is a spectral family, strongly continuous on the right.*

*Proof.* As before, we shall obtain an expression for $P_t$ in terms of a direct sum decomposition as in Theorem 2.6 and 2.7 of Chapter XIX. Suppose that

$$H = \widehat{\bigoplus} H_n \quad \text{and} \quad A = \widehat{\bigoplus} A_n,$$

where each $A_n = AH_n$ is bounded hermitian. Let $\{P_t^{(n)}\}$ be the spectral family of $A_n$ on $H_n$, so $P_t^{(n)} = \psi_t(A_n)$. Then by §1, we get

$$P_t = \sum_n P_t^{(n)}.$$

The first condition **SF 1** is obviously satisfied. For the second, fix $v = \sum v_n$. Then

$$\sum P_t^{(n)} v = \sum P_t^{(n)} v_n.$$

Select $N$ so large that

$$\sum_{n>N} |v_n|^2 < \varepsilon.$$

Then let $t \to -\infty$ to get the first limit. For $t \to \infty$ consider $v - P_t v$. Finally, we want to prove continuity from the right, i.e. for $v \in H$ we want to show

$$\lim_{\delta \to 0} (P_{t+\delta} - P_t)v = 0.$$

We look at

$$\left| \sum_n (P_{t+\delta}^{(n)} - P_t^{(n)}) v_n \right|^2 .$$

Again take $N$ so large that $\sum |v_n|^2 < \varepsilon$. We can then find $\delta$ so small that

$$\sum_{n=1}^{N} |[P_{t+\delta}^{(n)} - P_t^{(n)}] v_n|^2 < \varepsilon,$$

thus getting our continuity and proving the theorem.

## XX, §5. THE SPECTRAL INTEGRAL AS STIELTJES INTEGRAL

In Chapter IX, §7 we defined the Stieltjes integral with respect to an increasing real valued function. Such a function arises naturally from a spectral family, as follows. If $h$ is an increasing function, we again let $dh$ be the associated Stieltjes functional on $C_c(\mathbf{R})$.

Let $\{P_t\}$ be a spectral family, not necessarily associated with an opera-

tor. Let $v \in H$ and let $h = h_{P,v}$ be the function

$$h(t) = \langle P_t v, v \rangle.$$

Then $h$ is positive, increasing, bounded by 1.

**Theorem 5.1.** *Let $A$ be a self-adjoint operator, and let $\{P_t\}$ be the associated spectral family. Let $h(t) = \langle P_t v, v \rangle$ as above. Then for any function $\varphi$ in $C_c(\mathbf{R})$, we have*

$$\int \varphi \, dh = \int \varphi \, d\mu_v,$$

*where $\mu_v$ is the spectral measure associated with $A$ and $v$.*

*Proof.* We know from §4 that

$$P_t = \psi_t(A).$$

For a partition $T$ of sufficiently small size, the integral

$$\int \varphi \, dh$$

is approximated by a sum

$$\begin{aligned}\sum \varphi(c_k)\langle (P_{t_{k+1}} - P_{t_k})v, v\rangle &= \sum \varphi(c_k)\langle (\psi_{t_{k+1}}(A) - \psi_{t_k}(A))v, v\rangle \\ &= \int \sum \varphi(c_k)(\psi_{t_{k+1}} - \psi_{t_k})\, d\mu_v.\end{aligned}$$

But

$$\sum \varphi(c_k)(\psi_{t_{k+1}} - \psi_{t_k})$$

is an ordinary Riemann sum for $\varphi$, uniformly close to $\varphi$ if the partition has sufficiently small size. By the dominated convergence theorem with respect to the measure $\mu_v$ this last expression is therefore uniformly close to

$$\int \varphi \, d\mu_v,$$

thus proving the theorem.

## XX, §6. EXERCISES

Instead of starting with a self-adjoint operator as in the text, one may start with a spectral family, develop the functional calculus, and get back (unbounded) operators as follows.

1. Let $\{P_t\}$ be a spectral family, and let $v \in H$. Let

$$h(t) = \langle P_t v, v \rangle.$$

Show that there is a positive functional $\lambda_v$, bounded by 1, such that

$$\begin{aligned}\lambda_v(\varphi) &= \lim \sum \varphi(c_k)\langle (P_{t_{k+1}} - P_{t_k})v, v\rangle \\ &= \lim_{T,c} S(T, c, \varphi)\end{aligned}$$

the limit being taken in the same sense as in the text, for the size of the partition tending to 0. Deduce the existence and uniqueness of a measure $\mu_v$ such that

$$\lambda_v(\varphi) = \int \varphi \, d\mu_v.$$

In a similar way, obtain the complex measure $\mu_{v,w}$ such that

$$\lambda_{v,w}(\varphi) = \lim_{T,c} \sum \varphi(c_k)\langle (P_{t_{k+1}} - P_{t_k})v, w\rangle.$$

Show that

$$\left| \int_{\mathbf{R}} f \, d\mu_{v,w} \right| \leqq \|f\|_\infty |v||w|.$$

2. Conclude that there exists a unique bounded operator $f(P)$ for each $f \in BM(\mathbf{R})$ such that

$$\int_{\mathbf{R}} f \, d\mu_{v,w} = \langle f(P)v, w\rangle.$$

3. Show that the map $f \mapsto f(P)$ is a linear map from $BM(\mathbf{R})$ into the space of operators, satisfying the five properties **SPEC 1** through **SPEC 5**, except that $A$ is replaced by $P$.

4. **Theorem.** *Let $\{P_t\}$ be a spectral family. Let $f$ be a real Borel measurable function on* **R**. *Then there exists a unique self-adjoint operator $f(P)$ such that:*
   (i) *The domain of $f(P)$ consists of those $v \in H$ such that*

$$f \in \mathscr{L}^2(\mu_v).$$

   (ii) $\langle f(P)v, v\rangle = \int f \, d\mu_v$ *for all $v \in$ Domain of $f(P)$.*
   (iii) $|f(P)v|^2 = \int f^2 \, d\mu_v$.

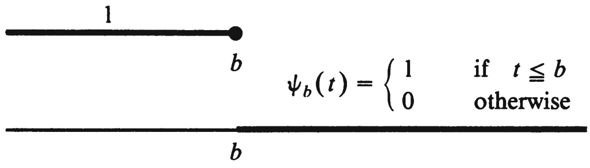
   [*Hint*: Follow step by step the proof given for the analogous theorem in the text, concerning $f(A)$ when we start with a self-adjoint operator $A$.]

Right continuity played no role in the above results. It is important only for uniqueness purposes, as shown in the next result.

5. **Theorem.** *Let $\{P_t\}$, $\{Q_t\}$ be spectral families, which are strongly continuous from the right, and such that they induce the same functional on $C_c(\mathbf{R})$. Then $P_t = Q_t$*

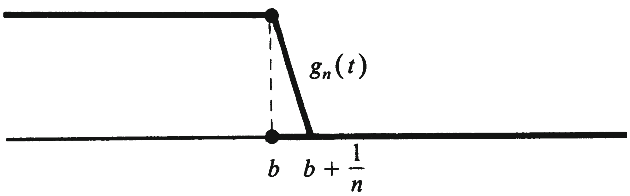
*for all $t$. If $b$ is a real number and $\psi_b$ is the function whose graph is drawn below, then*

$$\psi_b(P) = P_b.$$




$$\psi_b(t) = \begin{cases} 1 & \text{if } t \leqq b \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* From the assumption it follows that if $\varphi \in C_c(\mathbf{R})$, then $\varphi(P_t) = \varphi(Q_t)$ for all $t$. Let $\{g_n\}$ be a sequence decreasing to $\psi_b$ as shown.




Then for a fine partition,

$$\begin{aligned} \langle P_b v, v\rangle &= \sum_{t_{k+1} \leqq b} \langle (P_{t_{k+1}} - P_{t_k})v, v\rangle \\ &\leqq \int g_n \, d\mu_v \leqq \langle P_{b+1/n} v, v\rangle. \end{aligned}$$

Since

$$\int g_n \, d\mu_v \to \int \psi_b \, d\mu_v = \langle \psi_b(P)v, v\rangle$$

as $n \to \infty$, we get $P_b = \psi_b(P)$, thereby proving the theorem.

# PART SIX

# Global Analysis

One of the most attractive things that can be done with analysis is to mix it up with the global topology of geometric structures. For instance, whereas the local existence theorem for differential equations yields integral curves in an open set of say euclidean space, one may wish to see what happens if a differential equation is given on the sphere. In this case, the integral curves wind around the sphere and one investigates their behavior as time goes to infinity. Similarly, one can work on toruses, or arbitrarily complicated similar structures, which have one thing in common: locally, they look like euclidean space, but globally they turn and twist. The relations between the analytic properties, and the algebraic-topological invariants associated with the topological structure, constitute one of the central parts of mathematics. Our task here is but to lay down the most basic definitions to prepare readers for further readings, and to give them the flavor of global results, as distinguished from local ones in open sets of euclidean space.

We should add, however, that even on open sets of euclidean space, i.e. locally, we may be interested in certain objects and properties which are invariant under $C^p$ changes of coordinate systems, i.e. under $C^p$ isomorphisms. The language of manifolds provides the natural language for such properties. Thus we begin with the change of variables formula, which gives an example how the integral changes under $C^1$ isomorphisms. The change is of such a nature that we can associate with it an integral on manifolds. This is done in the last chapter, which includes the basic theorem of Stokes.

We don't do too much with differential equations besides defining the basic notions on manifolds. Readers can refer to [La 2] for further foundations. Smale's survey [Sm 2] is an excellent starting point for the

global analysis of ordinary differential equations. As for partial differential equations, Nirenberg's exposition of certain basic results in [Pr] gives an exceptionally attractive introduction for this part of global analysis. In fact, the whole proceedings [Pr] are highly recommended.

CHAPTER XXI

# Local Integration of Differential Forms

*Throughout this chapter, $\mu$ is Lebesgue measure on $\mathbf{R}^n$.*
*If $A$ is a subset of $\mathbf{R}^n$, we write $\mathscr{L}^1(A)$ instead of $\mathscr{L}^1(A, \mu, \mathbf{C})$.*

## XXI, §1. SETS OF MEASURE 0

We recall that a set has **measure** 0 in $\mathbf{R}^n$ if and only if, given $\varepsilon$, there exists a covering of the set by a sequence of rectangles $\{R_j\}$ such that $\sum \mu(R_j) < \varepsilon$. We denote by $R_j$ the closed rectangles, and we may always assume that the interiors $R_j^0 = \mathrm{Int}(R_j)$ cover the set, at the cost of increasing the lengths of the sides of our rectangles very slightly (an $\varepsilon/2^n$ argument). We shall prove here some criteria for a set to have measure 0. We leave it to the reader to verify that instead of rectangles, we could have used cubes in our characterization of a set of a measure 0 (a cube being a rectangle all of whose sides have the same length).

We recall that a map $f$ satisfies a **Lipschitz condition** on a set $A$ if there exists a number $C$ such that

$$|f(x) - f(y)| \leqq C|x - y|$$

for all $x, y \in A$. Any $C^1$ map $f$ satisfies locally at each point a Lipschitz condition, because its derivative is bounded in a neighborhood of each point, and we can then use the mean value estimate,

$$|f(x) - f(y)| \leqq |x - y| \sup|f'(z)|,$$

the sup being taken for $z$ on the segment between $x$ and $y$. We can take

the neighborhood of the point to be a ball, say, so that the segment between any two points is contained in the neighborhood.

**Lemma 1.1.** *Let $A$ have measure* 0 *in* $\mathbf{R}^n$ *and let* $f: A \to \mathbf{R}^n$ *satisfy a Lipschitz condition. Then* $f(A)$ *has measure* 0.

*Proof.* Let $C$ be a Lipschitz constant for $f$. Let $\{R_j\}$ be a sequence of cubes covering $A$ such that $\sum \mu(R_j) < \varepsilon$. Let $r_j$ be the length of the side of $R_j$. Then for each $j$ we see that $f(A \cap R_j)$ is contained in a cube $R'_j$ whose sides have length $\leqq 2Cr_j$. Hence

$$\mu(R'_j) \leqq 2^n C^n r_j^n = 2^n C^n \mu(R_j).$$

Our lemma follows.

**Lemma 1.2.** *Let $U$ be open in* $\mathbf{R}^n$ *and let* $f: U \to \mathbf{R}^n$ *be a* $C^1$ *map. Let $Z$ be a set of measure* 0 in $U$. *Then* $f(Z)$ *has measure* 0.

*Proof.* For each $x \in U$ there exists a rectangle $R_x$ contained in $U$ such that the family $\{R_x^0\}$ of interiors covers $Z$. Since $U$ is separable, there exists a denumerable subfamily covering $Z$, say $\{R_j\}$. It suffices to prove that $f(Z \cap R_j)$ has measure 0 for each $j$. But $f$ satisfies a Lipschitz condition on $R_j$ since $R_j$ is compact and $f'$ is bounded on $R_j$, being continuous. Our lemma follows from Lemma 1.1.

**Lemma 1.3.** *Let $A$ be a subset of* $\mathbf{R}^m$. *Assume that $m < n$. Let* $f: A \to \mathbf{R}^n$ *satisfy a Lipschitz condition. Then* $f(A)$ *has measure* 0.

*Proof.* We view $\mathbf{R}^m$ as embedded in $\mathbf{R}^n$ on the space of the first $m$ coordinates. Then $\mathbf{R}^m$ has measure 0 in $\mathbf{R}^n$, so that $A$ has also $n$-dimensional measure 0. Lemma 1.3 is therefore a consequence of Lemma 1.1.

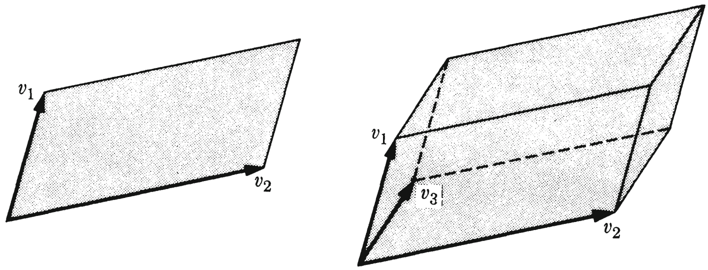
**Note.** All three lemmas may be viewed as stating that certain parametrized sets have measure 0. Lemma 1.3 shows that parametrizing a set by strictly lower dimensional spaces always yields an image having measure 0. The other two lemmas deal with a map from one space into another of the same dimension. Observe that Lemma 1.3 would be false if $f$ is only assumed to be continuous (Peano curves).

## XXI, §2. CHANGE OF VARIABLES FORMULA

We first deal with the simplest of cases. We consider vectors $v_1, \ldots, v_n$ in $\mathbf{R}^n$ and we define the **block** $B$ spanned by these vectors to be the set of points

$$t_1 v_1 + \cdots + t_n v_n$$

with $0 \leqq t_i \leqq 1$. We say that the block is **degenerate** (in $\mathbf{R}^n$) if the vectors $v_1, \ldots, v_n$ are linearly dependent. Otherwise, we say that the block is **non-degenerate**, or is a **proper block** in $\mathbf{R}^n$.




We see that a block in $\mathbf{R}^2$ is nothing but a parallelogram, and a block in $\mathbf{R}^3$ is nothing but a parallelepiped (when not degenerate).

We shall sometimes use the word volume instead of measure when applied to blocks or their images under maps, for the sake of geometry.

We denote by $\mathrm{Vol}(v_1, \ldots, v_n)$ the volume of the block $B$ spanned by $v_1, \ldots, v_n$. We define the **oriented volume**

$$\mathrm{Vol}^0(v_1, \ldots, v_n) = \pm \mathrm{Vol}(v_1, \ldots, v_n),$$

taking the $+$ if $\mathrm{Det}(v_1, \ldots, v_n) > 0$ and the $-$ if $\mathrm{Det}(v_1, \ldots, v_n) < 0$. The determinant is viewed as the determinant of the matrix whose column vectors are $v_1, \ldots, v_n$, in that order.

We recall the following characterization of determinants. Suppose that we have a product

$$(v_1, \ldots, v_n) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_n$$

which to each $n$-tuple of **vectors** associates a number such that the product is multilinear, alternating, and such that

$$e_1 \wedge \cdots \wedge e_n = 1$$

if $e_1, \ldots, e_n$ are the unit vectors. Then this product is necessarily the determinant, i.e. it is uniquely determined. "Alternating" means that if $v_i = v_j$ for some $i \neq j$ then

$$v_1 \wedge \cdots \wedge v_n = 0.$$

The uniqueness is easily proved, and we recall this short proof. We can write

$$v_i = a_{i1} e_1 + \cdots + a_{in} e_n$$

for suitable numbers $a_{ij}$, and then

$$\begin{aligned} v_1 \wedge \cdots \wedge v_n &= (a_{11}e_1 + \cdots + a_{1n}e_n) \wedge \cdots \wedge (a_{n1}e_1 + \cdots + a_{nn}e_n) \\ &= \sum_\sigma a_{1,\sigma(1)}e_{\sigma(1)} \wedge \cdots \wedge a_{n,\sigma(n)}e_{\sigma(n)} \\ &= \sum_\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)}. \end{aligned}$$

The sum is taken over all maps $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$, but because of the alternating property, whenever $\sigma$ is not a permutation the term corresponding to $\sigma$ is equal to 0. Hence the sum may be taken only over all permutations. Since

$$e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} = \varepsilon(\sigma) e_1 \wedge \cdots \wedge e_n$$

where $\varepsilon(\sigma) = 1$ or $-1$ is a sign depending only on $\sigma$, it follows that the alternating product is completely determined by its value $e_1 \wedge \cdots \wedge e_n$, and in particular is the determinant if this value is equal to 1.

**Theorem 2.1.** *We have*

$$\mathrm{Vol}^0(v_1,\dots,v_n) = \mathrm{Det}(v_1,\dots,v_n)$$

*and*

$$\mathrm{Vol}(v_1,\dots,v_n) = |\mathrm{Det}(v_1,\dots,v_n)|.$$

*Proof.* If $v_1, \dots, v_n$ are linearly dependent, then the determinant is equal to 0, and the volume is also equal to 0, for instance by Lemma 1.3. So our formula holds in this case. It is clear that

$$\mathrm{Vol}^0(e_1,\dots,e_n) = 1.$$

To show that $\mathrm{Vol}^0$ satisfies the characteristic properties of the determinant, all we have to do now is to show that it is linear in each variable, say the first. In other words, we must prove

$$(*) \qquad \mathrm{Vol}^0(cv, v_2,\dots,v_n) = c\,\mathrm{Vol}^0(v, v_2,\dots,v_n) \qquad \text{for } c \in \mathbf{R},$$

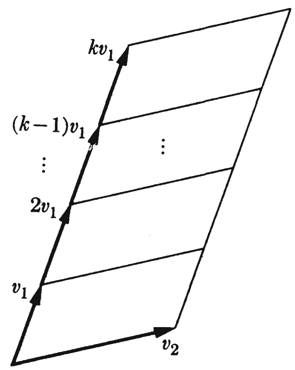
$$(**) \qquad \mathrm{Vol}^0(v + w, v_2,\dots,v_n) = \mathrm{Vol}^0(v, v_2,\dots,v_n) + \mathrm{Vol}^0(w, v_2,\dots,v_n).$$

As to the first assertion, suppose first that $c$ is some positive integer $k$. Let $B$ be the block spanned by $v, v_2, \dots, v_n$. We may assume without loss of generality that $v, v_2, \dots, v_n$ are linearly independent (otherwise, the relation is obviously true, both sides being equal to 0). We verify at once from the definition that if $B(v, v_2,\dots,v_n)$ denotes the block spanned by

$v, v_2, \ldots, v_n$ then $B(kv, v_2, \ldots, v_n)$ is the union of the two sets

$$B\big((k-1)v, v_2, \ldots, v_n\big) \qquad \text{and} \qquad B(v, v_2, \ldots, v_n) + (k-1)v,$$

which have only a set of measure 0 in common, as one verifies at once from the definitions.




Therefore, we find that

$$\begin{aligned}\mathrm{Vol}(kv, v_2, \ldots, v_n) &= \mathrm{Vol}\big((k-1)v, v_2, \ldots, v_n\big) + \mathrm{Vol}(v, v_2, \ldots, v_n)\\ &= (k-1)\mathrm{Vol}(v, v_2, \ldots, v_n) + \mathrm{Vol}(v, v_2, \ldots, v_n)\\ &= k\,\mathrm{Vol}(v, v_2, \ldots, v_n),\end{aligned}$$

as was to be shown.

Now let

$$v = v_1/k$$

for a positive integer $k$. Then applying what we have just proved shows that

$$\mathrm{Vol}\left(\frac{1}{k}v_1, v_2, \ldots, v_n\right) = \frac{1}{k}\mathrm{Vol}(v_1, \ldots, v_n).$$

Writing a positive rational number in the form $m/k = m \cdot 1/k$, we conclude that the first relation holds when $c$ is a positive rational number. If $r$ is a positive real number, we find positive rational numbers $c, c'$ such that $c \leqq r \leqq c'$. Since

$$B(cv, v_2, \ldots, v_n) \subset B(rv, v_2, \ldots, v_n) \subset B(c'v, v_2, \ldots, v_n),$$

we conclude that

$$c\,\mathrm{Vol}(v, v_2, \ldots, v_n) \leqq \mathrm{Vol}(rv, v_2, \ldots, v_n) \leqq c'\,\mathrm{Vol}(v, v_2, \ldots, v_n).$$

Letting $c$, $c'$ approach $r$ as a limit, we conclude that for any real number $r \geqq 0$ we have

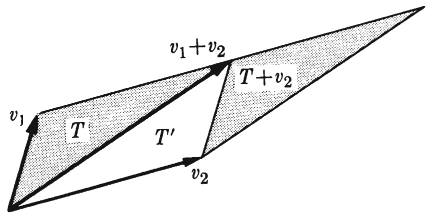
$$\mathrm{Vol}(rv, v_2, \ldots, v_n) = r\,\mathrm{Vol}(v, v_2, \ldots, v_n).$$

Finally, we note that $B(-v, v_2, \ldots, v_n)$ is the translation of

$$B(v, v_2, \ldots, v_n)$$

by $-v$ so that these two blocks have the same volume. This proves the first assertion.

As for the second, we look at the geometry of the situation, which is made clear by the following picture in case $v = v_1$, $w = v_2$.




The block spanned by $v_1$, $v_2$, ... consists of two "triangles" $T$, $T'$ having only a set of measure zero in common. The block spanned by $v_1 + v_2$ and $v_2$ consists of $T'$ and the translation $T + v_2$. It follows that these two blocks have the same volume. We conclude that for any number $c$,

$$\mathrm{Vol}^0(v_1 + cv_2, v_2, \ldots, v_n) = \mathrm{Vol}^0(v_1, v_2, \ldots, v_n).$$

Indeed, if $c = 0$ this is obvious, and if $c \neq 0$ then

$$\begin{aligned} c\,\mathrm{Vol}^0(v_1 + cv_2, v_2, \ldots, v_n) &= \mathrm{Vol}^0(v_1 + cv_2, cv_2, \ldots, v_n) \\ &= \mathrm{Vol}^0(v_1, cv_2, \ldots, v_n) = c\,\mathrm{Vol}^0(v_1, v_2, \ldots, v_n). \end{aligned}$$

We can then cancel $c$ to get our conclusion.

To prove the linearity of $\mathrm{Vol}^0$ with respect to its first variable, we may assume that $v_2, \ldots, v_n$ are linearly independent, otherwise both sides of (**) are equal to 0. Let $v_1$ be so chosen that $\{v_1, \ldots, v_n\}$ is a basis of $\mathbf{R}^n$. Then by induction, and what has been proved above,

$$\begin{aligned} \mathrm{Vol}^0(c_1 v_1 + \cdots + c_n v_n, v_2, \ldots, v_n) &= \mathrm{Vol}^0(c_1 v_1 + \cdots + c_{n-1} v_{n-1}, v_2, \ldots, v_n) \\ &= \mathrm{Vol}^0(c_1 v_1, v_2, \ldots, v_n) \\ &= c_1\,\mathrm{Vol}^0(v_1, \ldots, v_n). \end{aligned}$$

From this the linearity follows at once, and the theorem is proved.

**Corollary 2.2.** *Let $S$ be the unit cube spanned by the unit vectors in $\mathbf{R}^n$. Let $\lambda: \mathbf{R}^n \to \mathbf{R}^n$ be a linear map. Then*

$$\operatorname{Vol} \lambda(S) = |\operatorname{Det}(\lambda)|.$$

*Proof.* If $v_1, \ldots, v_n$ are the images of $e_1, \ldots, e_n$ under $\lambda$, then $\lambda(S)$ is the block spanned by $v_1, \ldots, v_n$. If we represent $\lambda$ by the matrix $A = (a_{ij})$, then

$$v_i = a_{1i}e_1 + \cdots + a_{ni}e_n$$

and hence $\operatorname{Det}(v_1, \ldots, v_n) = \operatorname{Det}(A) = \operatorname{Det}(\lambda)$. This proves the corollary.

**Corollary 2.3.** *If $R$ is any rectangle in $\mathbf{R}^n$ and $\lambda: \mathbf{R}^n \to \mathbf{R}^n$ is a linear map, then*

$$\operatorname{Vol} \lambda(R) = |\operatorname{Det}(\lambda)| \operatorname{Vol}(R).$$

*Proof.* After a translation, we can assume that the rectangle is a block. If $R = \lambda_1(S)$ where $S$ is the unit cube, then

$$\lambda(R) = \lambda \circ \lambda_1(S)$$

whence by Corollary 2.2,

$$\operatorname{Vol} \lambda(R) = |\operatorname{Det}(\lambda \circ \lambda_1)| = |\operatorname{Det}(\lambda) \operatorname{Det}(\lambda_1)| = |\operatorname{Det}(\lambda)| \operatorname{Vol}(R).$$

The next theorem extends Corollary 2.3 to the more general case where the linear map $\lambda$ is replaced by an arbitrary $C^1$-invertible map. The proof then consists of replacing the $C^1$ map by its derivative and estimating the error thus introduced. For this purpose, we define the **Jacobian determinant**

$$\Delta_f(x) = \operatorname{Det} J_f(x) = \operatorname{Det} f'(x)$$

where $J_f(x)$ is the Jacobian matrix, and $f'(x)$ is the derivative of the map $f: U \to \mathbf{R}^n$.

**Theorem 2.4.** *Let $R$ be a rectangle in $\mathbf{R}^n$, contained in some open set $U$. Let $f: U \to \mathbf{R}^n$ be a $C^1$ map, which is $C^1$-invertible on $U$. Then*

$$\mu(f(R)) = \int_R |\Delta_f| \, d\mu.$$

*Proof.* When $f$ is linear, this is nothing but Corollary 2.3 of the preceding theorem. We shall prove the general case by approximating $f$ by its derivative. Let us first assume that $R$ is a cube for simplicity.

Given $\varepsilon$, let $P$ be a partition of $R$, obtained by dividing each side of $R$ into $N$ equal segments for large $N$. Then $R$ is partitioned into $N^n$ subcubes which we denote by $S_j$ $(j = 1, \dots, N^n)$. We let $a_j$ be the center of $S_j$.

We have

$$\operatorname{Vol} f(R) = \sum_j \operatorname{Vol} f(S_j)$$

because the images $f(S_j)$ have only sets of measure 0 in common. We investigate $f(S_j)$ for each $j$. The derivative $f'$ is uniformly continuous on $R$. Given $\varepsilon$, we assume that $N$ has been taken so large that for $x \in S_j$ we have

$$f(x) = f(a_j) + \lambda_j(x - a_j) + \varphi(x - a_j)$$

where $\lambda_j = f'(a_j)$ and

$$|\varphi(x - a_j)| \leqq |x - a_j|\varepsilon.$$

To determine $\operatorname{Vol} f(S_j)$ we must therefore investigate $f(S)$ where $S$ is a cube centered at the origin, and $f$ has the form

$$f(x) = \lambda x + \varphi(x), \qquad |\varphi(x)| \leqq |x|\varepsilon$$

on the cube $S$. (We have made suitable translations which don't affect volumes.) We have

$$\lambda^{-1} \circ f(x) = x + \lambda^{-1} \circ \varphi(x),$$

so that $\lambda^{-1} \circ f$ is nearly the identity map. For some constant $C$, we have for $x \in S$:

$$|\lambda^{-1} \circ \varphi(x)| \leqq C\varepsilon.$$

From the lemma after the proof of the inverse mapping theorem, we conclude that $\lambda^{-1} \circ f(S)$ contains a cube of radius

$$(1 - C\varepsilon)(\text{radius } S)$$

and trivial estimates show that $\lambda^{-1} \circ f(S)$ is contained in a cube of radius

$$(1 + C\varepsilon)(\text{radius } S).$$

We apply $\lambda$ to these cubes, and determine their volumes. Putting indices $j$ on everything, we find that

$$\begin{aligned} &|\operatorname{Det} f'(a_j)| \operatorname{Vol}(S_j) - \varepsilon C_1 \operatorname{Vol}(S_j) \\ &\qquad \leqq \operatorname{Vol} f(S_j) \leqq |\operatorname{Det} f'(a_j)| \operatorname{Vol}(S_j) + \varepsilon C_1 \operatorname{Vol}(S_j) \end{aligned}$$

with some fixed constant $C_1$. Summing over $j$ and estimating $|\Delta_f|$, we see that our theorem follows at once, in case $R$ is a cube.

**Remark.** We assumed for simplicity that $R$ was a cube. Actually, by changing the norm on each side, multiplying by a suitable constant, and taking the sup of the adjusted norms, we see that this involves no loss of generality. Alternatively, we can approximate a given rectangle by cubes.

**Corollary 2.5.** *If $g$ is a continuous function on $f(R)$, then*

$$\int_{f(R)} g \, d\mu = \int_R (g \circ f)|\Delta_f| \, d\mu.$$

*Proof.* The functions $g$ and $(g \circ f)|\Delta_f|$ are uniformly continuous on $f(R)$ and $R$ respectively. Let us take a partition of $R$ and let $\{S_j\}$ be the subrectangles of this partition. If $\delta$ is the maximum length of the sides of the subrectangles of the partition, then $f(S_j)$ is contained in a rectangle whose sides have length $\leqq C\delta$ for some constant $C$. We have

$$\int_{f(R)} g \, d\mu = \sum_j \int_{f(S_j)} g \, d\mu.$$

We may assume $g$ real. The sup and inf of $g$ on $f(S_j)$ differ only by $\varepsilon$ if $\delta$ is taken sufficiently small. Using the theorem, applied to each $S_j$, and replacing $g$ by its minimum $m_j$ and maximum $M_j$ on $S_j$ we see that the corollary follows at once.

**Theorem 2.6 (Change of Variables Formula).** *Let $U$ be open in $\mathbf{R}^n$ and let $f\colon U \to \mathbf{R}^n$ be a $C^1$ map, which is $C^1$ invertible on $U$. Let $g \in \mathscr{L}^1(f(U))$. Then $(g \circ f)|\Delta_f|$ is in $\mathscr{L}^1(U)$ and we have*

$$\int_{f(U)} g \, d\mu = \int_U (g \circ f)|\Delta_f| \, d\mu.$$

*Proof.* Let $R$ be a closed rectangle contained in $U$. We shall first prove that the restriction of $(g \circ f)|\Delta_f|$ to $R$ is in $\mathscr{L}^1(R)$, and that the formula holds when $U$ is replaced by $R$. We know that $C_c(f(U))$ is $L^1$-dense in $\mathscr{L}^1(f(U))$ by Theorem 3.1 of Chapter IX. Hence there exists a sequence $\{g_k\}$ in $C_c(f(U))$ which is $L^1$-convergent to $g$. Using Theorem 5.2 of Chapter VI, we may assume that $\{g_k\}$ converges pointwise to $g$ except on a set $Z$ of measure 0 in $f(U)$. By Lemma 1.2, we know that $f^{-1}(Z)$ has measure 0.

Let $g_k^* = (g_k \circ f)|\Delta_f|$. Each function $g_k^*$ is continuous on $R$. The sequence $\{g_k^*\}$ converges almost everywhere to $(g \circ f)|\Delta_f|$ restricted to $R$. It is in fact an $L^1$-Cauchy sequence in $\mathscr{L}^1(R)$. To see this, we have by

the result for rectangles and continuous functions (corollary of the preceding theorem):

$$\int_R |g_k^* - g_m^*|\, d\mu = \int_{f(R)} |g_k - g_m|\, d\mu,$$

so the Cauchy nature of the sequence $\{g_k^*\}$ is clear from that of $\{g_k\}$. It follows that the restriction of $(g \circ f)|\Delta_f|$ to $R$ is the $L^1$-limit of $\{g_k^*\}$, and is in $\mathscr{L}^1(R)$. It also follows that the formula of the theorem holds for $R$, that is

$$\int_{f(A)} g\, d\mu = \int_A (g \circ f)|\Delta_f|\, d\mu$$

when $A = R$.

The theorem is now seen to hold for any measurable subset $A$ of $R$, since $f(A)$ is measurable, and since a function $g$ in $\mathscr{L}^1(f(A))$ can be extended to a function in $\mathscr{L}^1(f(R))$ by giving it the value 0 outside $f(A)$. From this it follows that the theorem holds if $A$ is a finite union of rectangles contained in $U$. We can find a sequence of rectangles $\{R_m\}$ contained in $U$ whose union is equal to $U$, because $U$ is separable. Taking the usual stepwise complementation, we can find a disjoint sequence of measurable sets

$$A_m = R_m - (R_1 \cup \cdots \cup R_{m-1})$$

whose union is $U$, and such that our theorem holds if $A = A_m$. Let

$$h_m = g_{f(A_m)} = g\chi_{f(A_m)} \qquad \text{and} \qquad h_m^* = (h_m \circ f)|\Delta_f|.$$

Then $\sum h_m$ converges to $g$ and $\sum h_m^*$ converges to $(g \circ f)|\Delta_f|$. Our theorem follows from Corollary 5.13 of the dominated convergence theorem, Chapter VI.

**Note.** In dealing with polar coordinates or the like, one sometimes meets a map $f$ which is invertible except on a set of measure 0. It is now trivial to recover a result covering this type of situation.

**Corollary 2.7.** *Let $U$ be open in $\mathbf{R}^n$ and let $f: U \to \mathbf{R}^n$ be a $C^1$ map. Let $A$ be a measurable subset of $U$ such that the boundary of $A$ has measure 0, and such that $f$ is $C^1$ invertible on the interior of $A$. Let $g$ be in $\mathscr{L}^1(f(A))$. Then $(g \circ f)|\Delta_f|$ is in $\mathscr{L}^1(A)$ and*

$$\int_{f(A)} g\, d\mu = \int_A (g \circ f)|\Delta_f|\, d\mu.$$

*Proof.* Let $U_0$ be the interior of $A$. The sets $f(A)$ and $f(U_0)$ differ only by a set of measure 0, namely $f(\partial A)$. Also the sets $A$, $U_0$ differ only

by a set of measure 0. Consequently we can replace the domains of integration $f(A)$ and $A$ by $f(U_0)$ and $U_0$, respectively. The theorem applies to conclude the proof of the corollary.

**Note.** Since step maps are dense in $\mathscr{L}^1(X, E)$ for a Banach space $E$, the preceding proof generalizes at once to the case of Banach valued maps.

The change of variables formula depends on a $C^1$ isomorphism $f: U \to V$ between open sets of $n$-space. It suggests that one should define some object which changes by multiplication of the Jacobian (or its absolute value) under such an isomorphism, and this is what we shall do in the next section, by defining differential forms. After that, we introduce a language, that of manifolds, which allows us to speak invariantly about these objects.

## XXI, §3. DIFFERENTIAL FORMS

We recall first two simple results from linear (or rather multilinear) algebra. We use the notation $E^{(r)} = E \times E \times \cdots \times E$, $r$ times.

**Theorem A.** *Let $E$ be a finite dimensional vector space over the reals of dimension $n$. For each positive integer $r$ with $1 \leqq r \leqq n$ there exists a vector space $\bigwedge^r E$ and a multilinear alternating map*

$$E^{(r)} \to \bigwedge^r E$$

*denoted by $(u_1, \ldots, u_r) \mapsto u_1 \wedge \cdots \wedge u_r$, having the following property. If $\{v_1, \ldots, v_n\}$ is a basis of $E$, then the elements*

$$\{v_{i_1} \wedge \cdots \wedge v_{i}\}, \qquad i_1 < i_2 < \cdots < i_r,$$

*form a basis of $\bigwedge^r E$.*

We recall that **alternating** means that $u_1 \wedge \cdots \wedge u_r = 0$ if $u_i = u_j$ for some $i \neq j$. We call $\bigwedge^r E$ the $r$-th **alternating** product (or **exterior** product) of $E$. If $r = 0$, we define $\bigwedge^0 E = \mathbf{R}$. Elements of $\bigwedge^r E$ which can be written in the form $u_1 \wedge \cdots \wedge u_r$ are called **decomposable**. Such elements generate $\bigwedge^r E$. If $r > \dim E$, we define $\bigwedge^r E = \{0\}$.

**Theorem B.** *For each pair of positive integers $(r, s)$, there exists a unique product (bilinear map)*

$$\bigwedge^r E \times \bigwedge^s E \to \bigwedge^{r+s} E$$

*such that if* $u_1, \ldots, u_r, w_1, \ldots, w_s \in E$ *then*

$$(u_1 \wedge \cdots \wedge u_r) \times (w_1 \wedge \cdots \wedge w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

*This product is associative.*

The proofs for these two statements will be briefly summarized in the appendix to this chapter.

Let $E^*$ be the dual space, $E^* = L(E, \mathbf{R})$. (We prefer here to use $E^*$ rather than $E'$, first because we shall use the prime for the derivative, and second because we want a certain notational consistency as in §4.) If $E = \mathbf{R}^n$ and $\lambda_1, \ldots, \lambda_n$ are the coordinate functions, then each $\lambda_i$ is an element of the dual space, and in fact $\{\lambda_1, \ldots, \lambda_n\}$ is a basis of this dual space.

Let $U$ be an open set in $\mathbf{R}^n$. By a **differential form** of degree $r$ on $U$ (or an $r$-form) we mean a map

$$\omega: U \to \bigwedge^r E^*$$

from $U$ into the $r$-th alternating product of $E^*$. We say that the form is of class $C^p$ if the map is of class $C^p$. (We view $\bigwedge^r E^*$ as a normed vector space, using any norm. It does not matter which, since all norms on a finite dimensional vector space are equivalent.)

Since $\{\lambda_1, \ldots, \lambda_n\}$ is a basis of $E^*$, we can express each differential form in terms of its coordinate functions with respect to the basis

$$\{\lambda_{i_1} \wedge \cdots \wedge \lambda_{i_r}\} \qquad (i_1 < \cdots < i_r),$$

namely for each $x \in U$ we have

$$\omega(x) = \sum_{(i)} f_{i_1 \cdots i_r}(x) \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_r}$$

where $f_{(i)} = f_{i_1 \cdots i_r}$ is a function on $U$. Each such function has the same order of differentiability as $\omega$. We call the preceding expression the **standard form** of $\omega$. We say that a form is **decomposable** if it can be written as just one term $f(x)\lambda_{i_1} \wedge \cdots \wedge \lambda_{i_r}$. Every differential form is a sum of decomposable ones.

We agree to the convention that functions are differential forms of degree 0.

It is clear that the differential forms of given degree $r$ form a vector space, denoted by $\Omega^r(U)$.

Let $E = \mathbf{R}^n$. Let $f$ be a function on $U$. For each $x \in U$ the derivative

$$f'(x): \mathbf{R}^n \to \mathbf{R}$$

is a linear map, and thus an element of the dual space. Thus

$$f' : U \to E^*$$

is a differential form of degree 1, which is usually denoted by $df$. If $f$ is of class $C^p$, then $df$ is class $C^{p-1}$.

Let $\lambda_i$ be the $i$-th coordinate function. Then we know that

$$d\lambda_i(x) = \lambda_i'(x) = \lambda_i$$

for each $x \in U$ because $\lambda'(x) = \lambda$ for any continuous linear map $\lambda$. Whenever $\{x_1, \ldots, x_n\}$ are used systematically for the coordinates of a point in $\mathbf{R}^n$, it is customary in the literature to use the notation

$$d\lambda_i(x) = dx_i.$$

This is slightly incorrect, but is useful in formal computations. We shall also use it in this book on occasions. Similarly, we also write (incorrectly)

$$\omega = \sum_{(i)} f_{(i)}\, dx_{i_1} \wedge \cdots \wedge dx_{i_r}$$

instead of the correct

$$\omega(x) = \sum_{(i)} f_{(i)}(x) \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_r}.$$

In terms of coordinates, the map $df$ (or $f'$) is given by

$$df(x) = f'(x) = D_1 f(x)\lambda_1 + \cdots + D_n f(x)\lambda_n$$

where $D_i f(x) = \partial f/\partial x_i$ is the $i$-th partial derivative. This is simply a restatement of the fact that if $h = (h_1, \ldots, h_n)$ is a vector, then

$$f'(x)h = \frac{\partial f}{\partial x_1} h_1 + \cdots + \frac{\partial f}{\partial x_n} h_n.$$

Thus in old notation, we have

$$df(x) = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Let $\omega$ and $\psi$ be forms of degrees $r$ and $s$ respectively, on the open set $U$. For each $x \in U$ we can then take the alternating product $\omega(x) \wedge \psi(x)$ and we define the **alternating product** $\omega \wedge \psi$ by

$$(\omega \wedge \psi)(x) = \omega(x) \wedge \psi(x).$$

If $f$ is a differential form of degree 0, that is a function, then we define

$$f \wedge \omega = f\omega$$

where $(f\omega)(x) = f(x)\omega(x)$. By definition, we then have

$$\omega \wedge f\psi = f\omega \wedge \psi.$$

We shall now define the **exterior derivative** $d\omega$ for any differential form $\omega$. We have already done it for functions. We shall do it in general first in terms of coordinates, and then show that there is a characterization independent of these coordinates. If

$$\omega = \sum_{(i)} f_{(i)}\, d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r}$$

we define

$$d\omega = \sum_{(i)} df_{(i)} \wedge d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r}.$$

**Example.** Suppose $n = 2$ and $\omega$ is a 1-form, given in terms of the two coordinates $(x, y)$ by

$$\omega(x, y) = f(x, y)\, dx + g(x, y)\, dy.$$

Then

$$\begin{aligned} d\omega(x, y) &= df(x, y) \wedge dx + dg(x, y) \wedge dy \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy\right) \wedge dy \\ &= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right) dy \wedge dx \end{aligned}$$

because the terms involving $dx \wedge dx$ and $dy \wedge dy$ are equal to 0.

**Theorem 3.1.** *The map $d$ is linear, and satisfies*

$$d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^r \omega \wedge d\psi$$

*if $r = \deg \omega$. The map $d$ is uniquely determined by these properties, and by the fact that for a function $f$, we have $df = f'$.*

*Proof.* The linearity of $d$ is obvious. Hence it suffices to prove the formula for decomposable forms. We note that for any function $f$ we

have

$$d(f\omega) = df \wedge \omega + f\, d\omega.$$

Indeed, if $\omega$ is a function $g$, then from the derivative of a product we get $d(fg) = f\,dg + g\,df$. If

$$\omega = g\, d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r}$$

where $g$ is a function, then

$$\begin{aligned} d(f\omega) &= d(fg\, d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r}) = d(fg) \wedge d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r} \\ &= (f\,dg + g\,df) \wedge d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r} \\ &= f\,d\omega + df \wedge \omega, \end{aligned}$$

as desired. Now suppose that

$$\begin{aligned} \omega &= f\, d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r} \qquad \text{and} \qquad & \psi &= g\, d\lambda_{j_1} \wedge \cdots \wedge d\lambda_{j_s} \\ &= f\tilde{\omega} & &= g\tilde{\psi} \end{aligned}$$

with $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_s$ as usual. If some $i_\nu = j_\mu$, then from the definitions we see that the expressions on both sides of the equality in the theorem are equal to 0. Hence we may assume that the sets of indices $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ have no element in common. Then $d(\tilde{\omega} \wedge \tilde{\psi}) = 0$ by definition, and

$$\begin{aligned} d(\omega \wedge \psi) = d(fg\tilde{\omega} \wedge \tilde{\psi}) &= d(fg) \wedge \tilde{\omega} \wedge \tilde{\psi} \\ &= (g\,df + f\,dg) \wedge \tilde{\omega} \wedge \tilde{\psi} \\ &= d\omega \wedge \psi + f\,dg \wedge \tilde{\omega} \wedge \tilde{\psi} \\ &= d\omega \wedge \psi + (-1)^r f\tilde{\omega} \wedge dg \wedge \tilde{\psi} \\ &= d\omega \wedge \psi + (-1)^r \omega \wedge d\psi, \end{aligned}$$

thus proving the desired formula, in the present case. (We used the fact that $dg \wedge \tilde{\omega} = (-1)^r \tilde{\omega} \wedge dg$, whose proof is left to the reader.) The formula in the general case follows because any differential form can be expressed as a sum of forms of the type just considered, and one can then use the bilinearity of the product. Finally, $d$ is uniquely determined by the formula, and its effect on functions, because any differential form is a sum of forms of type $f\, d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_r}$ and the formula gives an expression of $d$ in terms of its effect on forms of lower degree. By induction, if the value of $d$ on functions is known, its value can then be determined on forms of degree $\geqq 1$. This proves the theorem.

**Corollary 3.2.** *Let $\omega$ be a form of class $C^2$. Then $dd\omega = 0$.*

*Proof.* If $f$ is a function, then

$$df(x) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j$$

and

$$ddf(x) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j.$$

Using the fact that the partials commute, and the fact that for any two positive integers $r$, $s$ we have $dx_r \wedge dx_s = -dx_s \wedge dx_r$, we see that the preceding double sum is equal to 0. A similar argument shows that the theorem is true for 1-forms of type $g(x)\, dx_i$ where $g$ is a function, and thus for all 1-forms by linearity. We proceed by induction. It suffices to prove the formula in general for decomposable forms. Let $\omega$ be decomposable of degree $r$, and write

$$\omega = \eta \wedge \psi$$

where $\deg \psi = 1$. Using the formula of Theorem 3.1 twice, and the fact that $dd\psi = 0$ and $dd\eta = 0$ by induction, we see at once that $dd\omega = 0$, as was to be shown.

## XXI, §4. INVERSE IMAGE OF A FORM

We start with some algebra once more. Let $E$, $F$ be finite dimensional vector spaces over $\mathbf{R}$ and let $\lambda: E \to F$ be a linear map. If $\mu: F \to \mathbf{R}$ is an element of $F^*$, then we may form the composite linear map

$$\mu \circ \lambda: E \to \mathbf{R}$$

which we visualize as

$$E \xrightarrow{\lambda} F \xrightarrow{\mu} \mathbf{R}.$$

We denote this composite $\mu \circ \lambda$ by $\lambda^*(\mu)$. It is an element of $E^*$. We have a similar definition on the higher alternating products, and in the appendix, we shall prove:

**Theorem C.** *Let $\lambda: E \to F$ be a linear map. For each $r$ there exists a unique linear map*

$$\lambda^*: \bigwedge^r F^* \to \bigwedge^r E^*$$

*having the following properties*:

(i) $\lambda^*(\omega \wedge \psi) = \lambda^*(\omega) \wedge \lambda^*(\psi)$ *for* $\omega \in \bigwedge^r F^*$, $\psi \in \bigwedge^s F^*$.
(ii) *If* $\mu \in F^*$ *then* $\lambda^*(\mu) = \mu \circ \lambda$, *and* $\lambda^*$ *is the identity on* $\bigwedge^0 F^* = \mathbf{R}$.

**Remark.** If $\mu_{j_1}, \ldots, \mu_{j_r}$ are in $F^*$, then from the two properties of Theorem C, we conclude that

$$\lambda^*(\mu_{j_1} \wedge \cdots \wedge \mu_{j_r}) = (\mu_{j_1} \circ \lambda) \wedge \cdots \wedge (\mu_{j_r} \circ \lambda).$$

Now we can apply this to differential forms. Let $U$ be open in $E = \mathbf{R}^n$ and let $V$ be open in $F = \mathbf{R}^m$. Let $f\colon U \to V$ be a $C^p$ map, $p \geqq 1$. For each $x \in U$ we obtain the linear map

$$f'(x)\colon E \to F$$

to which we can apply the preceding discussion. Consequently, we can reformulate Theorem C for differential forms as follows:

**Theorem 4.1.** *Let* $f\colon U \to V$ *be a* $C^p$ *map,* $p \geqq 1$. *Then for each* $r$ *there exists a unique linear map*

$$f^*\colon \Omega^r(V) \to \Omega^r(U)$$

*having the following properties*:

(i) *For any differential forms* $\omega$, $\psi$ *on* $V$ *we have*

$$f^*(\omega \wedge \psi) = f^*(\omega) \wedge f^*(\psi).$$

(ii) *If* $g$ *is a function on* $V$ *then* $f^*(g) = g \circ f$, *and if* $\omega$ *is a* 1-*form then*

$$(f^*\omega)(x) = \omega(f(x)) \circ df(x).$$

We apply Theorem C to get Theorem 4.1 simply by letting $\lambda = f'(x)$ at a given point $x$, and we define

$$(f^*\omega)(x) = f'(x)^*\omega(f(x)).$$

Then Theorem 4.1 is nothing but Theorem C applied at each point $x$.

**Example 1.** Let $y_1, \ldots, y_m$ be the coordinates on $V$, and let $\mu_j$ be the $j$-th coordinate function, $j = 1, \ldots, m$, so that $y_j = \mu_j(y_1, \ldots, y_m)$. Let

$$f\colon U \to V$$

be the map with coordinate functions

$$y_j = f_j(x) = \mu_j \circ f(x).$$

If

$$\omega(y) = g(y)\, dy_{j_1} \wedge \cdots \wedge dy_{j_s}$$

is a differential form on $V$, then

$$\boxed{f^*\omega = (g \circ f)\, df_{j_1} \wedge \cdots \wedge df_{j_s}.}$$

Indeed, we have for $x \in U$:

$$(f^*\omega)(x) = g(f(x))(\mu_{j_1} \circ f'(x)) \wedge \cdots \wedge (\mu_{j_s} \circ f'(x))$$

and

$$f_j'(x) = (\mu_j \circ f)'(x) = \mu_j \circ f'(x) = df_j(x).$$

**Example 2.** Let $f\colon [a, b] \to \mathbf{R}^2$ be a map from an interval into the plane, and let $x$, $y$ be the coordinates of the plane. Let $t$ be the coordinate in $[a, b]$. A differential form in the plane can be written in the form

$$\omega(x, y) = g(x, y)\, dx + h(x, y)\, dy$$

where $g$, $h$ are functions. Then by definition,

$$f^*\omega(t) = g(x(t), y(t)) \frac{dx}{dt}\, dt + h(x(t), y(t)) \frac{dy}{dt}\, dt$$

if we write $f(t) = (x(t), y(t))$. Let $G = (g, h)$ be the vector field whose components are $g$ and $h$. Then we can write

$$f^*\omega(t) = G(f(t)) \cdot f'(t)\, dt,$$

which is essentially the expression which is integrated when defining the integral of a vector field along a curve.

**Example 3.** Let $U$, $V$ be both open sets in $n$-space, and let $f\colon U \to V$ be a $C^p$ map. If

$$\omega(y) = g(y)\, dy_1 \wedge \cdots \wedge dy_n,$$

where $y_j = f_j(x)$ is the $j$-th coordinate of $y$, then

$$\begin{aligned} dy_j &= D_1 f_j(x)\, dx_1 + \cdots + D_n f_j(x)\, dx_n, \\ &= \frac{\partial y_j}{\partial x_1} dx_1 + \cdots + \frac{\partial y_j}{\partial x_n} dx_n \end{aligned}$$

and consequently, expanding out the alternating product according to the usual multilinear and alternating rules, we find that

$$f^*\omega(x) = g\bigl(f(x)\bigr)\Delta_f(x)\, dx_1 \wedge \cdots \wedge dx_n.$$

As in §2, $\Delta_f$ is the determinant of the Jacobian matrix of $f$.

**Theorem 4.2.** *Let $f\colon U \to V$ and $g\colon V \to W$ be $C^p$ maps of open sets. If $\omega$ is a differential form on $W$, then*

$$(g \circ f)^*(\omega) = f^*\bigl(g^*(\omega)\bigr).$$

*Proof.* This is an immediate consequence of the definitions.

**Theorem 4.3.** *Let $f\colon U \to V$ be a $C^2$ map and let $\omega$ be a differential form of class $C^1$ on $V$. Then*

$$f^*(d\omega) = df^*\omega.$$

*In particular, if $g$ is a function on $V$, then*

$$f^*(dg) = d(g \circ f).$$

*Proof.* We first prove this last relation. From the definitions, we have $dg(y) = g'(y)$, whence by the chain rule,

$$\bigl(f^*(dg)\bigr)(x) = g'\bigl(f(x)\bigr) \circ f'(x) = (g \circ f)'(x)$$

and this last term is nothing else but $d(g \circ f)(x)$, whence the last relation follows. For a form of degree 1, say

$$\omega(y) = g(y)\, dy_1,$$

with $y_1 = f_1(x)$, we find

$$(f^*\, d\omega)(x) = \bigl(g'(f(x)) \circ f'(x)\bigr) \wedge df_1(x).$$

Using the fact that $ddf_1 = 0$, together with Theorem 3.1, we get

$$(df^*\omega)(x) = \big(d(g \circ f)\big)(x) \wedge df_1(x),$$

which is equal to the preceding expression. Any 1-form can be expressed as a linear combination of forms, $g_i\, dy_i$, so that our assertion is proved for forms of degree 1.

The general formula can now be proved by induction. Using the linearity of $f^*$, we may assume that $\omega$ is expressed as $\omega = \psi \wedge \eta$ where $\psi$, $\eta$ have lower degree. We apply Theorem 3.1, and (i) of Theorem 4.1 to

$$f^*\, d\omega = f^*(d\psi \wedge \eta) + (-1)^r f^*(\psi \wedge d\eta)$$

and we see at once that this is equal to $df^*\omega$, because by induction, $f^*\, d\psi = df^*\psi$ and $f^*\, d\eta = df^*\eta$. This proves the theorem.

Let $U$ be open in $n$-space, and let $\omega$ be a continuous differential form on $U$ of degree $n$. We can associate a positive measure with $\omega$ as follows. Let us write

$$\omega(x) = h(x)\, dx_1 \wedge \cdots \wedge dx_n.$$

If $g \in C_c(U)$, we define $|\omega|$ by

$$\langle g, |\omega| \rangle = \int_U g(x)|h(x)|\, dx_1 \cdots dx_n = \int_U g|h|\, d\mu.$$

Then $g \mapsto \langle g, |\omega| \rangle$ is a positive functional on $C_c(U)$. We know that there exists a unique regular measure associated with this functional, and we shall call this measure the **measure on $U$ associated with** $|\omega|$. We may denote it by $\mu_{|\omega|}$. It is characterized by the relation

$$\langle g, |\omega| \rangle = \int_U g\, d\mu_{|\omega|}.$$

We shall analyze this measure more closely in Chapter XXIII.

## XXI, §5. APPENDIX

We shall give brief reviews of the proofs of the algebraic theorems which have been quoted in this chapter.

We first discuss "formal linear combinations". Let $S$ be a set. We wish to define what we mean by expressions

$$c_1 s_1 + \cdots + c_n s_n$$

where $\{c_i\}$ are numbers, and $\{s_i\}$ are distinct elements of $S$. What do we wish such a "sum" to be like? Well, we wish it to be entirely determined by the "coefficients" $c_i$, and each "coefficient" $c_i$ should be associated with the element $s_i$ of the set $S$. But an association is nothing but a function. This suggests to us how to define "sums" as above.

For each $s \in S$ and each number $c$ we define the symbol

$$cs$$

to be the function which associates $c$ to $s$ and 0 to $z$ for any element $z \in S$, $z \neq s$. If $b$, $c$ are numbers, then clearly

$$b(cs) = (bc)s \qquad \text{and} \qquad (b + c)s = bs + cs.$$

We let $T$ be the set of all functions defined on $S$ which can be written in the form

$$c_1 s_1 + \cdots + c_n s_n$$

where $c_i$ are numbers, and $s_i$ are distinct elements of $S$. Note that we have no problem now about addition, since we know how to add functions.

We contend that if $s_1, \ldots, s_n$ are distinct elements of $S$, then

$$1s_1, \ldots, 1s_n$$

are linearly independent. To prove this, suppose $c_1, \ldots, c_n$ are numbers such that

$$c_1 s_1 + \cdots + c_n s_n = 0 \qquad \text{(the zero function).}$$

Then by definition, the left-hand side takes on the value $c_i$ at $s_i$ and hence $c_i = 0$. This proves the desired linear independence.

In practice, it is convenient to abbreviate the notation, and to write simply $s_i$ instead of $1s_i$. The elements of $T$, which are called **formal linear combinations of elements of** $S$, can be expressed in the form

$$c_1 s_1 + \cdots + c_n s_n,$$

and any given element has a *unique* such expression, because of the linear independence of $s_1, \ldots, s_n$. This justifies our terminology.

We now come to the statements concerning multilinear alternating products. Let $E$, $F$ be vector spaces over $\mathbf{R}$. As before, let

$$E^{(r)} = E \times \cdots \times E,$$

taken $r$ times. Let

$$f: E^{(r)} \to F$$

be an $r$-multilinear alternating map. Let $v_1, \ldots, v_n$ be linearly independent elements of $E$. Let $A = (a_{ij})$ be an $r \times n$ matrix and let

$$\begin{aligned} u_1 &= a_{11}v_1 + \cdots + a_{1n}v_n \\ &\vdots \\ u_r &= a_{r1}v_1 + \cdots + a_{rn}v_n. \end{aligned}$$

Then

$$\begin{aligned} f(u_1, \ldots, u_r) &= f(a_{11}v_1 + \cdots + a_{1n}v_n, \ldots, a_{r1}v_1 + \cdots + a_{rn}v_n) \\ &= \sum_\sigma f(a_{1,\sigma(1)}v_{\sigma(1)}, \ldots, a_{r,\sigma(r)}v_{\sigma(r)}) \\ &= \sum_\sigma a_{1,\sigma(1)} \cdots a_{r,\sigma(r)} f(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \end{aligned}$$

where the sum is taken over all maps $\sigma: \{1, \ldots, r\} \to \{1, \ldots, n\}$. In this sum, all terms will be 0 whenever $\sigma$ is not an injective mapping, that is whenever there is some pair $i, j$ with $i \neq j$ such that $\sigma(i) = \sigma(j)$, because of the alternating property of $f$. From now on, we consider only injective maps $\sigma$. Then $\{\sigma(1), \ldots, \sigma(r)\}$ is simply a permutation of some $r$-tuple $(i_1, \ldots, i_r)$ with $i_1 < \cdots < i_r$.

We wish to rewrite this sum in terms of a determinant.

For each subset $S$ of $\{1, \ldots, n\}$ consisting of precisely $r$ elements, we can take the $r \times r$ submatrix of $A$ consisting of those elements $a_{ij}$ such that $j \in S$. We denote by

$$\mathrm{Det}_S(A)$$

the determinant of this submatrix. We also call it the subdeterminant of $A$ corresponding to the set $S$. We denote by $P(S)$ the set of maps

$$\sigma: \{1, \ldots, r\} \to \{1, \ldots, n\}$$

whose image is precisely the set $S$. Then

$$\mathrm{Det}_S(A) = \sum_{\sigma \in P(S)} \varepsilon_S(\sigma) a_{1,\sigma(1)} \cdots a_{r,\sigma(r)},$$

where $\varepsilon_S(\sigma)$ is the sign of $\sigma$, depending only on $\sigma$. In terms of this notation, we can write our expression for $f(u_1, \ldots, u_r)$ in the form

$$(1) \qquad \boxed{f(u_1, \ldots, u_r) = \sum_S \mathrm{Det}_S(A) f(v_S)}$$

where $v_S$ denotes $(v_{i_1}, \ldots, v_{i_r})$ if $i_1 < \cdots < i_r$ are the elements of the set $S$. The first sum over $S$ is taken over all subsets of $1, \ldots, n$ having precisely $r$ elements.

**Theorem A.** *Let $E$ be a vector space over $\mathbf{R}$, of dimension $n$. Let $r$ be an integer $1 \leqq r \leqq n$. There exists a finite dimensional space $\bigwedge^r E$ and an $r$-multilinear alternating map $E^{(r)} \to \bigwedge^r E$ denoted by*

$$(u_1, \ldots, u_r) \mapsto u_1 \wedge \cdots \wedge u_r$$

*satisfying the following properties*:

**AP 1.** *If $F$ is a vector space over $\mathbf{R}$ and $g\colon E^{(r)} \to F$ is an $r$-multilinear alternating map, then there exists a unique linear map*

$$g_*\colon \textstyle\bigwedge^r E \to F$$

*such that for all $u_1, \ldots, u_r \in E$ we have*

$$g(u_1, \ldots, u_r) = g_*(u_1 \wedge \cdots \wedge u_r).$$

**AP 2.** *If $\{v_1, \ldots, v_n\}$ is a basis of $E$, then the set of elements*

$$v_{i_1} \wedge \cdots \wedge v_{i_r}, \qquad 1 \leqq i_1 < \cdots < i_r \leqq n,$$

*is a basis of $\bigwedge^r E$.*

*Proof.* For each subset $S$ of $\{1, \ldots, n\}$ consisting of precisely $r$ elements, we select a letter $t_S$. As explained at the beginning of the section, these letters $t_S$ form a basis of a vector space whose dimension is equal to the binomial coefficient $\binom{n}{r}$. It is the space of formal linear combinations of these letters. Instead of $t_S$, we could also write $t_{(i)} = t_{i_1 \ldots i_r}$ with $i_1 < \cdots < i_r$. Let $\{v_1, \ldots, v_n\}$ be a basis of $E$ and let $u_1, \ldots, u_r$ be elements of $E$. Let $A = (a_{ij})$ be the matrix of numbers such that

$$\begin{aligned} u_1 &= a_{11}v_1 + \cdots + a_{1n}v_n \\ &\;\vdots \\ u_r &= a_{r1}v_1 + \cdots + a_{rn}v_n. \end{aligned}$$

*Define*

$$u_1 \wedge \cdots \wedge u_r = \sum_S \operatorname{Det}_S(A) t_S.$$

We contend that this product has the required properties.

The fact that it is multilinear and alternating simply follows from the corresponding property of the determinant.

We note that if $S = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$, then

$$t_S = v_{i_1} \wedge \cdots \wedge v_{i_r}.$$

A standard theorem on linear maps asserts that there always exists a unique linear map having prescribed values on basis elements. In particular, if $g\colon E^{(r)} \to F$ is a multilinear alternating map, then there exists a unique linear map

$$g_*\colon \bigwedge^r E \to F$$

such that for each set $S$, we have

$$g_*(t_S) = g(v_{i_1}, \ldots, v_{i_r})$$

if $i_1, \ldots, i_r$ are as above. By formula (1), it follows that

$$g(u_1, \ldots, u_r) = g_*(u_1 \wedge \cdots \wedge u_r)$$

for all elements $u_1, \ldots, u_r$ of $E$. This proves **AP 1**.

As for **AP 2**, let $\{w_1, \ldots, w_n\}$ be a basis of $E$. From the expansion of (1), it follows that the elements $\{w_S\}$, i.e. the elements $\{w_{i_1} \wedge \cdots \wedge w_{i_r}\}$ with all possible choices of $r$-tuples $(i_1, \ldots, i_r)$ satisfying $i_1 < \cdots < i_r$ are generators of $\bigwedge^r E$. The number of such elements is precisely $\binom{n}{r}$. Hence they must be linearly independent, and form a basis of $\bigwedge^r E$, as was to be shown.

**Theorem B.** *For each pair of positive integers* $(r, s)$ *there exists a unique bilinear map*

$$\bigwedge^r E \times \bigwedge^s E \to \bigwedge^{r+s} E$$

*such that if* $u_1, \ldots, u_r, w_1, \ldots, w_s \in E$ *then*

$$(u_1 \wedge \cdots \wedge u_r) \times (w_1 \wedge \cdots \wedge w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

*This product is associative.*

*Proof.* For each $r$-tuple $(u_1, \ldots, u_r)$ consider the map of $E^{(s)}$ into $\bigwedge^{r+s} E$ given by

$$(w_1, \ldots, w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

This map is obviously $s$-multilinear and alternating. Consequently, by **AP 1** of Theorem A, there exists a unique linear map

$$g_{(u)} = g_{u_1, \ldots, u_r}\colon \bigwedge^s E \to \bigwedge^{r+s} E$$

such that for any elements $w_1, \ldots, w_s \in E$ we have

$$g_{(u)}(w_1 \wedge \cdots \wedge w_s) = u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

Now the association $(u) \mapsto g_{(u)}$ is clearly an $r$-multilinear alternating map of $E^{(r)}$ into $L(\bigwedge^s E, \bigwedge^{r+s} E)$, and again by **AP 1** of Theorem A, there exists a unique linear map

$$g_*\colon \bigwedge^r E \to L(\bigwedge^s E, \bigwedge^{r+s} E)$$

such that for all elements $u_1, \ldots, u_r \in E$ we have

$$g_{u_1, \ldots, u_r} = g_*(u_1 \wedge \cdots \wedge u_r).$$

To obtain the desired product $\bigwedge^r E \times \bigwedge^s E \to \bigwedge^{r+s} E$, we simply take the association

$$(\omega, \psi) \mapsto g_*(\omega)(\psi).$$

It is bilinear, and is uniquely determined since elements of the form $u_1 \wedge \cdots \wedge u_r$ generate $\bigwedge^r E$, and elements of the form $w_1 \wedge \cdots \wedge w_s$ generate $\bigwedge^s E$. This product is associative, as one sees at once on decomposable elements, and then on all elements by linearity. This proves Theorem B.

Let $E$, $F$ be vector spaces, finite dimensional over **R**, and let $\lambda\colon E \to F$ be a linear map. If $\mu\colon F \to \mathbf{R}$ is an element of the dual space $F^*$, i.e. a linear map of $F$ into **R**, then we may form the composite linear map

$$\mu \circ \lambda\colon E \to \mathbf{R}$$

which we visualize as

$$E \xrightarrow{\lambda} F \xrightarrow{\mu} \mathbf{R}.$$

We denote this composite $\mu \circ \lambda$ by $\lambda^*(\mu)$. It is an element of $E^*$.

**Theorem C.** *Let $\lambda\colon E \to F$ be a linear map. For each $r$ there exists a unique linear map*

$$\lambda^*\colon \bigwedge^r F^* \to \bigwedge^r E^*$$

*having the following properties*:

(i) $\lambda^*(\omega \wedge \psi) = \lambda^*(\omega) \wedge \lambda^*(\psi)$, *for* $\omega \in \bigwedge^r F^*$, $\psi \in \bigwedge^s F^*$.
(ii) *If* $\mu \in F^*$ *then* $\lambda^*(\mu) = \mu \circ \lambda$, *and* $\lambda^*$ *is the identity on* $\bigwedge^0 F^* = \mathbf{R}$.

*Proof.* The composition of mappings

$$F^* \times \cdots \times F^* = F^{*(r)} \to E^* \times \cdots \times E^* = E^{*(r)} \to \bigwedge^r E^*$$

given by

$$(\mu, \ldots, \mu_r) \mapsto (\mu_1 \circ \lambda, \ldots, \mu_r \circ \lambda) \mapsto (\mu_1 \circ \lambda) \wedge \cdots \wedge (\mu_r \circ \lambda)$$

is obviously multilinear and alternating. Hence there exists a unique linear map $\bigwedge^r F^* \to \bigwedge^r E^*$ such that

$$\mu_1 \wedge \cdots \wedge \mu_r \mapsto \lambda^*(\mu_1) \wedge \cdots \wedge \lambda^*(\mu_r).$$

Property (i) now follows by linearity and the fact that decomposable elements $\mu_1 \wedge \cdots \wedge \mu_r$ generate $\bigwedge^r F^*$. Property (ii) comes from the definition. This proves Theorem C.

CHAPTER XXII

# Manifolds

## XXII, §1. ATLASES, CHARTS, MORPHISMS

Let $X$ be a set. An **atlas** of class $C^p$ ($p \geqq 0$) on $X$ is a family of pairs $\{(U_i, \varphi_i)\}$ ($i \in I$) satisfying the following conditions:

**AT 1.** *Each $U_i$ is a subset of $X$ and the $U_i$ cover $X$.*

**AT 2.** *Each $\varphi_i$ is a bijection of $U_i$ onto an open subset of a euclidean space $E$, and for every pair $i$, $j$ of indices, the set $\varphi_i(U_i \cap U_j)$ is open in $E$.*

**AT 3.** *The map*

$$\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

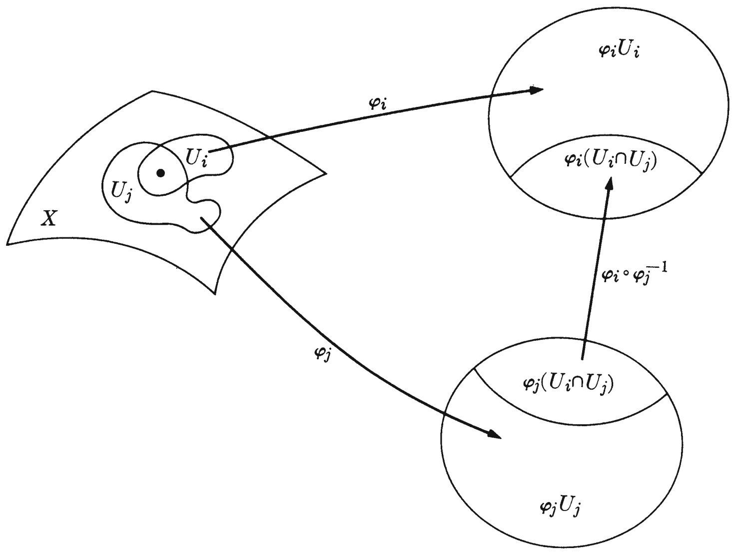
*is a $C^p$-isomorphism for each pair of indices $i$, $j$.*

The space $E$ is assumed to be the same for all $i$. If its dimension is $n$, we say that the atlas is $n$-dimensional. All that is done in this chapter would go over to the Banach case, but the principal applications we have in mind in the next chapter are strictly finite dimensional, and so for a first introduction to manifolds here we make the finite dimensionality assumption at once. Readers are referred to [La 2] for the general development, in a systematic way. They will note that there is essentially no change from the partial development given here.

Each pair $(U_i, \varphi_i)$ will be called a **chart** of the atlas. We see that the inverse map

$$\varphi_i^{-1}\colon \varphi_i U_i \to U_i$$

may be interpreted as a parametrization of a portion of $X$ by an open set in euclidean space. Thus in particular, $X$ is a set which can be covered by subsets, each of which is so parametrized. The extra condition **AT 3** is one which will allow us to speak of differentiability relative to $X$ itself.




Since $\varphi: U \to \mathbf{R}^n$ is a map into $n$-space, we can represent $\varphi$ by coordinate functions, and we can write for $x \in U$,

$$\varphi(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big) = (x_1, \ldots, x_n).$$

We call $(x_1, \ldots, x_n)$ the **local coordinates** of $x$ in the chart $(U, \varphi)$. The notation here is already somewhat concise, but useful. If readers feel the need for it, they may extend this notation as follows. Denote a point of $X$ by $P$. Then in a chart $\varphi: U \to \mathbf{R}^n$ at $P$, we have coordinates $\big(x_1(P), \ldots, x_n(P)\big)$ for the point $\varphi(P)$, $P \in U$, and we abbreviate this $n$-tuple by $x(P)$. In most cases, it is a useful abbreviation to do away with the extra letter.

Let $U$ be a subset of $X$ and let $\varphi: U \to \varphi U$ be a bijection of $U$ onto an open subset of $E$. We say that the pair $(U, \varphi)$ is **compatible** with the atlas $\{(U_i, \varphi_i)\}$ if each map $\varphi_i \varphi^{-1}$ (defined on a suitable intersection as in **AT 3**) is a $C^p$-isomorphism. Two atlases are said to be **compatible** if each chart of one is compatible with the other atlas. The relation of compatibility between atlases is immediately verified to be an equivalence relation. An equivalence class of $C^p$-atlases on $X$ is said to define a structure of **$C^p$-manifold** on $X$. The number $n$ being fixed, we say that $X$ is then an $n$-dimensional manifold.

If $(U, \varphi)$ and $(V, \psi)$ are two charts of a manifold, then we shall call the map $\varphi \circ \psi^{-1}$ (defined on $\psi(U \cap V)$, whenever $U \cap V$ is not empty) a **transition map**.

So far we have not assumed that $X$ has a topology. In many cases, a topology is first given, and then to make the atlases topologically compatible with this topology, one can require the additional condition that the maps $\varphi$ of the charts be homeomorphisms. However, it is also useful not to do this and deal with the more general situation when $X$ is merely a set, and we shall in fact have an important application later when we deal with the tangent bundle.

We shall now see how to define a topology on $X$ by means of the atlases. Let $\{(U_i, \varphi_i)\}$ be an atlas. A subset $U$ of $X$ is defined to be open if and only if the intersection $U \cap U_i$ with each open set of the atlas is such that $\varphi_i(U \cap U_i)$ is open, in $E$ of course. It is a trivial exercise to verify that this defines a topology. Furthermore, if $\{(V_j, \psi_j)\}$ is an equivalent atlas, then the two topologies coincide. We leave the formal verification to the reader. We note merely that the basic reason is that if a point $x$ lies in charts $U_i$ and $V_j$, then there is a subset $W$ containing $x$ such that $\varphi_i W$ and $\psi_j W$ are open. Since a topology is really determined locally (i.e. an open set is a union of open neighborhoods of its points) one sees at once that a set is open relative to one atlas if and only if it is open relative to the other.

Let $X$ be a manifold, and $U$ an open subset of $X$. Then it is possible, in the obvious way, to induce a manifold structure on $U$, by taking as atlases the intersections

$$\big(U_i \cap U, \varphi_i|(U_i \cap U)\big).$$

**Example 1.** Any open set in euclidean space is a manifold, the charts being the obvious ones: $C^p$-isomorphisms of open subsets onto other open sets in euclidean space.

**Example 2.** We speak of $\mathbf{R}/\mathbf{Z}$ as the circle group. Then $\mathbf{R}/\mathbf{Z}$ is a compact manifold, for which we can find an atlas consisting of two charts. The open interval $(0, 1)$ maps bijectively onto an open subset of $\mathbf{R}/\mathbf{Z}$ (by assigning to each real number its equivalence class modulo $\mathbf{Z}$), and the open interval $(-\frac{1}{2}, \frac{1}{2})$ also maps bijectively onto an open subset of $\mathbf{R}/\mathbf{Z}$. Readers will verify at once that these two maps are the charts of a $C^\infty$ atlas.

**Example 3.** Instead of $\mathbf{R}/\mathbf{Z}$ we can take $\mathbf{R}^n/\mathbf{Z}^n$, the $n$-dimensional torus, and define charts similarly.

**Example 4.** Let $S^n$ be the $n$-sphere in $\mathbf{R}^{n+1}$, i.e. the set of all points $(x_1, \ldots, x_{n+1})$ such that

$$x_1^2 + \cdots + x_{n+1}^2 = 1.$$

Then $S^n$ is a manifold, if we define charts as follows. Let

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2.$$

The sphere is the set of points $x$ such that $f(x) = 1$. For any point $a \in S^n$, $a = (a_1, \ldots, a_{n+1})$, some coordinate is not equal to 0, say $a_1$. Then

$$D_1 f(a) \neq 0,$$

and we can apply the implicit function theorem, so that there is a $C^\infty$ map $\varphi_1$ defined on an open neighborhood $U$ of $(a_2, \ldots, a_{n+1})$ such that

$$f(\varphi_1(x), x_2, \ldots, x_{n+1}) = 1$$

and $\varphi_1(a_2, \ldots, a_{n+1}) = a_1$. Furthermore, if we take $U$ small enough, then $\varphi_1$ is uniquely determined. Let

$$\varphi(x_2, \ldots, x_{n+1}) = (\varphi_1(x), x_2, \ldots, x_{n+1}).$$

It is an exercise to verify that the collection of all similar pairs $(\varphi U, \varphi^{-1})$ is a $C^\infty$ atlas for $S^n$. Actually, we shall obtain some theorems below which will prove this, and give general criteria showing that certain subsets of euclidean space are manifolds.

In our definition of a manifold, it was convenient to take the charts as maps from the set $X$ into the vector space. In our examples, we actually defined their inverses. We may visualize a manifold as a set which is parametrized locally by open subsets of some euclidean space. The parametrizing maps are the inverse maps of the charts. The whole point of condition **AT 3** is to ensure that the parametrizations are compatible with a certain order of differentiability.

**Example 5.** Let $X = \mathbf{R}$ and let $\varphi: X \to \mathbf{R}$ be the map $\varphi(x) = x^3$. Then $(X, \varphi)$ is a chart defining an atlas. We therefore get a differentiable structure on $\mathbf{R}$, but the identity map is not $C^1$ compatible with this atlas, because the map $x \mapsto x^{1/3}$ is not differentiable at 0.

Let $X$, $Y$ be manifolds. Then the product $X \times Y$ is a manifold in an obvious way. If $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ are atlases for $X$, $Y$, respectively, then

$$\{(U_i \times V_j, \varphi_i \times \psi_j)\}$$

is an atlas for the product, and the product of compatible atlases gives rise to compatible atlases, so that we do get a well-defined product manifold.

We know what it means for a map from an open set in euclidean space into another euclidean space to be differentiable, or of class $C^p$. Since our definition of a manifold is based locally on open sets in euclidean space, we can now define the notion of a $C^p$ map from one manifold into another. Let $X$, $Y$ be $C^p$-manifolds and $f: X \to Y$, a map. We say that $f$ is a $C^p$ map if given $x \in X$ there exists a chart $(U, \varphi)$ at $x$ and a chart $(V, \psi)$ at $f(x)$ such that $f(U) \subset V$ and such that the map

$$f_{U,V} = \psi \circ f \circ \varphi^{-1}: \varphi U \to \psi V$$

is a $C^p$ map. If this holds, then this same condition holds for any choice of charts $(U, \varphi)$ at $x$ and $(V, \psi)$ at $f(x)$ such that $f(U) \subset V$.

It is clear that the composite of two $C^p$ maps is itself a $C^p$ map (because it is true for open subsets of euclidean space).

It should be noted that $C^p$ manifolds and maps are useful with $p$ finite because Banach space techniques can be applied to sets of mappings. Indeed, the $C^p$-bounded maps of one open set of euclidean space into another form a Banach space. Manifold theory goes through if instead of euclidean space we take a Banach space in the definition of a manifold, and one can then give a manifold structure to the set of $C^p$ maps of one manifold into another. We don't go into this aspect of manifold theory in this book, but readers should keep this possibility in mind for future applications.

We shall deal with a fixed $p$ throughout a discussion. Thus it is convenient to call a $C^p$ map $f: X \to Y$ by a neutral name, and we call such maps **morphisms**. (If the order of differentiability needs to be specified, we can always add the $C^p$ prefix.) By a **$C^p$-isomorphism**, or simply isomorphism $f: X \to Y$ we mean a morphism for which there exists an inverse morphism, i.e. a morphism $g: Y \to X$ such that $g \circ f$ and $f \circ g$ are the identity mappings of $X$ and $Y$, respectively. This is the same terminology which we used with respect to open sets in Euclidean spaces. Similarly, we have the notion of a **local $C^p$-isomorphism** at a point $x \in X$, meaning that $f$ induces an isomorphism of an open neighborhood of $x$ in $X$ onto an open neighborhood of $f(x)$ in $Y$.

## XXII, §2. SUBMANIFOLDS

A manifold may arise like the torus, not embedded in any particular euclidean space, or it may be given as a subset of some euclidean space like the sphere. We now study this second possibility.

Let $X$ be a topological space and $Y$ a subspace. We say that $Y$ is **locally closed** in $X$ if every point $y \in Y$ has an open neighborhood $U$ in $X$ such that $Y \cap U$ is closed in $U$. We leave it to the reader to verify that a locally closed subset of $X$ is the intersection of an open set and a

closed set in $X$. For instance any open subset of $X$ is locally closed, and any open interval is locally closed in the plane.

Let $X$ be a manifold and $Y$ a subset. We shall say that $Y$ is a **submanifold** if, roughly speaking at each point $y \in Y$ there exists a chart such that in this chart, the points of $Y$ correspond to a factor in a product space. We now make this condition precise as follows. For each $y \in Y$ there exists a chart $(V, \varphi)$ at $y$ such that $\varphi$ gives an isomorphism

$$\varphi: V \to V_1 \times V_2,$$

where $V_1$ is open in some space $E_1$, and $V_2$ is open in some space $E_2$, and such that

$$\varphi(Y \cap V) = V_1 \times \{a_2\}$$

for some point $a_2 \in V_2$. If we make a translation on $V_2$, it is clear that we can always adjust $V_2$ such that $a_2 = 0$.

If we let $E = E_1 \times E_2$, then the coordinates split up naturally, namely we can write $\mathbf{R}^n = \mathbf{R}^m \times \mathbf{R}^q$, and

$$(x_1, \ldots, x_n) = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+q}).$$

If $a_2 = 0$ in our preceding definition, then we see that the points of $Y$ in the given chart $(V, \psi)$ are precisely the points having coordinates

$$(x_1, \ldots, x_m, 0, \ldots, 0).$$

All of this explains what we said about $Y$ being locally at each point a factor in a product.

We observe that if $Y$ is a submanifold of $X$, then $Y$ is locally closed in $X$. We must also justify our terminology by showing that $Y$ is a manifold in its own right. Indeed, if $(V, \varphi)$ is a chart at $y$ as in our definition, then $\varphi$ induces a bijection

$$\varphi_1: Y \cap V \to V_1.$$

*The collection of pairs $\{(Y \cap V, \varphi_1)\}$ obtained in the above manner constitutes an atlas for $Y$, of class $C^p$.*

The proof of this statement is essentially a triviality, and consists merely in keeping the definitions straight. We give it in full. Let

$$\psi: W \to W_1 \times W_2$$

be another chart at $y$ such that

$$\psi(Y \cap W) = W_1 \times \{b_2\}$$

for some point $b_2 \in W_2$. Then we get the bijection $\psi_1: Y \cap W \to W_1$. Furthermore,

$$\varphi(Y \cap V \cap W) \text{ is open in } V_1 \times \{a_2\} \text{ and thus equal to } V_1' \times \{a_2\}$$

for some open $V_1'$ in $V_1$. Similarly,

$$\psi(Y \cap V \cap W) \text{ is open in } W_1 \times \{b_2\} \text{ and thus equal to } W_1' \times \{b_2\}$$

for some open $W_1'$ in $W_1$. We have isomorphisms

$$V \cap W \xrightarrow{\varphi'} V' \subset V_1 \times V_2 \qquad \text{and} \qquad V \cap W \xrightarrow{\psi'} W' \subset W_1 \times W_2$$

under $\varphi$ and $\psi$, respectively, and thus

$$\varphi' \circ \psi'^{-1}: W' \to V'$$

is an isomorphism. If we look at the effect of this isomorphism on the part of $W'$ corresponding to $Y$, we see that it simply induces by restriction a map

$$W_1' \times \{a_2\} \to V_1' \times \{b_2\}$$

whence a map $W_1' \to V_1'$ which is of class $C^p$, and has a $C^p$ inverse, induced by $\psi' \circ \varphi'^{-1}: V' \to W'$. This proves what we wanted, i.e. that the family of all pairs $\{(Y \cap V, \varphi)\}$ is an atlas for $Y$.

The proof is based on the following obvious fact, which it is useful to keep in mind when dealing with submanifolds.

*Let $V_1$, $V_2$, $W_1$, $W_2$ be open subsets of euclidean spaces, and let*

$$g: V_1 \times V_2 \to W_1 \times W_2$$

*be a $C^p$ map. Let $a_2 \in V_2$, $b_2 \in W_2$, and assume that $g$ maps $V_1 \times \{a_2\}$ into $W_1 \times \{b_2\}$. Then the induced map*

$$g_1: V_1 \to W_1$$

*is also a $C^p$ map.*

Indeed, it is obtained as a composite map

$$V_1 \to V_1 \times V_2 \to W_1 \times W_2 \to W_1,$$

the first map being an injection of $V_1$ as a factor, and the third map a projection on the first factor.

The following statement has a proof based on the same principle.

*Let $Y$ be a submanifold of $X$ and let $f\colon Z \to X$ be a map from a manifold $Z$ into $X$ such that $f(Z)$ is contained in $Y$. Let $f_Y\colon Z \to Y$ be the induced map. Then $f$ is a morphism if and only if $f_Y$ is a morphism.*

We leave the proof to the reader.

We observe that if $Y$ is a submanifold of $X$, then the inclusion map of $Y$ into $X$ is a morphism. If $Y$ is also a closed subset of $X$, then we say that $Y$ is a **closed submanifold**.

**Theorem 2.1.** *Let $U$ be open in $\mathbf{R}^n$ and let $f\colon U \to \mathbf{R}$ be a $C^p$ function, $p \geqq 1$. Let*

$$a = (a_1, \ldots, a_n) \in U,$$

*and assume that $D_n f(a) \neq 0$. Then the map*

$$\varphi\colon (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, f(x))$$

*is a local $C^p$ isomorphism at $a$, i.e. is a chart at $a$. If $f(a) = c$, then locally at $a$, the inverse image $f^{-1}(c)$ is a submanifold of $\mathbf{R}^n$.*

*Proof.* The Jacobian matrix of $\varphi$ at $a$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ D_1 f(a) & \cdots & \cdots & \cdots & \cdots & D_n f(a) \end{pmatrix}$$

and its determinant is $D_n f(a) \neq 0$. The inverse mapping theorem shows that $\varphi$ is a local $C^p$ isomorphism at $a$, thus proving Theorem 2.1.

**Corollary 2.2.** *Let $Y$ be the subset of $U$ consisting of all points $x$ such that $f(x) = c$. Then there exists an open neighborhood $V$ of $a$ such that $Y \cap V$ is a submanifold of $X$.*

*Proof.* We take $V$ such that $\varphi$ is a $C^p$ isomorphism on $V$. Those points of $Y$ correspond to the points such that

$$\varphi(x) = (x_1, \ldots, x_{n-1}, c).$$

If $g$ is the inverse mapping of $\varphi$, then the map

$$(x_1, \ldots, x_{n-1}) \mapsto g(x_1, \ldots, x_{n-1}, c)$$

is the inverse parametrization of a chart, and $\varphi$ restricted to $V \cap Y$ maps $V \cap Y$ in a factor of a product, as desired.

**Example.** The map $f(x) = x_1^2 + \cdots + x_n^2$ and $c = 1$ give the sphere in the preceding corollary, so that the sphere is a submanifold of $\mathbf{R}^n$. For any point of the sphere, some coordinate is not equal to 0, and the partial derivative of $f$ at that point is not 0, so that the corollary applies.

The argument proving Theorem 2.1 can easily be generalized to cover other cases in which we can prove that a certain subset is a submanifold. We shall formulate these criteria, which involve the derivative as linear map. We shall not use them later in this book, and the reader may omit them without harm. For further applications and terminology concerning this, cf. books on differentiable manifolds, e.g. [La 2].

*Let $U$ be open in $E$ and let $f: U \to F$ be a $C^p$ morphism with $p \geqq 1$. Let $x_0 \in U$. Assume that $f'(x_0)$ is a linear isomorphism of $E$ onto a subspace $F_1$ of $F$. Let $F = F_1 \oplus F_2$. Then there exists a local $C^p$ isomorphism $g: F \to F_1 \times F_2$ at $f(x_0)$ and an open subset $U_1$ of $U$ containing $x_0$ such that the composite map $g \circ f$ induces a $C^p$ isomorphism of $U_1$ onto an open subset of $F_1$.*

*Proof.* Consider the map

$$\varphi: U \times F_2 \to F_1 \times F_2$$

given by

$$\varphi(x, y_2) = (f(x), 0) + (0, y_2).$$

Then

$$\varphi'(x_0, y_2) = \begin{pmatrix} f'(x_0) & 0 \\ 0 & I_2 \end{pmatrix}$$

where we use the matrix representation of a linear map of $E \times F_2$ into $F_1 \times F_2$. For $v_1 \in E$ and $v_2 \in F_2$ we have

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_{11} v_1 + \lambda_{12} v_2 \\ \lambda_{21} v_1 + \lambda_{22} v_2 \end{pmatrix}.$$

In this representation, we view $f'(x_0)$ as a linear map of $E$ onto $F_1$. We see that $\varphi'(x_0, 0)$ is a linear isomorphism between $E \times F_2$ and $F_1 \times F_2$. By the inverse mapping theorem, we conclude that $\varphi$ is a local $C^p$ isomorphism at $(x_0, 0)$. We let $\psi$ be its local $C^p$ inverse. Then it is obvious that $\psi$ induces a map $g$ to satisfy our requirements.

In the preceding result, we may view $f$ as parametrizing a subset of $F$, say locally at $x_0$ by $U_1 \to f(U_1)$. The lemma shows that there is a chart

at $f(x_0)$ in $f(U_1)$ which maps $f(U_1)$ into a factor in the product $F_1 \times F_2$. Note that if we write $E = \mathbf{R}^q$ and $F = \mathbf{R}^n$, then the subspace $F_1$ of $F$ is not necessarily equal to $\mathbf{R}^q$ in its usual embedding in $\mathbf{R}^n$ as the space of the first $q$ coordinates. The subspace $F_1$ can be quite arbitrary. However, we can find a complementary subspace $F_2$, and then a basis of $F_1$ and of $F_2$ in such a way that if we take coordinates with respect to this basis, then the coordinates of $g(f(U_1))$ are precisely the coordinates $(x_1, \ldots, x_q, 0, \ldots, 0)$. In our geometric terminology, we can say that $f(U_1)$ is a submanifold of $F$.

The next result deals with the dual situation, where instead of an injection we deal with a projection. If we have a map

$$V_1 \times V_2 \xrightarrow{\pi} F,$$

then we shall say that this map is a projection (on the first factor) if this map can be expressed as a composite

$$V_1 \times V_2 \to V_1 \to F$$

of the actual projection on the first factor, followed by a map of $V_1$ into $F$.

*Let $U$ be open in $E$ and let $a$ be a point of $U$. Let $f: U \to F$ be a $C^p$ map, $p \geqq 1$. Assume that the derivative $f'(a): E \to F$ is surjective. Let $E_2$ be a subspace of $E$ such that $f'(a)$ induces a linear isomorphism of $E_2$ with $F$, and let $E_1$ be a complementary subspace to $E_2$ in $E$, that is $E = E_1 \times E_2$. Then the map*

$$(x_1, x_2) \mapsto (x_1, f(x_1, x_2))$$

*is a local $C^p$ isomorphism at $a$.*

*Proof.* The derivative of the map at $a$ is represented by the matrix

$$\begin{pmatrix} I_1 & 0 \\ D_1 f(a) & D_2 f(a) \end{pmatrix}$$

and is therefore invertible at $a = (a_1, a_2)$ because $D_2 f(a)$ by definition is a linear isomorphism of $E_2$ with $F$. The inverse mapping theorem shows that our map is locally $C^p$ invertible at $a$, as was to be shown.

In particular, let $c \in F$, and consider those points $x \in U$ such that $f(x) = c$, i.e. such that $f$ has constant value $c$. If $v_2 \in E_2$ is such that $f(v_2) = c$ (and $v_2$ is close to $a_2$), then the inverse image $f^{-1}(c)$ corresponds to a factor $V_1 \times \{v_2\}$ in $E_1 \times E_2$ locally near $a$. One of the

most important examples is that of a function, which we treated in Theorem 2.1.

## XXII, §3. TANGENT SPACES

Let $X$ be a $C^p$ manifold ($p \geqq 1$). Let $x$ be a point of $X$. We then have a representation of $x$ in every chart at $x$, which maps an open neighborhood of $x$ into a euclidean space $E$. We consider triples $(U, \varphi, v)$ where $(U, \varphi)$ is a chart at $x$, and $v$ is a vector in the vector space in which $\varphi U$ lies. We say that two such triples $(U, \varphi, v)$ and $(V, \psi, w)$ are **equivalent** if the derivative of $\psi \circ \varphi^{-1}$ at $\varphi x$ maps $v$ on $w$. The formula reads

$$(\psi \circ \varphi^{-1})'(\varphi x)v = w.$$

This is obviously an equivalence relation by the chain rule. An equivalence class of such triples is called a **tangent vector** of $X$ at $x$. Thus we represent a tangent vector much the same way that we represent a point of $X$, by its representation relative to charts. The set of such tangent vectors is called the **tangent space** of $X$ at $x$ and is denoted by $T_x(X)$. Each chart $(U, \varphi)$ determines a bijection of $T_x(X)$ on a euclidean space, namely the equivalence class of $(U, \varphi, v)$ corresponds to the vector $v$.

Suppose that $X$ is a manifold. Then each derivative

$$(\psi \circ \varphi^{-1})'(\varphi x): E \to E$$

is an invertible linear map. Let $v_1$, $v_2$ be vectors representing tangent vectors $\bar{v}_1$, $\bar{v}_2$ in the chart $(U, \varphi)$, and let $w_1$, $w_2$ represent the same tangent vectors in $(V, \psi)$. Then by definition

$$w_i = (\psi \circ \varphi^{-1})'(\varphi x)v_i.$$

From this we see that $v_1 + v_2$ and $w_1 + w_2$ represent the same tangent vector, and that if $c \in \mathbf{R}$, then $cv_1$ and $cw_1$ represent the same tangent vector. Thus we can define addition and multiplication by numbers in $T_x(X)$ in such a way that

$$\bar{v}_1 + \bar{v}_2 = \overline{v_1 + v_2} \qquad \text{and} \qquad c\bar{v}_1 = \overline{cv_1}.$$

Then $T_x(X)$ is a vector space, and the map

$$v \mapsto \bar{v}$$

is a linear isomorphism of $E$ onto $T_x(X)$.

The derivative of a map defined on open sets of Euclidean spaces can

now be interpreted on manifolds. Let $f: X \to Y$ be a $C^p$ map and let $x \in X$. We define the **tangent map** at $x$,

$$T_x f: T_x(X) \to T_{f(x)}(Y)$$

as the unique linear map having the following property: If $(U, \varphi)$ is a chart at $x$ and $(V, \psi)$ is a chart at $f(x)$ such that $f(U) \subset V$, and $\bar{v}$ is a tangent vector at $x$ represented by $v$ in the chart $(U, \varphi)$, then

$$T_x f(\bar{v})$$

is the tangent vector at $f(x)$ represented by $Df_{U,V}(x)v$. It is immediately verified that there does exist such a unique linear map. The tangent linear map is also occasionally denoted by $df_x$, and is also called the **differential** of $f$ at $x$. The representation of $T_x f$ on the spaces of charts can be given in the form of a diagram.

$$\begin{array}{ccc} T_x(X) & \longrightarrow & E \\ {\scriptstyle T_x f}\big\downarrow & & \big\downarrow{\scriptstyle f'_{U,V}(x)} \\ T_{f(x)}(Y) & \longrightarrow & F \end{array}$$

Here of course, $F$ is the space in which $\psi(V)$ lies.

If $f: X \to Y$ and $g: Y \to Z$ are two $C^p$ maps, then the chain rule can be expressed by the formula

$$T_x(g \circ f) = (T_{f(x)}g) \circ (T_x f).$$

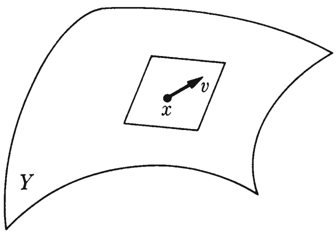
In particular, suppose that $Y$ is a submanifold of $X$, and let $x \in Y$ be a point of $Y$. Then we have the inclusion map

$$j: Y \to X$$

which induces an injective linear map

$$T_x j: T_x(Y) \to T_x(X),$$

whose image is a subspace of $T_x(X)$. This is the situation which is usually depicted by the following picture:



Here $X = E$ is the whole vector space. Suppose that $Y$ is a submanifold of $E$ and let $x \in Y$. Let

$$\psi_1 \colon V_1 \to Y$$

be a local isomorphism of some open $V_1$ in a space $F$ with $Y$, at a point $y_1 \in V_1$ such that $\psi_1(y) = x$. Let us view $\psi_1$ as a map of $V_1$ into $E$. Then

$$\psi_1'(y_1) \colon F \to E$$

is an injective linear map, whose image is a subspace $E_0$ of $E$. One can verify directly, or from the abstract fact that $T_y\psi$ is defined, that if

$$\psi_2 \colon V_2 \to Y$$

is a local isomorphism of some open $V_2$ in $F$ with $Y$, at a point $y_2 \in V_2$ such that $\psi_2(y_2) = x$ also, then the image of $\psi_2'(y_2)$ is in fact equal to $E_0$. This subspace $E_0$ is the translation of the "tangent space" drawn on the picture. In fact, the tangent space drawn on the picture consists of all pairs $(x, v)$ with $v \in E_0$. We view each such pair $(x, v)$ as a located vector, starting at $x$ and ending at $x + v$.

The collection of tangent spaces, namely the union of all $T_x(X)$ for all $x \in X$, will be called the **tangent bundle** of $X$, and will be denoted by $T(X)$. We can in fact make $T(X)$ into a $C^{p-1}$ manifold by giving natural charts for it as follows.

We have a natural map

$$\pi \colon T(X) \to X$$

which maps each tangent space $T_x(X)$ on the point $x$ of $X$. We call $\pi$ the natural **projection**. Let $(U, \varphi)$ be a chart of $X$, with $\varphi U$ is open in $E$. We then obtain a map

$$\tau_\varphi \colon \pi^{-1}(U) \to \varphi U \times E$$

defined by

$$\tau_\varphi(\bar{v}) = (\varphi x, v)$$

if $\pi(\bar{v}) = x$ and $\bar{v}$ is a tangent vector at $x$, represented by $v$ in $E$, with respect to the chart. In fact, it is clear that $\tau_\varphi$ is a bijection.

Let $(U, \varphi)$ and $(V, \psi)$ be two charts. We have

$$\pi^{-1}(U) \cap \pi^{-1}(V) = \pi^{-1}(U \cap V).$$

We obtain a transition mapping

$$\tau_\psi \circ \tau_\psi^{-1} \colon \varphi(U \cap V) \times E \to \psi(U \cap V) \times E,$$

by

$$(\varphi x, v) \mapsto \left(\psi x, D(\psi \circ \varphi^{-1})(x)v\right)$$

for $x \in U \cap V$ and $v \in E$. Since the derivative $D(\psi \circ \varphi^{-1})$ is of class $C^{p-1}$ and is a linear isomorphism at $x$, we conclude that our family of maps $\{\tau_\varphi\}$, for $(U, \varphi)$ ranging over all charts of $X$, is an atlas for $T(X)$, and therefore that $T(X)$ is a $C^{p-1}$ manifold, as we predicted it would be.

We call each chart $(\pi^{-1}U, \tau_\varphi)$ a **trivializing chart** of $T(X)$, over the open set $U$. Locally, we see that each such trivializing chart for $T(X)$ gives an isomorphism of the tangent bundle over $U$ with a product $\varphi U \times E$.

Let $f: X \to Y$ be a $C^p$ morphism, $p \geqq 1$. We can then define a tangent map

$$Tf: T(X) \to T(Y)$$

to be simply the map equal to

$$T_x f: T_x(X) \to T_{f(x)}(Y)$$

on the tangent space at $x$. It is immediately clear from the way in which we defined the charts for the tangent bundle that $Tf$ is a $C^{p-1}$ morphism. Over an open set $U$ of $X$ with chart $(U, \varphi)$, suppose that $f$ maps $U$ into an open set $V$ of $Y$, with chart $(V, \psi)$. We can represent $Tf$ as the derivative as on the following diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tau_\varphi} & \varphi U \times E \\ {\scriptstyle Tf}\downarrow & & \downarrow \\ \pi^{-1}(V) & \xrightarrow[\tau_\psi]{} & \psi V \times F. \end{array}$$

The map on the right can be viewed as the pair $(f_{U,V}, f'_{U,V})$.

## XXII, §4. PARTITIONS OF UNITY

Let $X$ be a $C^p$ manifold, $p \geqq 0$. By a $C^p$ **function** on $X$ we shall always mean a morphism of $X$ into $\mathbf{R}$ of class $C^p$ (unless otherwise specified, when we take complex valued functions). The functions form a ring $C^p(X)$. As usual, the **support** of a function is the closure of the set of points $x$ such that $f(x) \neq 0$.

Let $X$ be a topological space. A covering of $X$ is called **locally finite** if every point of $X$ has a neighborhood which intersects only finitely many elements of the covering. A **refinement** $\{V_j\}$ of a covering $\{U_i\}$ of $X$ is a covering such that each $V_j$ is contained in some $U_i$. We also say that the covering $\{V_j\}$ is **subordinated to the covering** $\{U_i\}$.

A **partition of unity** (of class $C^p$) on a manifold $X$ consists of an open covering $\{U_i\}$ of $X$ and a family of $C^p$ functions

$$\psi_i: X \to \mathbf{R}$$

satisfying the following conditions:

**PU 1.** *For all $x \in X$, we have $\psi_i(x) \geqq 0$.*

**PU 2.** *The support of $\psi_i$ is contained in $U_i$.*

**PU 3.** *The covering is locally finite.*

**PU 4.** *For each point $x \in X$ we have $\sum \psi_i(x) = 1$.*

(The sum is taken over all $i$, but is in fact finite for any given point $x$ in view of **PU 3**.)

As a matter of notation, we often write that $\{(U_i, \psi_i)\}$ is a partition of unity if it satisfies the previous four conditions.

**Theorem 4.1.** *Let $X$ be a manifold which is Hausdorff and whose topology has a countable base. Given an open covering $\mathscr{U}$ of $X$, there exists an atlas $\{(V_k, \varphi_k)\}$ such that the covering $\{V_k\}$ is locally finite and subordinated to the given covering $\mathscr{U}$, such that $\varphi_k V_k$ is the open ball $B_3(0)$ of radius 3 centered at 0, and such that the open sets $W_k = \varphi_k^{-1}(B_1)$ cover $X$ (where $B_1$ is the open ball of radius 1 centered at 0).*

*Proof.* Let $U_1, U_2, \ldots$ be a basis for the open sets of $X$ such that each $\overline{U}_i$ is compact. We can find such a basis since $X$ is locally compact. We construct inductively a sequence $A_1, A_2, \ldots$ of compact sets whose union is $X$, such that $A_i$ is contained in the interior of $A_{i+1}$. We start with $A_1 = \overline{U}_1$. Suppose that we have constructed $A_1, \ldots, A_i$. Let $j$ be the smallest integer such that $A_i$ is contained in $U_1 \cup \cdots \cup U_j$. We let $A_{i+1}$ be the closed and compact set

$$\overline{U}_1 \cup \cdots \cup \overline{U}_j \cup \overline{U}_{i+1}.$$

This gives our desired sequence of compact sets.

For each point $x \in X$ we can find an arbitrarily small chart $(V_x, \varphi_x)$ at $x$ such that $\varphi_x V_x$ is the open ball of radius 3 centered at 0. We can therefore assume that $V_x$ is contained in some open set of the covering $\mathscr{U}$. As for the statement concerning the ball of radius 3, we can always shrink our open set $V_x$ so that its image is a ball, and then adjust the image by a translation and multiplication by a positive number to make the image exactly equal to $B_3(0)$. For each $i$ and each $x$ in the open set

$$\operatorname{Int}(A_{i+2}) - A_{i-1}$$

we select $V_x$ to be contained in this open set. We let $W_x = \varphi_x^{-1}(B_1)$ be the inverse image of the ball of radius 1. We can cover the compact set (annulus)

$$A_{i+1} - \operatorname{Int}(A_i)$$

by a finite number of sets $W_{x_1}, \ldots, W_{x_m}$. Let $\mathscr{B}_i$ denote the family $\{V_{x_1}, \ldots, V_{x_m}\}$, and let $\mathscr{B}$ be the union of all $\mathscr{B}_i$ for all $i = 1, 2, \ldots$. Then $\mathscr{B}$ is an open covering of $X$, is locally finite, and is subordinated to our given covering $\mathscr{U}$. It also satisfies the other requirements of the theorem.

**Corollary 4.2.** *Let $X$ be a $C^p$ manifold which is Hausdorff, and whose topology has a countable base. Then $X$ has $C^p$ partitions of unity subordinated to a given covering $\mathscr{U}$.*

*Proof.* Let $\{(V_k, \varphi_k)\}$ be as in the theorem, and let $W_k = \varphi_k^{-1}(B_1)$ be as in the theorem. We can find a function $\psi_k$ of class $C^p$ such that $0 \leqq \psi_k \leqq 1$, such that $\psi_k(x) = 1$ for $x \in W_k$, and $\psi_k(x) = 0$ for $x \notin V_k$. (The proof is recalled below.) We now let

$$\psi = \sum \psi_k.$$

The sum is finite at each point, and we let $\gamma_k = \psi_k/\psi$. Then $\{(V_k, \gamma_k)\}$ is the desired partition of unity.

We now recall the argument giving the function $\psi_k$. If $0 \leqq a < b$, then the function defined by

$$\exp \frac{-1}{(t-a)(b-t)}$$

in the open interval $a < t < b$ and 0 outside the interval determines a bell-shaped $C^\infty$ function from $\mathbf{R}$ to $\mathbf{R}$. Its integral from $-\infty$ to $t$ divided by the area under the bell yields a function which lies strictly between 0 and 1 in the interval $a < t < b$, is equal to 0 for $t \leqq a$ and is equal to 1 for $t \geqq b$.

We can therefore find a real valued function of a real variable, say $\eta(t)$, such that $\eta(t) = 1$ for $|t| < 1$ and $\eta(t) = 0$ for $|t| \geqq 1 + \delta$ with small $\delta$, and such that $0 \leqq \eta \leqq 1$. Then $\eta(|x|)^2 = \psi(x)$ gives us a function which is equal to 1 on the ball of radius 1 and 0 outside the ball of radius $1 + \delta$. (We denote by $|\ |$ the euclidean norm.) This function can then be transported to the manifold by any given chart whose image is the ball of radius 3.

**Corollary 4.3.** *Let $A$, $B$ be disjoint closed subsets of $\mathbf{R}^n$, or of a manifold $X$ admitting $C^p$ partitions of unity subordinate to any given open covering. Then there exists a $C^p$ function $f$ such that*

$$0 \leqq f \leqq 1, \qquad f = 1 \text{ on } A, \quad f = 0 \text{ on } B.$$

*Proof.* For each $x \in A$ let $U_x$ be an open neighborhood of $x$ not intersecting $B$. Let $\{\alpha_i\}$ be a partition of unity subordinate to the cover-

ing consisting of $\mathscr{C}A$ and $\{U_x\}_{x \in A}$. Let $J$ be the set of those indices $j$ such that supp $\alpha_j \subset U_{x(j)}$ for some $x(j) \in A$. Let

$$f = \sum_{j \in J} \alpha_j.$$

For any $x \in X$ there is only a finite number of functions $\alpha_j$ such that $\alpha_j(x) \neq 0$, so our sum expressing $f$ is actually a finite sum. If $x \in A$ and $\alpha_i$ has support in $\mathscr{C}A$, then $\alpha_i(x) = 0$. Hence for each $x \in A$ we have $f(x) = 1$. If $x \in B$, then $\alpha_j(x) = 0$ for each $j \in J$ so $f(x) = 0$. For any $x \in X$ we have $0 \leqq f(x) \leqq 1$ because of the definition of a partition of unity and the fact that we take our sum for $f$ only over a subset of the indices $\{j\}$. This proves our corollary.

**Remark.** In some cases, one wants a function $f$ as in the corollary with certain bounds on its derivative. One can achieve such bounds by being more careful in selecting the $\alpha_i$. For an example of the kind of technique used, cf. the end of Chapter XXIII, §6.

## XXII, §5. MANIFOLDS WITH BOUNDARY

In our applications, we need manifolds with boundary. Let $\lambda: E \to \mathbf{R}$ be a functional on $E$. For instance, if $E = \mathbf{R}^n$ we may consider $\lambda = \lambda_n$ to be the projection on the $n$-th coordinate. We denote by $E_\lambda^0$ the kernel of $\lambda$, and by $E_\lambda^+$ (resp. $E_\lambda^-$) the set of points $x \in E$ such that $\lambda x \geqq 0$ (resp. $\lambda x \leqq 0$). We call $E_\lambda^0$ the **hyperplane** determined by $\lambda$, and we call $E_\lambda^+$ or $E_\lambda^-$ a **half space**. The terminology is justified by the natural pictures.

*If $\mu$ is another functional, and $E_\lambda^+ = E_\mu^+$, then there exists a number $c > 0$ such that $\lambda = c\mu$.*

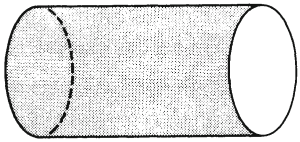
This is easily proved. Indeed, we see at once that the kernels of $\lambda$ and $\mu$ must be equal. Suppose that $\lambda \neq 0$. Let $x_0$ be such that $\lambda(x_0) > 0$. Then $\mu(x_0) > 0$ also. The functional

$$\lambda - c\mu$$

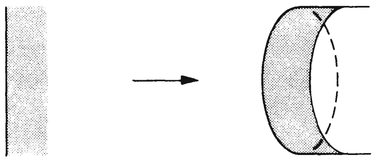
where $c = \lambda(x_0)/\mu(x_0)$ vanishes on the kernel of $\lambda$ (or $\mu$), and also on $x_0$. Therefore it is the 0 functional and $c$ satisfies our requirement.

In practice, we shall use mostly coordinate functions as functionals, and especially the $n$-th coordinate function. However, it is reasonable to use a slightly more invariant language which exhibits more directly the

geometric nature of the forthcoming constructions. We shall be interested in figures like the shell of a cylinder:



If we exclude the two end circles, then what is left is just an ordinary manifold. However, if we include the two end circles, then we have an object which, at each point of one of the end circles, does not look like some open set in 2-space, but rather looks like a point at the boundary of a half plane. In other words, we have a parametrization as indicated by the following picture:



We shall formulate the definitions and lemmas which allow us to give a formal development for such parametrizations.

Let $E$, $F$ be euclidean spaces, and let $E^+_\lambda$ and $F^+_\mu$ be two half spaces in $E$ and $F$, respectively. Let $U$, $V$ be open subsets of these half spaces respectively. We shall say that a mapping

$$f\colon U \to V$$

is of class $C^p$ if the following condition is satisfied. Given $x \in U$, there exists an open neighborhood $U_1$ of $x$ in $E$, and an open neighborhood $V_1$ of $f(x)$ in $F$, and a $C^p$ map $f_1\colon U_1 \to V_1$ such that the restriction of $f_1$ to $U_1 \cap U$ is equal to $f$. As usual, we take $p \geqq 1$.

We can now define a **manifold with boundary** in a manner entirely similar to the one used to define manifolds, namely by conditions **AT 1, AT 2, AT 3**, except that we take the $U_i$ of an atlas to be open subsets of half spaces. The notion of $C^p$-isomorphism is defined as usual by the condition of having a $C^p$-inverse.

We must make some remarks concerning the boundary, and we need some lemmas, e.g. to show that the boundary is a "differentiable invariant".

**Lemma 5.1.** *Let $U$ be open in $E$, and let $f: U \to F$ and $g: U \to F$ be two $C^p$ maps ($p \geqq 1$). Assume that $f$ and $g$ have the same restriction to $U \cap E_\lambda^+$ for some half space $E_\lambda^+$, and let $x \in U \cap E_\lambda^+$. Then $f'(x) = g'(x)$.*

*Proof.* After considering the difference of $f$ and $g$, we may assume without loss of generality that the restriction of $f$ to $U \cap E_\lambda^+$ is 0. It then follows that $f'(x) = 0$ because the directions of the half space span the whole space.

**Lemma 5.2.** *Let $U$ be open in $E$. Let $\mu$ be a non-zero functional on $F$, and let $f: U \to F_\mu^+$ be a $C^p$ map with $p \geqq 1$. Let $x$ be a point of $U$ such that $f(x)$ lies in $F_\mu^0$. Then $f'(x)$ maps $E$ into $F_\mu^0$.*

*Proof.* Without loss of generality, after translations, we may assume that $x = 0$ and $f(x) = 0$. Let $W$ be a given bounded neighborhood of 0 in $F$. Suppose that we can find an element $v \in E$ such that $\mu f'(0)v \neq 0$. We can write (for small $t > 0$):

$$f(tv) = tf'(0)v + o(t)w_t$$

with some element $w_t \in W$. By assumption, $f(tv)$ lies in $F_\mu^+$. Applying $\mu$, we get

$$t\mu f'(0)v + o(t)\mu(w_t) \geqq 0.$$

Dividing by $t$, this yields

$$\mu f'(0)v \geqq \frac{o(t)}{t}\mu(w_t).$$

Replacing $t$ by $-t$ we get a similar inequality on the other side. Letting $t$ tend to 0 shows that $\mu f'(0)v = 0$, a contradiction.

Let $U$ be open in some half plane $E_\lambda^+$. We define the **boundary** of $U$ (written $\partial U$) to be the intersection of $U$ with $E_\lambda^0$. We define the **interior** of $U$, written $\operatorname{Int}(U)$, to be the complement of $\partial U$ in $U$. Then $\operatorname{Int}(U)$ is open in $E$.

**Example.** Let $E_\lambda^+$ be a half space, with $\lambda \neq 0$. Then from our definition, we see that this half space is $C^\infty$ isomorphic to a product

$$E_\lambda^+ \approx E_\lambda^0 \times \mathbf{R}^+,$$

where $\mathbf{R}^+$ is the set of real numbers $\geqq 0$. The boundary in this case is $E_\lambda^0 \times \{0\}$.

**Lemma 5.3.** *Let $\lambda$ be a functional on $E$ and $\mu$ a functional on $F$. Let $U$ be open in $E_\lambda^+$ and $V$ open in $F_\mu^+$. Assume that $U \cap E_\lambda^0$ and $V \cap F_\mu^0$ are not empty. Let $f: U \to V$ be a $C^p$-isomorphism ($p \geqq 1$). Then $\lambda \neq 0$ if and only if $\mu \neq 0$. If $\lambda \neq 0$, then $f$ induces a $C^p$-isomorphism of* $\operatorname{Int}(U)$ *on* $\operatorname{Int}(V)$ *and of $\partial U$ on $\partial V$.*

*Proof.* For each $x \in U$, we conclude from the chain rule that $f'(x)$ is invertible. Our first assertion then follows from Lemma 5.2. We also see that no interior point of $U$ maps on a boundary point of $V$ and conversely. Thus $f$ induces a bijection of $\partial U$ and $\partial V$, and a bijection of $\operatorname{Int}(U)$ on $\operatorname{Int}(V)$. Since these interiors are open in their respective spaces, it follows that $f$ induces an isomorphism between them. As for the boundary, it is a submanifold of the full space, and locally, our definition of the derivative, together with the product structure, shows that the restriction of $f$ to $\partial U$ must be an isomorphism on $\partial V$.

We see that Lemma 5.3 gives us the invariance of the boundary under $C^p$ maps ($p \geqq 1$), first for open subsets of half spaces, but then also immediately for the boundary of a manifold since the property reduces at once to such subsets, under charts.

We can then describe local coordinates at a point in a manifold with boundary as follows. If the point is not a boundary point, then a neighborhood of this point is described by coordinates $(x_1, \ldots, x_n)$ in some open set of $\mathbf{R}^n$, which we may even take to contain 0, and such that 0 corresponds to the given point.

If the point is a boundary point, then an open neighborhood can be described by coordinates $(x_1, \ldots, x_n)$ satisfying

$$c_1 > x_n \geqq a, \qquad \text{and} \qquad x_1, \ldots, x_{n-1} \text{ lying in some open set of } \mathbf{R}^{n-1}.$$

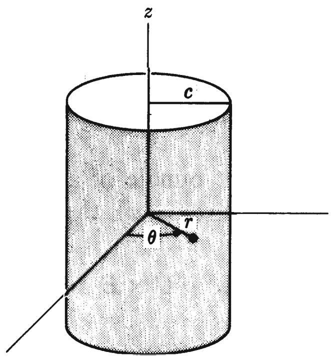
After a translation, we can even achieve an inequality $x_n \geqq 0$ instead of $x_n \geqq a$. The points with coordinates $x_n = 0$ are precisely those on the boundary. This comes from the fact that after a suitable choice of basis of $\mathbf{R}^n$, we can always achieve the result that a functional is simply the projection on the first coordinate of a suitable basis.

Similarly, we can define an embedded $k$-dimensional submanifold with boundary in $\mathbf{R}^n$, in terms of coordinates. Namely, we say that a subset $X$ of $\mathbf{R}^n$ is such a submanifold if for each $x \in X$ there exists an open set $U$ in $\mathbf{R}^n$ containing $x$, an open set $V$ in $\mathbf{R}^n$ and a $C^p$ isomorphism $\varphi: U \to V$ such that

$$\varphi(U \cap X) = V \cap (H^k \times \{c\}),$$

where $H^k$ is, say, the half plane in $\mathbf{R}^k$ defined by $x_k \geqq a$, and $c$ is a point $(c_{k+1}, \ldots, c_n)$ in $\mathbf{R}^{n-k}$.

**Example.** Consider the cylinder, conveniently placed vertically as follows:




We can define a chart for part of the cylinder in terms of the three coordinates $(r, \theta, z)$ satisfying the inequalities:

$$0 < \theta < \pi,$$

$$r = c,$$

$$0 < z \leqq a.$$

The map $\psi$ such that

$$\psi(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$$

is the inverse of the map $\varphi$ in the previous definition.

**The Tangent Spaces.** These can be defined much as for a manifold without boundary, since the charts and the spaces in which they lie can be used as before. For the equivalence classes between vectors, we needed the derivative of maps defining the changes of charts, but these are well defined independently of the manner in which one extends such changes of charts from half spaces to a full open set in the vector space.

**Partitions of Unity.** The theorem proved in §4 goes over without essential change to manifolds with boundary. Of course, the open balls in Theorem 4.1 have to be replaced with their intersections with half spaces.

## XXII, §6. VECTOR FIELDS AND GLOBAL DIFFERENTIAL EQUATIONS

Let $X$ be a manifold without boundary, of class $C^p$ with $p \geqq 2$. We assume that $X$ is Hausdorff. Let $\pi$: $T(X) \to X$ be the natural map of its tangent bundle onto $X$. We know that $T(X)$ is a manifold of class $C^{p-1}$, and that $\pi$ is of class $C^{p-1}$.

By a **vector field** on $X$ we mean a morphism (of class $C^{p-1}$)

$$\xi: X \to T(X)$$

such that $\xi(x)$ lies in the tangent space $T_x(X)$ for each $x \in X$, or in other words, such that $\pi \circ \xi = \mathrm{id}_X$. Thus a vector field assigns a tangent vector to each point.

When we identify the tangent bundle of an open set $U$ in $E$ with the product $U \times E$ relative to a chart $(U, \varphi)$, then we see that a vector field corresponds to a map

$$U \to U \times E$$

such that

$$\xi(x) = (x, f(x))$$

where $f: U \to E$ is a $C^{p-1}$ map. Thus a vector field is completely determined by the map $f$, which has been studied in Chapter XIV. We call $f$ the **local representation** of the vector field $\xi$ in the chart $(U, \varphi)$.

Let $J$ be an open interval of $\mathbf{R}$. The tangent bundle of $J$ is then naturally identifiable with $J \times \mathbf{R}$, since the identity map of $J$ is a global chart for $J$. In particular, we can view the number 1 as a tangent vector at each point, and we have a constant vector field over $J$ which takes this value 1 at all points.

Let $\alpha: J \to X$ be a curve, i.e. a map from an open interval $J$ into $X$. Assume that $\alpha$ is of class $C^1$. We want to take the derivative of $\alpha$. Locally at each point of $J$, we can shrink the domain of definition of $\alpha$ to a subinterval $J_0$ such that $\alpha(J_0)$ is contained in the domain of definition $U$ of a chart $(U, \varphi)$. Then the composite $\varphi \circ \alpha$ is a curve into $E$,

$$J_0 \xrightarrow{\alpha} U \xrightarrow{\varphi} \varphi U \subset E.$$

We can then take the derivative $(\varphi \circ \alpha)'(t)$ for $t \in J_0$ as a vector in $E$. This vector represents a tangent vector in $T_{\alpha(t)}(X)$, and it is immediately clear that if we change the chart to another $(V, \psi)$, then $(\psi \circ \alpha)'(t)$ represents the same tangent vector. In this way we obtain a curve which we shall denote by $\alpha'$, into the tangent bundle, namely

$$\alpha': J \to T(X),$$

which is such that $\alpha'(t)$ lies in $T_{\alpha(t)}(X)$. We shall also write $d\alpha/dt$ instead of $\alpha'(t)$, following standard notation, consistent with previous notation when we studied vector fields on open sets of vector spaces.

We say that $\alpha$ is an **integral curve** for the vector field $\xi$ if we have

$$\alpha'(t) = \xi(\alpha(t))$$

for all $t \in J$. If $J$ contains 0 and $\alpha(0) = x_0$, we say that $x_0$ is the **initial condition** of $\alpha$. The theorems on differential equations proved in Chapter XIV, §3, §4, §5 can now be formulated on manifolds.

*Let $\alpha_1: J_1 \to X$ and $\alpha_2: J_2 \to X$ be two integral curves of the vector field $\xi$ on $X$, with the same initial condition $x_0$. Then $\alpha_1$ and $\alpha_2$ are equal on $J_1 \cap J_2$.*

*Proof.* The proof is identical with that of Theorem 3.3 of Chapter XIV.

The preceding result allows us to define an integral curve with given initial condition $x$ on a maximal interval $J(x)$. Of course, the local existence theorem proved in Chapter XIV shows that such an integral curve exists. As before, we let $\mathfrak{D}(\xi)$ be the subset of $\mathbf{R} \times X$ consisting of all points $(t, x)$ such that $t \in J(x)$. We define a **global flow** for $\xi$ to be the map

$$\alpha: \mathfrak{D}(\xi) \to X$$

such that for each $x \in X$ the map $\alpha_x: J(x) \to X$ given by

$$\alpha_x(t) = \alpha(t, x)$$

is an integral curve for $\xi$ with initial condition $x$. When we select a chart at a point $x$ of $X$, then we see that this definition of flow coincides with the definition we gave for open sets in euclidean spaces for the local representation of our vector field. As in Chapter XIV, we abbreviate $\alpha(t, x)$ by $tx$.

**Theorem 6.1.** *Let $\xi$ be a vector field on $X$ and $\alpha$ its flow. Let $x \in X$. If $t_0$ lies in $J(x)$, then*

$$J(t_0 x) = J(x) - t_0$$

*and we have for all $t$ in $J(x) - t_0$:*

$$t(t_0 x) = (t + t_0)x.$$

*Proof.* Just like the proof of Theorem 5.1 of Chapter XIV.

**Theorem 6.2.** *Let $\xi$ be a vector field of class $C^{p-1}$ on the $C^p$ manifold $X$ $(2 \leqq p \leqq \infty)$. Then the domain $\mathfrak{D}(\xi)$ is open in $\mathbf{R} \times X$ and the flow $\alpha$ for $\xi$ is a $C^{p-1}$ morphism.*

*Proof.* Identical with the proof of Theorem 5.2, Chapter XIV.

**Corollary 6.3.** *For each* $t \in \mathbf{R}$, *the set of* $x \in X$ *such that* $(t, x)$ *is contained in the domain* $\mathfrak{D}(\xi)$ *is open in* $X$.

**Corollary 6.4.** *Let* $\mathfrak{D}_t(\xi)$ *be the set of points* $x$ *of* $X$ *such that* $(t, x)$ *lies in* $\mathfrak{D}(\xi)$. *Then* $\mathfrak{D}_t(\xi)$ *is open for each* $t \in \mathbf{R}$, *and* $\alpha_t$ *is a* $C^p$-*isomorphism of* $\mathfrak{D}_t(\xi)$ *onto an open subset of* $X$. *In fact,* $\alpha_t(\mathfrak{D}_t) = \mathfrak{D}_{-t}$ *and* $\alpha_t^{-1} = \alpha_{-t}$.

*Proof.* Immediate from Theorems 4.1 and 6.1.

**Corollary 6.5.** *If* $x_0$ *is a point of* $X$ *and* $t$ *is in* $J(x_0)$, *then there exists an open neighborhood* $U$ *of* $x_0$ *such that* $t$ *lies in* $J(x)$ *for all* $x \in U$, *and the map*

$$x \mapsto tx$$

*is an isomorphism of* $U$ *onto an open neighborhood of* $tx_0$.

In the present section, we have given the terminology which allows us to discuss differential equations on manifolds.

CHAPTER XXIII

# Integration and Measures on Manifolds

*Throughout this chapter, unless otherwise specified, we use the word manifold to denote manifolds possibly having boundaries. From §3 to the end, we let $X$ be a manifold of class $C^p$ with $p \geqq 1$, which is Hausdorff and has a countable base. These last two assumptions are to ensure that $X$ admits $C^p$ partitions of unity, subordinated to any given open covering.*

## XXIII, §1. DIFFERENTIAL FORMS ON MANIFOLDS

Let $X$ be a $C^p$ manifold, with $p$ always $\geqq 1$. To each tangent space $T_x(X) = T_x$ we can associate the dual space $T_x^*$, and the alternating product $\bigwedge^r T_x^*$. We form the union

$$\bigcup_{x \in X} \bigwedge^r T_x^* \qquad \text{denoted by} \qquad \bigwedge^r T^*(X).$$

By a **differential form** on $X$ (of degree $r$) we shall mean a map

$$\omega\colon X \to \bigwedge^r T^*(X)$$

such that for each $x$ the value $\omega(x)$ lies in $\bigwedge^r T_x^*$. (We shall add differentiability conditions in a moment.) The set of differential forms is a vector space denoted by $\Omega^r(X)$.

If $f\colon X \to Y$ is a $C^p$ map of manifolds, then we obtain an induced map

$$f^*\colon \Omega^r(Y) \to \Omega^r(X),$$

just as in the case of subsets of euclidean spaces, and arising from the induced linear map at each point,

$$T_x f\colon T_x \to T_{f(x)}.$$

(Cf. Theorem C of the Appendix, Chapter XXI, and Theorem 4.1 of Chapter XXI.)

Essentially as we did with tangent vectors, we can find local representations of differential forms in the corresponding euclidean space $\mathbf{R}^n = E$ of the manifold $X$. Indeed, let $x \in X$. Let

$$\varphi\colon U \to \varphi U \subset \mathbf{R}^n$$

be a chart at $x$. We have an isomorphism

$$\sigma_\varphi\colon \mathbf{R}^n \to T_x,$$

which to each vector $v \in \mathbf{R}^n$ associates the class of $(U, \varphi, v)$. If $\lambda$ is a functional on $T_x$, then $\lambda \circ \sigma_\varphi$ is a functional on $\mathbf{R}^n$. Let $\omega$ be a 1-form on $U$, and $\omega_x$ the value of $\omega$ at $x$. Then $\omega_x$ can be pulled back to $\mathbf{R}^n$, to obtain the form

$$\sigma_\varphi^*(\omega_x) = \omega_x \circ \sigma_\varphi$$

on $\mathbf{R}^n$. If $\omega$ is an $r$-form, and $x_1, \ldots, x_n$ are the coordinates of $x$ in $\mathbf{R}^n$, then there exist functions $g_{(i)}$ on $\varphi U$ such that

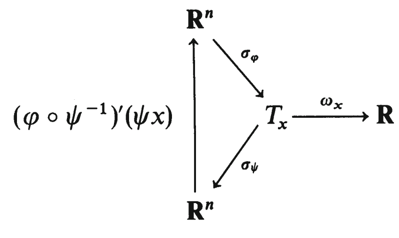
$$\sigma_\varphi^*(\omega_x) = \sum g_{(i)}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

and we say that the expression on the right is the **local expression** of $\omega$ determined by the chart, or corresponding to the chart $\varphi$. We shall also commit the abuse of notation, writing $g_{(i)}(x)$ instead of $g_{(i)}(x_1, \ldots, x_n)$. We abbreviate

$$\omega_x^\varphi = \sigma_\varphi^*(\omega_x)$$

for simplicity.

If $(V, \psi)$ is another chart such that $U \cap V$ is not empty (so that our two charts $(U, \varphi)$ and $(V, \psi)$ may be viewed as charts at a common point), then we obtain a representation of $\omega$ determined by $\psi$. If $\omega$ is a 1-form, then $\omega_x$ is a functional on $T_x$, and the pull backs of $\omega_x$ to $\mathbf{R}^n$ by $\sigma_\varphi$ and $\sigma_\psi$ respectively can be visualized in the following diagram:



The vertical map on the left is simply the derivative $(\varphi \circ \psi^{-1})'(\psi x)$, i.e. the derivative at $\psi x$ of the transition map $\varphi \circ \psi^{-1}$ giving the change of charts. In terms of local coordinates, the change in the local representation of $\omega_x$ is given in terms of partial derivatives, which are of class $C^{p-1}$. Similarly, for any $r$-form the change in the local representation is given by certain subdeterminants of the Jacobian matrix of $(\varphi \circ \psi^{-1})'(\psi x)$, and is again of class $C^{p-1}$. The most important case is that of an $n$-form, and we can then write $\omega$ locally as

$$\omega_x^{\varphi} = g(x)\, dx_1 \wedge \cdots \wedge dx_n.$$

If $(V, \psi)$ is the other chart, then we have explicitly

$$\omega_y^{\psi} = g(f(y))\Delta_f(y)\, dy_1 \wedge \cdots \wedge dy_n,$$

if $x = f(y)$ and $f = \varphi \circ \psi^{-1}$. As usual, $\Delta_f$ is the Jacobian determinant.

We say that $\omega$ is of **class** $C^{p-1}$ at a point, if in some local representation relative to a chart at that point, the functions $g_{(i)}$ as above are of class $C^{p-1}$. The remark in the preceding paragraph then shows that this will then be true for any local representation relative to any chart at that point. We say that $\omega$ is $C^{p-1}$ if it is of class $C^{p-1}$ at every point.

Theorems 3.1, 4.1, 4.2, 4.3 of Chapter XXI, concerning the operation $\omega \mapsto d\omega$, and the inverse image of a form now extend immediately to manifolds. In fact, the theorems of Chapter XXI give the expression for these operations on the local representation of forms. We define the wedge product $\omega \wedge \eta$ of two forms just as in the local case, according to the general algebraic result of Theorem B in the Appendix to Chapter XXI. We shall now repeat the statements of the theorems loc. cit. on manifolds.

*There exists a unique family of linear maps*

$$d\colon \Omega^r(X) \to \Omega^{r+1}(X) \qquad (r = 0, 1, 2, \ldots)$$

*defined on the space of r-forms (of class $C^q$, into the space of forms of class $C^{q-1}$), satisfying the properties that if* $\deg \omega = r$, *then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta,$$

*and $df = Tf$ if $f$ is a function, i.e. a form of degree* 0.

*If $f\colon X \to Y$ is a $C^p$ map, $p \geqq 1$, then for each r there exists a unique linear map*

$$f^*\colon \Omega^r(Y) \to \Omega^r(X)$$

*having the following properties*:

(i) *For any differential forms* $\omega$, $\eta$ *on* $Y$ *we have*

$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta).$$

(ii) *If* $g$ *is a function on* $Y$, *then* $f^*(g) = g \circ f$, *and if* $\omega$ *is a* 1-*form then*

$$(f^*\omega)(x) = \omega(f(x)) \circ T_x f.$$

*If* $g: Y \to Z$ *is a* $C^p$ *map, then*

$$(g \circ f)^* = f^* \circ g^*.$$

*Let* $f: X \to Y$ *be a* $C^2$ *map and* $\omega$ *a differential form of class* $C^1$ *on* $Y$. *Then*

$$f^*(d\omega) = df^*\omega.$$

*In particular, if* $g$ *is a* $C^1$ *function on* $Y$, *then*

$$f^*(dg) = d(g \circ f).$$

Observe that the operation $d$ loses one order of differentiability, and that if $f$ is of class $C^p$, then $Tf$ is of class $C^{p-1}$, so that $f^*\omega$ has the order of differentiability equal to the minimum of that of $Tf$ and $\omega$.

Our definition of the local representation of a differential form in terms of local coordinates is compatible with this operation of inverse image taken with respect to the map of a chart. In fact, if $(U, \varphi)$ is a chart, so that

$$\varphi: U \to \varphi U$$

is a $C^p$-isomorphism of $U$ onto an open set in a half space, and if $\omega$ is a differential form on $U$, then we can take

$$(\varphi^{-1})^*(\omega)$$

which is a differential form on $\varphi U$. The expression in local coordinates of $\omega$ is nothing but the expression of $(\varphi^{-1})^*(\omega)$ taken with respect to the identity chart of $\varphi U$ as a subset of $\mathbf{R}^n$. In the case of isomorphisms like charts, it is useful to use the notation $\varphi_*\omega$ instead of the inverse image we have just written.

We can define the **support** of a differential form as we define the support of a function. It is the closure of the set of all $x \in X$ such that $\omega(x) \neq 0$. If $\omega$ is a form of class $C^q$ and $\alpha$ is a $C^q$ function on $X$, then we

can form the product $\alpha\omega$, which is the form whose value at $x$ is $\alpha(x)\omega(x)$. If $\alpha$ has compact support, then $\alpha\omega$ has compact support. Later, we shall study the integration of forms, and reduce this to a local problem by means of partitions of unity, in which we multiply a form by functions.

If $X$ is a manifold and $Y$ a submanifold, then any differential form on $X$ induces a form on $Y$. We can view this as a very special case of the inverse image of a form, under the embedding (injection) map

$$\mathrm{id}\colon Y \to X.$$

In particular, if $Y$ has dimension $n - 1$, and if $(x_1, \ldots, x_n)$ is a system of coordinates for $X$ at some point of $Y$ such that the points of $Y$ correspond to those coordinates satisfying $x_j = c$ for some fixed number $c$, and index $j$, and if the form on $X$ is given in terms of these coordinates by

$$\omega(x) = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n,$$

then the restriction of $\omega$ to $Y$ (or the form induced on $Y$) has the representation

$$f(x_1, \ldots, c, \ldots, x_n)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where the roof over $dx_j$ means $dx_j$ is omitted. We should denote this induced form by $\omega_Y$, although occasionally we omit the subscript $Y$. We shall use such an induced form especially when $Y$ is the boundary of a manifold $X$.

## XXIII, §2. ORIENTATION

Let $U$, $V$ be open sets in half spaces of $\mathbf{R}^n$ and let $\varphi\colon U \to V$ be a $C^1$ isomorphism. We shall say that $\varphi$ is **orientation preserving** if the Jacobian determinant $\Delta_\varphi(x)$ is $> 0$, all $x \in U$. If the Jacobian determinant is negative, then we say that $\varphi$ is orientation **reversing**.

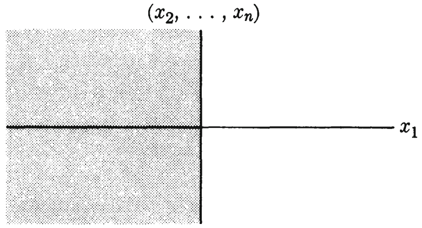
Let $X$ be a $C^p$ manifold, $p \geqq 1$, and let $\{(U_i, \varphi_i)\}$ be an atlas. We say that this atlas is **oriented** if all transition maps $\varphi_j \circ \varphi_i^{-1}$ are orientation preserving. Two atlases $\{(U_i, \varphi_i)\}$ and $\{(V_\alpha, \psi_\alpha)\}$ are said to **define the same orientation**, or to be **orientation equivalent**, if their union is oriented. We can also define locally a chart $(V, \psi)$ to be **orientation compatible** with the oriented atlas $\{(U_i, \varphi_i)\}$ if all transition maps $\varphi_i \circ \psi^{-1}$ (defined whenever $U_i \cap V$ is not empty) are orientation preserving. An orientation equivalence class of oriented atlases is said to define an **oriented** manifold, or to be an **orientation** of the manifold. It is a simple exercise to verify that if a connected manifold has an orientation, then it has two distinct orientations.

The standard examples of the Moebius strip or projective plane show that not all manifolds admit orientations. We shall now see that the boundary of an oriented manifold with boundary can be given a natural orientation.

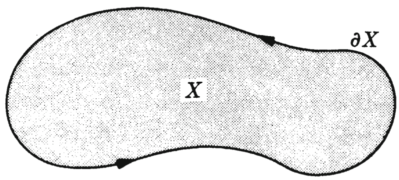
*Let $\varphi: U \to \mathbf{R}^n$ be an oriented chart at a boundary point of $X$, such that:*

(1) *if $(x_1, \ldots, x_n)$ are the local coordinates of the chart, then the boundary points correspond to those points in $\mathbf{R}^n$ satisfying $x_1 = 0$; and*
(2) *the points of $U$ not in the boundary have coordinates satisfying $x_1 < 0$.*

Then $(x_2, \ldots, x_n)$ are the local coordinates for a chart of the boundary, namely the restriction of $\varphi$ to $\partial X \cap U$, and the picture is as follows.




We may say that we have considered a chart $\varphi$ such that the manifold lies to the left of its boundary. If readers think of a domain in $\mathbf{R}^2$, having a smooth curve for its boundary, as on the following picture, they will see that our choice of chart corresponds to what is usually visualized as "counterclockwise" orientation.




*The collection of all pairs $(U \cap \partial X, \varphi|(U \cap \partial X))$, chosen according to the criteria described above, is obviously an atlas for the boundary $\partial X$, and we contend that it is an oriented atlas.*

We prove this easily as follows. If

$$(x_1, \ldots, x_n) = x \quad \text{and} \quad (y_1, \ldots, y_n) = y$$

are coordinate systems at a boundary point corresponding to choices of charts made according to our specifications, then we can write $y =$

$f(x)$ where $f = (f_1, \ldots, f_n)$ is the transition mapping. Since we deal with oriented charts for $X$, we know that $\Delta_f(x) > 0$ for all $x$. Since $f$ maps boundary into boundary, we have

$$f_1(0, x_2, \ldots, x_n) = 0$$

for all $x_2, \ldots, x_n$. Consequently the Jacobian matrix of $f$ at a point $(0, x_2, \ldots, x_n)$ is equal to

$$\begin{bmatrix} D_1 f_1(0, x_2, \ldots, x_n) & 0 \cdots 0 \\ * & \\ * & \Delta_g^{(n-1)} \\ * & \end{bmatrix},$$

where $\Delta_g^{(n-1)}$ is the Jacobian matrix of the transition map $g$ induced by $f$ on the boundary, and given by

$$\begin{aligned} y_2 &= f_2(0, x_2, \ldots, x_n), \\ &\vdots \qquad \vdots \\ y_n &= f_n(0, x_2, \ldots, x_n). \end{aligned}$$

However, we have

$$D_1 f_1(0, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f_1(h, x_2, \ldots, x_n)}{h},$$

taking the limit with $h < 0$ since by prescription, points of $X$ have coordinates with $x_1 < 0$. Furthermore, for the same reason we have

$$f_1(h, x_2, \ldots, x_n) < 0.$$

Consequently

$$D_1 f_1(0, x_2, \ldots, x_n) > 0.$$

From this it follows that $\Delta_g^{(n-1)}(x_2, \ldots, x_n) > 0$, thus proving our assertion that the atlas we have defined for $\partial X$ is oriented.

*From now on, when we deal with an oriented manifold, it is understood that its boundary is taken with orientation described above, and called the* **induced orientation.**

## XXIII, §3. THE MEASURE ASSOCIATED WITH A DIFFERENTIAL FORM

*Let $X$ be a manifold of class $C^p$ with $p \geqq 1$. We assume from now on that $X$ is Hausdorff and has a countable base. Then we know that $X$ admits $C^p$ partitions of unity, subordinated to any given open covering.*

(Actually, instead of the conditions we assumed, we could just as well have assumed the existence of $C^p$ partitions of unity, which is the precise condition to be used in the sequel.)

**Theorem 3.1.** *Let* $\dim X = n$ *and let* $\omega$ *be an n-form on* $X$ *of class* $C^0$, *i.e. continuous. Then there exists a unique positive functional* $\lambda$ *on* $C_c(X)$ *having the following property. If* $(U, \varphi)$ *is a chart and*

$$\omega(x) = f(x)\, dx_1 \wedge dx_n$$

*is the local representation of* $\omega$ *in this chart, then for any* $g \in C_c(X)$ *with support in* $U$, *we have*

$$\lambda g = \int_{\varphi U} g_\varphi(x) |f(x)|\, dx, \tag{1}$$

*where* $g_\varphi$ *represents* $g$ *in the chart* [*i.e.* $g_\varphi(x) = g(\varphi^{-1}(x))$], *and* $dx$ *is Lebesgue measure.*

*Proof.* The integral in (1) defines a positive functional on $C_c(U)$. The change of variables formula shows that if $(U, \varphi)$ and $(V, \psi)$ are two charts, and if $g$ has support in $U \cap V$, then the value of the functional is independent of the choice of charts. Thus we get a positive functional by the general localization theorem for measures or functionals (Theorem 5.1 of Chapter IX, §5), using partitions of unity.

The positive measure corresponding to the functional in Theorem 3.1 will be called the **measure associated with** $|\omega|$, and can be denoted by $\mu_{|\omega|}$.

Theorem 3.1 does not need any orientability assumption. With such an assumption, we have a similar theorem, obtained without taking the absolute value.

**Theorem 3.2.** *Let* $\dim X = n$ *and assume that* $X$ *is oriented. Let* $\omega$ *be an n-form on* $X$ *of class* $C^0$. *Then there exists a unique functional* $\lambda$ *on* $C_c(X)$ *having the following property. If* $(U, \varphi)$ *is an oriented chart and*

$$\omega(x) = f(x)\, dx_1 \wedge \cdots \wedge dx_n$$

*is the local representation of* $\omega$ *in this chart, then for any* $g \in C_c(X)$ *with support in* $U$, *we have*

$$\lambda g = \int_{\varphi U} g_\varphi(x) f(x)\, dx,$$

*where* $g_\varphi$ *represents* $g$ *in the chart, and* $dx$ *is Lebesgue measure.*

*Proof.* Since the Jacobian determinant of transition maps belonging to oriented charts is positive, we see that Theorem 3.2 follows like Theorem 3.1 from the change of variables formula (in which the absolute value sign now becomes unnecessary) and the existence of partitions of unity.

If $\lambda$ is the functional of Theorem 3.2, we shall call it the functional associated with $\omega$. For any function $g \in C_c(X)$, we define

$$\int_X g\omega = \lambda g.$$

If $\mu_{|\omega|}(X)$ is finite, then we know by general theory that we can extend $\lambda$ by continuity to $\mathscr{L}^1(|m|)$, where $m$ is the regular complex Borel measure associated with $\lambda$ (cf. Theorem 4.2, Chapter IX and also Exercise 9 of that chapter). If in particular $\omega$ has compact support, we can also proceed directly as follows. Let $\{\alpha_i\}$ be a partition of unity over $X$ such that each $\alpha_i$ has compact support. We define

$$\int_X \omega = \sum_i \int_X \alpha_i \omega,$$

all but a finite number of terms in this sum being equal to 0. As usual, it is immediately verified that this sum is in fact independent of the choice of partition of unity, and in fact, we could just as well use only a partition of unity over the support of $\omega$. Alternatively, if $\alpha$ is a function in $C_c(X)$ which is equal to 1 on the support of $\omega$, then we could also define

$$\int_X \omega = \int_X \alpha\omega.$$

It is clear that these two possible definitions are equivalent.

For an interesting theorem at the level of this chapter, see J. Moser's paper "On the volume element on a manifold," *Trans. Amer. Math. Soc.* **120** (December 1965) pp. 286–294.

## XXIII, §4. STOKES' THEOREM FOR A RECTANGULAR SIMPLEX

Let

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

be a rectangle in $n$-space, i.e. a product of $n$ closed intervals. The set-theoretic boundary in $R$ consists of the union over all $i = 1, \ldots, n$ of the

pieces

$$R_i^0 = [a_1, b_1] \times \cdots \times \{a_i\} \times \cdots \times [a_n, b_n],$$
$$R_i^1 = [a_1, b_1] \times \cdots \times \{b_i\} \times \cdots \times [a_n, b_n].$$

If

$$\omega(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

is an $(n-1)$-form, and the roof over anything means that this thing is to be omitted, then we define

$$\int_{R_i^0} \omega = \int_{a_1}^{b_1} \cdots \widehat{\int_{a_i}^{b_i}} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, a_i, \ldots, x_n)\, dx_1 \cdots \widehat{dx_i} \cdots dx_n,$$

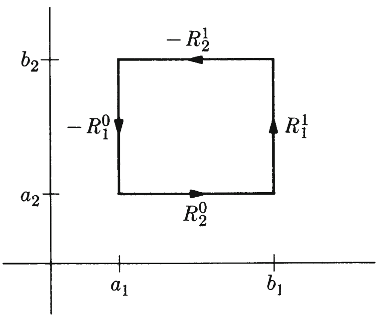
if $i = j$, and 0 otherwise. And similarly for the integral over $R_i^1$. We define the integral over the oriented boundary to be

$$\int_{\partial^0 R} = \sum_{i=1}^{n} (-1)^i \left[ \int_{R_i^0} - \int_{R_i^1} \right].$$

**Stokes' Theorem for Rectangles.** *Let $R$ be a rectangle in an open set $U$ in $n$-space. Let $\omega$ be an $(n-1)$-form on $U$. Then*

$$\int_R d\omega = \int_{\partial^0 R} \omega.$$

*Proof.* In two dimensions, the picture looks like this:




It suffices to prove the assertion when $\omega$ is a decomposable form, say

$$\omega(x) = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

We then evaluate the integral over the boundary of $R$. If $i \neq j$, then it is

clear that

$$\int_{R_i^0} \omega = 0 = \int_{R_i^1} \omega,$$

so that

$$\int_{\partial^0 R} \omega = (-1)^j \int_{a_1}^{b_1} \cdots \widehat{\int_{a_j}^{b_j}} \cdots \int_{a_n}^{b_n} [f(x_1, \ldots, a_j, \ldots, x_n) \\ - f(x_1, \ldots, b_j, \ldots, x_n)]\, dx_1 \cdots \widehat{dx_j} \cdots dx_n.$$

On the other hand, from the definitions we find that

$$\begin{aligned} d\omega(x) &= \left(\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n\right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

(The $(-1)^{j-1}$ comes from interchanging $dx_j$ with $dx_1, \ldots, dx_{j-1}$. All other terms disappear by the alternation rule.)

Integrating $d\omega$ over $R$, we may use repeated integration and integrate $\partial f/\partial x_j$ with respect to $x_j$ first. Then the fundamental theorem of calculus for *one* variable yields

$$\int_{a_j}^{b_j} \frac{\partial f}{\partial x_j} dx_j = f(x_1, \ldots, b_j, \ldots, x_n) - f(x_1, \ldots, a_j, \ldots, x_n).$$

We then integrate with respect to the other variables, and multiply by $(-1)^{j-1}$. This yields precisely the value found for the integral of $\omega$ over the oriented boundary $\partial^0 R$, and proves the theorem.

**Remark.** Stokes' theorem for a rectangle extends at once to a version in which we parametrize a subset of some space by a rectangle. Indeed, if $\sigma: R \to V$ is a $C^1$ map of a rectangle of dimension $n$ into an open set $V$ in $\mathbf{R}^N$, and if $\omega$ is an $(n-1)$-form in $V$, we may define

$$\int_\sigma d\omega = \int_R \sigma^* \, d\omega.$$

One can define

$$\int_{\partial\sigma} \omega = \int_{\partial^0 R} \sigma^* \omega,$$

and then we have a formula

$$\boxed{\int_\sigma d\omega = \int_{\partial\sigma} \omega.}$$

In the next section, we prove a somewhat less formal result.

## XXIII, §5. STOKES' THEOREM ON A MANIFOLD

**Theorem 5.1.** *Let $X$ be an oriented manifold of class $C^2$, dimension $n$, and let $\omega$ be an $(n-1)$-form on $X$, of class $C^1$. Assume that $\omega$ has compact support. Then*

$$\int_X d\omega = \int_{\partial X} \omega.$$

*Proof.* Let $\{\alpha_i\}_{i\in I}$ be a partition of unity, of class $C^2$. Then

$$\sum_{i\in I} \alpha_i\omega = \omega,$$

and this sum has only a finite number of non-zero terms since the support of $\omega$ is compact. Using the additivity of the operation $d$, and that of the integral, we find:

$$\int_X d\omega = \sum_{i\in I} \int_X d(\alpha_i\omega).$$

Suppose that $\alpha_i$ has compact support in some open set $V_i$ of $X$ and that we can prove

$$\int_{V_i} d(\alpha_i\omega) = \int_{V_i\cap\partial X} \alpha_i\omega,$$

in other words we can prove Stokes' theorem locally in $V_i$. We can write

$$\int_{V_i\cap\partial X} \alpha_i\omega = \int_{\partial X} \alpha_i\omega,$$

and similarly

$$\int_{V_i} d(\alpha_i\omega) = \int_X d(\alpha_i\omega).$$

Using the additivity of the integral one more, we get

$$\int_X d\omega = \sum_{i\in I} \int_X d(\alpha_i\omega) = \sum_{i\in I} \int_{\partial X} \alpha_i\omega = \int_{\partial X} \omega,$$

which yields Stokes' theorem on the whole manifold. Thus our argument with partitions of unity reduces Stokes' theorem to the local case, namely it suffices to prove that for each point of $X$ there exists an open neighborhood $V$ such that if $\omega$ has compact support in $V$, then Stokes' theorem holds with $X$ replaced by $V$. We now do this.

If the point is not a boundary point, we take an oriented chart $(U, \varphi)$ at the point, containing an open neighborhood $V$ of the point, satisfying the following conditions: $\varphi U$ is an open ball, and $\varphi V$ is the interior of a rectangle, whose closure is contained in $\varphi U$. If $\omega$ has compact support in $V$, then its local representation in $\varphi U$ has compact support in $\varphi V$. Applying Stokes' theorem for rectangles as proved in the preceding section, we find that the two integrals occurring in Stokes' formula are equal to 0 in this case (the integral over an empty boundary being equal to 0 by convention).

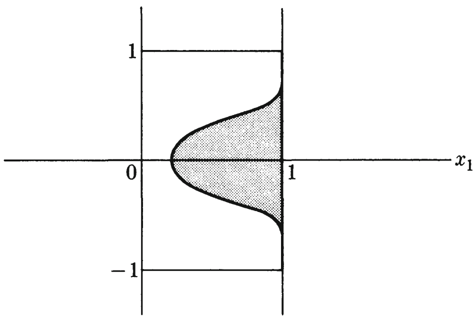
Now suppose that we deal with a boundary point. We take an oriented chart $(U, \varphi)$ at the point, having the following properties. First, $\varphi U$ is described by the following inequalities in terms of local coordinates $(x_1, \ldots, x_n)$:

$$-2 < x_1 \leqq 1 \qquad \text{and} \qquad -2 < x_j < 2 \qquad \text{for} \quad j = 2, \ldots, n.$$

Next, the given point has coordinates $(1, 0, \ldots, 0)$, and that part of $U$ on the boundary of $X$, namely $U \cap \partial X$, is given in terms of these coordinates by the equation $x_1 = 1$. We then let $V$ consist of those points whose local coordinates satisfy

$$0 < x_1 \leqq 1 \qquad \text{and} \qquad -1 < x_j < 1 \qquad \text{for} \quad j = 2, \ldots, n.$$

If $\omega$ has compact support in $V$, then $\omega$ is equal to 0 on the boundary of the rectangle $R$ equal to the closure of $\varphi V$, except on the face given by $x_1 = 1$, which defines that part of the rectangle corresponding to $\partial X \cap V$. Thus the support of $\omega$ looks like the shaded portion of the following picture.




In the sum giving the integral over the boundary of a rectangle as in the previous section, only one term will give a non-zero contribution, corre-

sponding to $i = 1$, which is

$$(-1)\left[\int_{R_1^0} \omega - \int_{R_1^1} \omega\right].$$

Furthermore, the integral over $R_1^0$ will also be 0, and in the contribution of the integral over $R_1^1$, the two minus signs will cancel, and yield the integral of $\omega$ over the part of the boundary lying in $V$, because our charts are so chosen that $(x_2, \ldots, x_n)$ is an oriented system of coordinates for the boundary. Thus we find

$$\int_V d\omega = \int_{V \cap \partial X} \omega,$$

which proves Stokes' theorem locally in this case, and concludes the proof of Theorem 5.1.

For any number of reasons, some of which we consider in the next section, it is useful to formulate conditions under which Stokes' theorem holds even when the form $\omega$ does not have compact support. We shall say that $\omega$ has **almost compact support** if there exists a decreasing sequence of open sets $\{U_k\}$ in $X$ such that the intersection

$$\bigcap_{k=1}^{\infty} U_k$$

is empty, and a sequence of $C^1$ functions $\{g_k\}$, having the following properties:

**AC 1.** *We have* $0 \leqq g_k \leqq 1$, $g_k = 1$ *outside* $U_k$, *and* $g_k \omega$ *has compact support.*

**AC 2.** *If* $\mu_k$ *is the measure associated with* $|dg_k \wedge \omega|$ *on* $X$, *then*

$$\lim_{k \to \infty} \mu_k(\overline{U}_k) = 0.$$

We then have the following application of Stokes' theorem.

**Corollary 5.2.** *Let* $X$ *be a* $C^2$ *oriented manifold, of dimension* $n$, *and let* $\omega$ *be an* $(n-1)$*-form on* $X$, *of class* $C^1$. *Assume that* $\omega$ *has almost compact support, and that the measures associated with* $|d\omega|$ *on* $X$ *and* $|\omega|$ *on* $\partial X$ *are finite. Then*

$$\int_X d\omega = \int_{\partial X} \omega.$$

*Proof.* By our standard form of Stokes' theorem we have

$$\int_{\partial X} g_k\omega = \int_X d(g_k\omega) = \int_X dg_k \wedge \omega + \int_X g_k\, d\omega.$$

We estimate the left-hand side by

$$\left|\int_{\partial X} \omega - \int_{\partial X} g_k\omega\right| = \left|\int_{\partial X} (1 - g_k)\omega\right| \leqq \mu_{|\omega|}(U_k \cap \partial X).$$

Since the intersection of the sets $U_k$ is empty, it follows for a purely measure-theoretic reason that

$$\lim_{k\to\infty} \int_{\partial X} g_k\omega = \int_{\partial X} \omega.$$

Similarly,

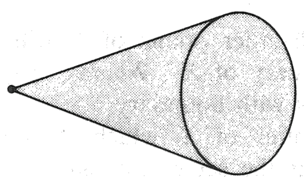
$$\lim_{k\to\infty} \int_X g_k\, d\omega = \int_X d\omega.$$

The integral of $dg_k \wedge \omega$ over $X$ approaches 0 as $k \to \infty$ by assumption, and the fact that $dg_k \wedge \omega$ is equal to 0 on the complement of $\overline{U}_k$ since $g_k$ is constant on this complement. This proves our corollary.

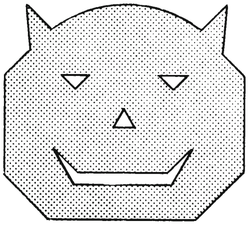
The above proof shows that the second condition **AC 2** is a very natural one to reduce the integral of an arbitrary form to that of a form with compact support. In the next section, we relate this condition to a question of singularities when the manifold is embedded in some bigger space.

## XXIII, §6. STOKES' THEOREM WITH SINGULARITIES

If $X$ is a compact manifold, then of course every differential form on $X$ has compact support. However, the version of Stokes' theorem which we have given is useful in contexts when we start with an object which is not a manifold, say as a subset of $\mathbf{R}^n$, but is such that when we remove a portion of it, what remains is a manifold. For instance, consider a cone (say the solid cone) as illustrated in the next picture:



The vertex and the circle surrounding the base disc prevent the cone from being a submanifold of $\mathbf{R}^3$. However, if we delete the vertex and this circle, what remains is a submanifold with boundary embedded in $\mathbf{R}^3$. The boundary consists of the conical shell, and of the base disc (without its surrounding circle). Another example is given by polyhedra, as on the following figure.



The idea is to approximate a given form by a form with compact support, to which we can apply Theorem 5.1, and then take the limit. We shall indicate one possible technique to do this.

The word "boundary" has been used in two senses: the sense of point set topology, and the sense of boundary of a manifold. Up to now, they were used in different contexts so no confusion could arise. We must now make a distinction, and therefore use the word boundary only in its manifold sense. If $X$ is a subset of $\mathbf{R}^N$, we denote its closure by $\overline{X}$ as usual. We call the set theoretic difference $\overline{X} - X$ the **frontier** of $X$ in $\mathbf{R}^N$, and denote it by $\mathrm{fr}(X)$.

Let $X$ be a submanifold without boundary of $\mathbf{R}^N$, of dimension $n$. We know that this means that at each point of $X$ there exists a chart for an open neighborhood of this point in $\mathbf{R}^N$ such that the points of $X$ in this chart correspond to a factor in a product, just as in Chapter XXII, §2. A point $P$ of $\overline{X} - X$ will be called a **regular** frontier point of $X$ if there exists a chart at $P$ in $\mathbf{R}^N$ with local coordinates $(x_1, \ldots, x_N)$ such that $P$ has coordinates $(0, \ldots, 0)$; the points of $X$ are those with coordinates

$$x_{n+1} = \cdots = x_N = 0 \qquad \text{and} \qquad x_n < 0;$$

and the points of the frontier of $X$ which lie in the chart are those with coordinates satisfying

$$x_n = x_{n+1} = \cdots = x_N = 0.$$

The set of all regular frontier points of $X$ will be denoted by $\partial X$, and will be called the **boundary** of $X$. We may say that $X \cup \partial X$ is a submanifold of $\mathbf{R}^N$, possibly with boundary.

A point of the frontier of $X$ which is not regular will be called **singular**. It is clear that the set of singular points is closed in $\mathbf{R}^N$. We

now formulate a version of Theorem 5.1 when $\omega$ does not necessarily have compact support in $X \cup \partial X$. Let $S$ be a subset of $\mathbf{R}^N$. By a **fundamental sequence** of open neighborhoods of $S$ we shall mean a sequence $\{U_k\}$ of open sets containing $S$ such that, if $W$ is an open set containing $S$, then $U_k \subset W$ for all sufficiently large $k$.

Let $S$ be the set of singular frontier points of $X$ and let $\omega$ be a form defined on an open neighborhood of $\bar{X}$, and having compact support. The intersection of $\operatorname{supp} \omega$ with $(X \cup \partial X)$ need not be compact, so that we cannot apply Theorem 5.1 as it stands. The idea is to find a fundamental sequence of neighborhoods $\{U_k\}$ of $S$, and a function $g_k$ which is 0 on a neighborhood of $S$ and 1 outside $U_k$ so that $g_k\omega$ differs from $\omega$ only inside $U_k$. We can then apply Theorem 5.1 to $g_k\omega$ and we hope that taking the limit yields Stokes' theorem for $\omega$ itself. However, we have

$$\int_X d(g_k\omega) = \int_X dg_k \wedge \omega + \int_X g_k\, d\omega.$$

Thus we have an extra term on the right, which should go to 0 as $k \to \infty$ if we wish to apply this method. In view of this, we make the following definition.

Let $S$ be a closed subset of $\mathbf{R}^N$. We shall say that $S$ is **negligible for** $X$ if there exists an open neighborhood $U$ of $S$ in $\mathbf{R}^N$, a fundamental sequence of open neighborhoods $\{U_k\}$ of $S$ in $U$, with $\bar{U}_k \subset U$, and a sequence of $C^1$ functions $\{g_k\}$, having the following properties.

**NEG 1.** *We have* $0 \leqq g_k \leqq 1$. *Also*, $g_k(x) = 0$ *for* $x$ *in some open neighborhood of* $S$, *and* $g_k(x) = 1$ *for* $x \notin U_k$.

**NEG 2.** *If* $\omega$ *is an* $(n-1)$*-form of class* $C^1$ *on* $U$, *and* $\mu_k$ *is the measure associated with* $|dg_k \wedge \omega|$ *on* $U \cap X$, *then* $\mu_k$ *is finite for large* $k$, *and*

$$\lim_{k\to\infty} \mu_k(U \cap X) = 0.$$

From our first condition, we see that $g_k\omega$ vanishes on an open neighborhood of $S$. Since $g_k = 1$ on the complement of $\bar{U}_k$, we have $dg_k = 0$ on this complement, and therefore our second condition implies that the measures induced on $X$ near the singular frontier by $|dg_k \wedge \omega|$ (for $k = 1, 2, \ldots$), are concentrated on shrinking neighborhoods and tend to 0 as $k \to \infty$.

**Theorem 6.1 (Stokes' Theorem with Singularities).** *Let* $X$ *be an oriented,* $C^2$ *submanifold without boundary of* $\mathbf{R}^N$. *Let* $\dim X = n$. *Let* $\omega$ *be an* $(n-1)$*-form of class* $C^1$ *on an open neighborhood of* $\bar{X}$ *in* $\mathbf{R}^N$, *and with compact support. Assume that:*

(i) *If* $S$ *is the set of singular points in the frontier of* $X$, *then* $S \cap \operatorname{supp} \omega$ *is negligible for* $X$.

(ii) *The measures associated with* $|d\omega|$ *on* $X$, *and* $|\omega|$ *on* $\partial X$, *are finite. Then*

$$\int_X d\omega = \int_{\partial X} \omega.$$

*Proof.* Let $U$, $\{U_k\}$, and $\{g_k\}$ satisfy conditions **NEG 1** and **NEG 2**. Then $g_k\omega$ is 0 on an open neighborhood of $S$, and since $\omega$ is assumed to have compact support, one verifies immediately that

$$(\operatorname{supp} g_k\omega) \cap (X \cup \partial X)$$

is compact. Thus Theorem 5.1 is applicable, and we get

$$\int_{\partial X} g_k\omega = \int_X d(g_k\omega) = \int_X dg_k \wedge \omega + \int_X g_k\, d\omega.$$

We have

$$\left|\int_{\partial X} \omega - \int_{\partial X} g_k\omega\right| \leqq \left|\int_{\partial X} (1 - g_k)\omega\right|$$

$$\leqq \int_{U_k \cap \partial X} 1\, d\mu_{|\omega|} = \mu_{|\omega|}(U_k \cap \partial X).$$

Since the intersection of all sets $U_k \cap \partial X$ is empty, it follows for purely measure theoretic reasons that the limit of the right-hand side is 0 as $k \to \infty$. Thus

$$\lim_{k\to\infty} \int_{\partial X} g_k\omega = \int_{\partial X} \omega.$$

For similar reasons, we have

$$\lim_{k\to\infty} \int_X g_k\, d\omega = \int_X d\omega.$$

Our second assumption **NEG 2** guarantees that the integral of $dg_k \wedge \omega$ over $X$ approaches 0. This proves our theorem.

We shall now give criteria for a set to be negligible.

**Criterion 1.** *Let* $S$, $T$ *be compact negligible sets for a submanifold* $X$ *of* $\mathbf{R}^N$ (*assuming* $X$ *without boundary*). *Then the union* $S \cup T$ *is negligible for* $X$.

*Proof.* Let $U$, $\{U_k\}$, $\{g_k\}$ and $V$, $\{V_k\}$, $\{h_k\}$ be triples associated with $S$ and $T$, respectively, as in conditions **NEG 1** and **NEG 2** (with $V$ replacing $U$ and $h$ replacing $g$ when $T$ replaces $S$). Let

$$W = U \cup V, \qquad W_k = U_k \cup V_k, \qquad \text{and} \qquad f_k = g_k h_k.$$

Then the open sets $\{W_k\}$ form a fundamental sequence of open neighborhoods of $S \cup T$ in $W$, and **NEG 1** is trivially satisfied. As for **NEG 2**, we have

$$d(g_k h_k) \wedge \omega = h_k\, dg_k \wedge \omega + g_k\, dh_k \wedge \omega,$$

so that **NEG 2** is also trivially satisfied, thus proving our criterion.

**Criterion 2.** *Let $X$ be an open set, and let $S$ be a compact subset in $\mathbf{R}^n$. Assume that there exists a closed rectangle $R$ of dimension $m \leqq n-2$ and a $C^1$ map $\sigma: R \to \mathbf{R}^n$ such that $S = \sigma(R)$. Then $S$ is negligible for $X$.*

Before giving the proof, we make a couple of simple remarks. First, we could always take $m = n - 2$, since any parametrization by a rectangle of dimension $< n - 2$ can be extended to a parametrization by a rectangle of dimension $n - 2$ simply by projecting away extra coordinates. Second, by our first criterion, we see that a finite union of sets as described above, i.e. parametrized smoothly by rectangles of codimension $\geqq 2$, is negligible. Third, our Criterion 2, combined with the first criterion, shows that negligibility in this case is local, i.e. we can subdivide a rectangle into small pieces.

We now prove Criterion 2. Composing $\sigma$ with a suitable linear map, we may assume that $R$ is a unit cube. We cut up each side of the cube into $k$ equal segments and thus get $k^m$ small cubes. Since the derivative of $\sigma$ is bounded on a compact set, the image of each small cube is contained in an $n$-cube in $\mathbf{R}^n$ of radius $\leqq C/k$ (by the mean value theorem), whose $n$-dimensional volume is $\leqq (2C)^n/k^n$. Thus we can cover the image by small cubes such that the sum of their $n$-dimensional volumes is $\leqq (2C)^n/k^{n-m} \leqq (2C)^n/k^2$.

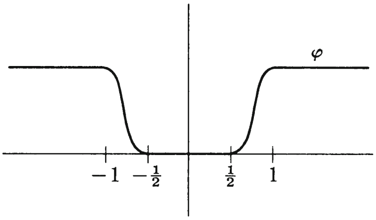
**Lemma 6.2.** *Let $S$ be a compact subset of $\mathbf{R}^n$. Let $U_k$ be the open set of points $x$ such that $d(x, S) < 2/k$. There exists a $C^\infty$ function $g_k$ on $\mathbf{R}^n$ which is equal to $0$ in some open neighborhood of $S$, equal to $1$ outside $U_k$, $0 \leqq g_k < 1$, and such that all partial derivatives of $g_k$ are bounded by $C_1 k$, where $C_1$ is a constant depending only on $n$.*

*Proof.* Let $\varphi$ be a $C^\infty$ function such that $0 \leqq \varphi \leqq 1$, and

$$\varphi(x) = 0 \quad \text{if} \quad 0 \leqq \|x\| \leqq 1/2,$$

$$\varphi(x) = 1 \quad \text{if} \quad 1 \leqq \|x\|.$$

We use $\| \ \|$ for the sup norm in $\mathbf{R}^n$. The graph of $\varphi$ looks like this:




For each positive integer $k$, let $\varphi_k(x) = \varphi(kx)$. Then each partial derivative $D_i\varphi_k$ satisfies the bound

$$|D_i\varphi_k| \leqq k|D_i\varphi|,$$

which is thus bounded by a constant times $k$. Let $L$ denote the lattice of integral points in $\mathbf{R}^n$. For each $l \in L$, we consider the function

$$x \mapsto \varphi_k\left(x - \frac{l}{2k}\right).$$

This function has the same shape as $\varphi_k$ but is translated to the point $l/2k$. Consider the product

$$g_k(x) = \prod \varphi_k\left(x - \frac{l}{2k}\right)$$

taken over all $l \in L$ such that $d(l/2k, S) \leqq 1/k$. If $x$ is a point of $\mathbf{R}^n$ such that $d(x, S) < 1/4k$, then we pick an $l$ such that

$$d(x, l/2k) \leqq 1/2k.$$

For this $l$ we have $d(l/2k, S) < 1/k$, so that this $l$ occurs in the product, and

$$\varphi_k(x - l/2k) = 0.$$

Therefore $g_k$ is equal to 0 in an open neighborhood of $S$. If on the other hand we have $d(x, S) > 2/k$ and if $l$ occurs in the product, that is

$$d(l/2k, S) \leqq 1/k,$$

then

$$d(x, l/2k) > 1/k$$

and hence $g_k(x) = 1$. The partial derivatives of $g_k$ are bounded in the

desired manner. This is easily seen, for if $x_0$ is a point where $g_k$ is not identically 1 in a neighborhood of $x_0$, then $\|x_0 - l_0/2k\| \leqq 1/k$ for some $l_0$. All other factors $\varphi_k(x - l/2k)$ will be identically 1 near $x_0$ unless $\|x_0 - l/2k\| \leqq 1/k$. But then $\|l - l_0\| \leqq 4$ whence the number of such $l$ is bounded as a function of $n$ (in fact by $9^n$). Thus when we take the derivative, we get a sum of at most $9^n$ terms, each one having a derivative bounded by $C_1 k$ for some constant $C_1$. This proves our lemma.

We return to the proof of Criterion 2. We observe that when an $(n-1)$-form $\omega$ is expressed in terms of its coordinates,

$$\omega(x) = \sum f_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

then the coefficients $f_j$ are bounded on a compact neighborhood of $S$. We take $U_k$ as in the lemma. Then for $k$ large, each function

$$x \mapsto f_j(x) D_j g_k(x)$$

is bounded on $U_k$ by a bound $C_2 k$, where $C_2$ depends on a bound for $\omega$, and on the constant of the lemma. The Lebesgue measure of $U_k$ is bounded by $C_3/k^2$, as we saw previously. Hence the measure of $U_k$ associated with $|dg_k \wedge \omega|$ is bounded by $C_3/k$, and tends to 0 as $k \to \infty$. This proves our criterion.

As an example, we now state a simpler version of Stokes' theorem, applying our criteria.

**Theorem 6.3.** *Let $X$ be an open subset of $\mathbf{R}^n$. Let $S$ be the set of singular points in the closure of $X$, and assume that $S$ is the finite union of $C^1$ images of m-rectangles with $m \leqq n-2$. Let $\omega$ be an $(n-1)$-form defined on an open neighborhood of $\overline{X}$. Assume that $\omega$ has compact support, and that the measures associated with $|\omega|$ on $\partial X$ and with $|d\omega|$ on $X$ are finite. Then*

$$\int_X d\omega = \int_{\partial X} \omega.$$

*Proof.* Immediate from our two criteria and Theorem 6.1.

We can apply Theorem 6.3 when, for instance, $X$ is the interior of a polyhedron, whose interior is open in $\mathbf{R}^n$. When we deal with a submanifold $X$ of dimension $n$, embedded in a higher dimensional space $\mathbf{R}^N$, then one can reduce the analysis of the singular set to Criterion 2 provided that there exists a finite number of charts for $X$ near this singular set on which the given form $\omega$ is bounded. This would for instance be the case with the surface of our cone mentioned at the

beginning of the section. Criterion 2 is also the natural one when dealing with manifolds defined by algebraic inequalities. By using the resolution of singularities due to Hironaka one can parametrize a compact set of algebraic singularities as in Criterion 2.

Finally, we note that the condition that $\omega$ have compact support in an open neighborhood of $\bar{X}$ is a very mild condition. If for instance $X$ is a bounded open subset of $\mathbf{R}^n$, then $\bar{X}$ is compact. If $\omega$ is any form on some open set containing $\bar{X}$, then we can find another form $\eta$ which is equal to $\omega$ on some open neighborhood of $\bar{X}$ and which has compact support. The integrals of $\eta$ entering into Stokes' formula will be the same as those of $\omega$. To find $\eta$, we simply multiply $\omega$ with a suitable $C^\infty$ function which is 1 in a neighborhood of $\bar{X}$ and vanishes a little further away. Thus Theorem 6.3 provides a reasonably useful version of Stokes' theorem which can be applied easily to all the cases likely to arise naturally.

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# Table of Notation

| | |
|---|---|
| $A^*$: | adjoint, p. 106 |
| $B(X, E)$: | bounded mappings of $X$ into $E$, p. 19 |
| $\mathrm{BV}(\mathbf{R})$: | space of functions of bounded variation on $\mathbf{R}$, p. 284 |
| $C^p$: | $p$-times continuously differentiable mappings, p. 346 |
| $C_c(X)$: | space of continuous functions with compact support on $X$, p. 252 |
| $C_c^\infty(X)$: | infinitely differentiable functions with compact support on $X$, pp. 167, 296 |
| $D_A$: | domain of an unbounded operator $A$, p. 469 |
| $\mathrm{End}(E)$: | continuous linear maps of $E$ into itself, p. 73 |
| $f^\wedge$ or $\hat{f}$: | Fourier transform of $f$, pp. 238, 288 |
| $f_A$: | the function or mapping which is equal to $f$ on $A$ and 0 outside $A$, p. 4 The same notation is also used in the theory of the double Fourier transform, for a truncation of the double Fourier transform, p. 288 |
| $f * g$: | convolution, p. 223 |
| $\mathrm{Hilb}(E)$: | norm-preserving automorphisms of a Hilbert space $E$, p. 439 |
| Int: | interior, pp. 22, 497 |
| $L(E, F)$: | space of continuous linear maps of $E$ into $F$, p. 65 |
| $\mathrm{Laut}(E)$: | continuous linear automorphisms of $E$, p. 67 |
| $\mathrm{Lis}(E, F)$: | continuous linear isomorphisms of $E$ with $F$, p. 67 |
| $\mathscr{L}^1$: | space of absolutely integrable functions or mappings, p. 128 |
| $L^1$: | equivalence classes of the above, p. 128 |
| $L^p$: | similar to the above, p. 209 |
| $M^1(\mathscr{M}, E)$: | $E$-valued measures, p. 199 |
| $M^1(\mu, E)$: | $\mu$-continuous $E$-valued measures, p. 205 |

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